Solutions to Week 2 Assessments with some hints to the solutions.

1. $X = \mathbb{R}, \mathcal{F} = \{A \subset \mathbb{R} \mid A \text{ is countable or } A^c \text{ is countable}\}$. Let

$$\mu(A) = \begin{cases} 1 & \text{if } A^c \text{ is countable} \\ 0 & \text{if } A \text{ is countable} \end{cases}$$

Let $f : (X, \mathcal{F}, \mu) \to \mathbb{R}$ be measurable. Which of the following are always true?

(a) f is constant ae.(μ)

(b) f is a non-constant continuous function

(c) f is a non constant polynomial

(d) $f(x) = 0 \; \forall \; x \in \mathbb{R}$

Solutions : (a)
$f^{-1}([n, n+1])$ is uncountable for some integer n. If this is true for 2 integers say m and n with $m < n$, then $n = m + 1$ and hence $f^{-1}(n)$ is co-countable. Then we are done with $f = n \; ae$. Now if the integer n is unique, then divide the interval $[n, n+1]$ to 2 and repeat the same arguments. The we get a decreasing sequence of compact intervals with length decreases to 0. Then we have an element in the intersection which is the required constant.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space and let E be a proper subset of $X, E \in \mathcal{F}$ and $0 < \mu(E) < \mu(X)$.

$$f_n = \begin{cases} \chi_E & \text{if } n \text{ is odd} \\ 1 - \chi_E & \text{if } n \text{ is even} \end{cases}$$

Which of the following are true?

(a) $\int_X \liminf f_n d\mu \;<\; \liminf \int_X f_n d\mu$

(b) $\int_X \liminf f_n d\mu \;=\; \liminf \int_X f_n d\mu$

(c) $\int_X \limsup f_n d\mu \;<\; \limsup \int_X f_n d\mu$

(d) $\int_X \limsup f_n d\mu \;=\; \limsup \int_X f_n d\mu$

Solutions : (a)
$\liminf f_n = 0$ and $\limsup f_n = 1$. So their integrals are 0 and $\mu(E)$ respectively. While limit of integrals are $\mu(E)$ and $\mu(E^C)$

3. Consider the space $\mathbb{N}$ with power set sigma algebra and counting measure μ. Let $f : \mathbb{N} \to \mathbb{R}$ be measurable and zero ae(μ). Which of the following are always true?

(a) $f(n) = 0 \; \forall n \in \mathbb{N}$

(b) $f(1) \neq 0, f(n) = 0 \; \forall n > 1$

(c) $f(n) = 0$ except for finitely many $n \in \mathbb{N}$

(d) $f(n) = 0$ only when n is a prime number

Solutions : (a) and (c)
Since singleton sets has measure 1 f cannot be non zero at any point.

4. Let X be a non empty set and $A \subset X$ be a proper subset. Consider the sigma algebra $\mathcal{F} = \{\phi, X, A, A^c\}$. Let $f : (X, (F) \to \mathbb{R})$ be measurable. Which of the following are always true?

   (a) $f = \alpha\chi_A + \beta\chi_{A^c}$ for some $\alpha, \beta \in \mathbb{R}$

   (b) $f = \alpha\chi_A$ for some $\alpha \in \mathbb{R}$

   (c) $f = \beta\chi_{A^c}$ for some $\beta \in \mathbb{R}$

   (d) $f \equiv 0$

   Solutions : (a)
   Suppose f is a constant say α, then $f = \alpha\chi_A + \alpha\chi_{A^c}$. Now if f is not constant, f takes atleast 2 values say α and β, then $f^{-1}(\alpha), f^{-1}(\beta)$ must be in $\mathcal{F}$. Let $f^{-1}(\alpha) = \chi_A$ and $f^{-1}(\beta) = \chi_{A^c}$. Then $f = \alpha\chi_A + \beta\chi_{A^c}$

5. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $A_n \in \mathcal{F}$ be such that $A_1 \subset A_2 \subset A_3 \subset \cdots$ and $\cup_{n=1}^{\infty} A_n = X$. Let $f : (X, \mathcal{F}, \mu) \to \mathbb{R}$ be a measurable function and $f(x) \geq 0$ ae(μ). Which of the following are always true?

   (a) $f\chi_{A_n} \uparrow f$ ae

   (b) $\int_{A_n} f d\mu \uparrow \int_X f d$

   (c