(1) Let P be any arbitrary partition of $[0,1]$.

i.e. $P = \{0 = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = 1\}$

Since, rationals and irrationals are dense in $\mathbb{R}$.

$[x_{k-1}, x_k)$ contains infinitely many rationals and irrationals for all $k = 1, 2, \dots, n$.

$$\therefore U(P,f) = \sum_{k=1}^{n} \sup_{t\in[x_{k-1},x_k]} f(t).(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} 1.(x_k - x_{k-1}) \quad (\because (x_{k-1}, x_k) \text{ contains infinitely many irrationals})$$

$$= 1 \quad \text{, for any partition } P$$

$$\& \ L(P,f) = \sum_{k=1}^{n} \inf_{t\in[x_{k-1},x_k]} f(t)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} 0.(x_k - x_{k-1}) \quad (\because (x_{k-1}, x_k) \text{ contains infinitely many rationals})$$

$$= 0 \quad \text{, for any partition } P.$$

$\Rightarrow$ Limit of lower sum of f is $\sup_P L(P,f) = \sup_P 0 = 0$

So, option (D) is correct

Limit of upper sum of f is $\inf_P U(P,f) = \inf_P 1 = 1$

So, option (C) is correct.

Since, the upper sum & lower sum are not equal, f is not Riemann integrable.

So, option (A) is correct & (B) is not correct.

② Given that $A_n \subseteq X, \quad \forall n \in \mathbb{N}$

$x \in \limsup A_n \iff x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$

$\iff x \in \bigcup_{k=n}^{\infty} A_k, \ \forall n$

$\iff \forall n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ (depending on n) such that $m \geq n$ and $x \in A_m$.

$\iff x \in A_k$, for infinitely many k

So, option (A) is correct.

options (B), (C) & (D) are not correct.

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③ $x \in \liminf A_n \iff x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$

$\iff x \in \bigcap_{k=n}^{\infty} A_k$, for some $n \in \mathbb{N}$

$\iff$ ~~for some xe~~

$\iff$ there exist $n \in \mathbb{N}$, such that $x \in A_k, \ \forall k \geq n$.

$\iff x \in A_k$ for all but finitely many k.

So, option (D) is correct.

Options (A), (B) & (C) are not correct.

(4)

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$= \bigcap_{n=1}^{\infty} (A \cup B) \quad (\because \text{By definition of } A_k)$$

$$= A \cup B$$

So, option (A) is correct.

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

$$= \bigcup_{n=1}^{\infty} (A \cap B) \quad (\because \text{by definition of } A_k)$$

$$= A \cap B$$

So, option (D) is correct.

Since, A & B are proper distinct subsets of X,

$A \cap B \subsetneq A \cup B$.

so, options (B) & (C) are not correct.

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(5) (A) $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$

since, $\{q\}$ is Borel measurable, $\forall q \in \mathbb{Q}$. & $\mathbb{Q}$ is countable.

$\mathbb{Q}$ is the countable union of Borel measurable sets.

$\therefore \mathbb{Q} \in \mathcal{B}(\mathbb{R})$.

Option (A) is correct.

(B) Let $X = \{x : 0 \le x \le 1 \text{ \& } x \text{ is irrational}\}$.

Then $X = [0,1] \cap \mathbb{Q}^c$

since, $[0,1], \mathbb{Q} \in \mathcal{B}(\mathbb{R})$, $\mathbb{Q}^c \in \mathcal{B}(\mathbb{R})$

$\therefore X = [0,1] \cap \mathbb{Q}^c \in \mathcal{B}(\mathbb{R})$

is $X \in \mathcal{B}(\mathbb{R})$.

$\therefore$ option (B) is correct.

(C) $$[a,b) = \bigcap_{n=1}^{\infty} (a-1/n, b) \quad \cancel{\text{where}}\ b>a$$

$[a,b)$ is the countable intersection of open sets.

$\therefore [a,b) \in \mathcal{B}(\mathbb{R})$.

$\therefore$ option (C) is correct.

(D). Since $[0,1]\times[0,1]$ is a closed set,

$$[0,1]\times[0,1] \in \mathcal{B}(\mathbb{R}^2)$$

option (D) is correct.

---

(6) The sets which are given in the options (A), (B), (C) & (D) are closed.

$\therefore$ ~~Then~~ All the sets are Borel measurable.

$\therefore$ Options (A), (B), (C) & (D) are correct.

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(7) (A) Let $\mathcal{F}$ be the σ-algebra generated by open balls of rational radii.

Let $r>0$ and (r_m) be an increasing sequence of rationals converging to r.

Then $$B(a,r) = \bigcup_{m=1}^{\infty} B(a, r_m)$$

$\therefore \mathcal{F}$ contains all the open balls.

Since, $\mathcal{B}(\mathbb{R}^n)$ is the smallest σ-algebra containing all the open sets & it can be generated by open balls.

$$\mathcal{F} = \mathcal{B}(\mathbb{R}^n).$$

$\therefore$ Option (A) is correct.

(B)

Let $\mathcal{F}$ be the σ-algebra generated by closed balls of rational radii.

Let $r>0$ & (r_m) be the increasing sequence of rationals converging to r

Then $B(a,r) = \bigcup_{m=1}^{\infty} \overline{B(a,r_m)}$

$\therefore$ $\mathcal{F}$ contains all the open balls.

$\therefore$ The arguement similar to option (A) gives us

$$\mathcal{F} = \mathcal{B}(\mathbb{R}^n).$$

Option (B) is correct.

(C) Let $\mathcal{F}$ be the σ-alagebra generated by singletons.

Then $\mathcal{F} = \{A \subseteq \mathbb{R}^n : \text{either } A \text{ is countable or } A^c \text{ is countable}\}$.

& $B = \{(x_1, x_2, x_n) : x_i > 0, i = 1,2,.,n\} \in \mathcal{B}(\mathbb{R}^n)$

& $B \notin \mathcal{F}$.

$\therefore$ $\mathcal{B}(\mathbb{R}^n)$ is not generated by singletons.

$\therefore$ Option (C) is not correct.

(D) Let $\mathcal{F}$ be the σ-algebra generated by $\{[a,b) : a,b \in \mathbb{Q}\}$

consider the open set $(a,b), a<b$.

Let (a_n) be a ~~sequence~~ decreasing sequence of rationals converging to a and (b_n) be a increasing sequence of rationals converging to b such that $a_n \in (a, \frac{a+b}{2})$ & $b_n \in (\frac{a+b}{2}, b)$, $\forall n$.

Then $(a,b) = \bigcup_{n=1}^{\infty} [a_n, b_n)$, $[a_n, b_n) \in \mathcal{F}$, $\forall n$.

$\Rightarrow (a,b) \in \mathcal{F}$

$\therefore$ The arguement similar to option (A) gives us $\mathcal{F} = \mathcal{B}(\mathbb{R})$.

$\therefore$ option (D) is correct.

---

(8)

(A)

Let $\mathcal{F} = \{A \subset \mathbb{R} : A \text{ is countable or } A^c \text{ is countable}\}$

clearly, $\phi, \mathbb{R} \in \mathcal{F}$.

Let (A_n) be a sequence of sets in $\mathcal{F}$.

If A_n is countable, $\forall n$, then $\cup A_n$ is countable.

If A_m is not countable for some $m \in \mathbb{N}$, then $\left(\bigcup_{n=1}^{\infty} A_n\right)^c$ is countable. Since, $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$ & A_m^c is countable.

In both the cases, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

$\Rightarrow \mathcal{F}$ is an σ-algebra.

Option (A) is correct.

(B) $\mathcal{F} = \{A \subset \mathbb{R} : A \text{ is finite or } A^c \text{ is finite}\}$

claim is $\mathcal{F}$ is not an σ-algebra

Let us take $\{n\}$, $n \in \mathbb{Z}$. $\{n\} \in \mathcal{F}$, $\forall n \in \mathbb{Z}$

But $\mathbb{Z} = \bigcup_{n \in \mathbb{Z}} \{n\} \notin \mathcal{F}$

$\therefore \mathcal{F}$ is not an σ-algebra.

(C) $\mathcal{F} = \{A \subseteq \mathbb{R} : A$ is a closed interval$\}$

Since, $[0,1], [2,3] \in \mathcal{F}$ but $[0,1] \cup [2,3] \notin \mathcal{F}$

$\mathcal{F}$ is not an σ-algebra.

(D) $\mathcal{F} = \{A : A \subseteq [0,1] \ \& \ A \in \mathcal{B}(\mathbb{R})\}$

Since, $\mathcal{F}$ is the restriction of the σ-algebra $\mathcal{B}(\mathbb{R})$ to $[0,1]$

$\mathcal{F}$ is an σ-algebra.

---

(9) (A) $\mu(\phi) = 0$ (By defn of μ)

Let (A_n) be a ~~disjoint~~ sequence of disjoint sets in $\mathcal{B}(\mathbb{R})$.

There are two cases $\mu(\bigcup_{n=1}^{\infty} A_n) = \infty$ and $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$.

case (i):

~~Suppose if~~ $\mu(\bigcup_{n=1}^{\infty} A_n) = \infty$.

Then either $\mu(A_n) < \infty, \forall n$ or $\mu(A_m) = \infty$, for some $m \in \mathbb{N}$

($\because$ A_n's are disjoint)

~~If $\mu(A_n) < \infty, \forall n$, the all the A_n's have~~ contains ~~atleast~~

If $\mu(A_n) < \infty$, $\forall n$, then infinitely many A_n's contains atleast one rational number.

$\Rightarrow \sum_n \mu(A_n) = \infty$.

or if, $\mu(A_m) = \infty$, for some $m \in \mathbb{N}$, then $\sum_n \mu(A_n) = \infty$.

In both the cases $\sum_{n=1}^{\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$.

Case (ii):

Suppose $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$

In this case only finitely many A_n's contains finitely many rationals.

$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

$\therefore$ μ is a measure.

(B) $\mu(\phi) = 0$

Let (A_n) be a disjoint sequence in $\mathcal{B}(\mathbb{R})$.

There are two possible cases, either $1 \in \bigcup_{n=1}^{\infty} A_n$ or $1 \notin \bigcup_{n=1}^{\infty} A_n$

case (i): $1 \in \bigcup_{n=1}^{\infty} A_n$.

Since A_n's are disjoint, $1 \in A_m$, for some only one $m \in \mathbb{N}$.

$\therefore \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 = \mu(A_m) = \sum_{n=1}^{\infty} \mu(A_n)$

case (ii): $1 \notin \bigcup_{n=1}^{\infty} A_n$

$\therefore 1 \notin A_n, \forall n \in \mathbb{N}$.

$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0 = \sum_{n=1}^{\infty} \mu(A_n)$.

In both the cases $\sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

$\therefore \mu$ is a measure.

(C) Arguemet similar to that of option (A) will show that μ is a measure.

(D) $\mu(\phi) =$ number of rational in $\mathbb{R} = \infty$.

$\therefore \mu$ is not a measure.

---

(10) Since, X is finite, $f : X \to \mathbb{R}$ is a simple function.

$\therefore$ By definition, $\int_X f \, d\mu = \sum_{k \in X} f(k)$

$\therefore$ Option (A) is the only correct option.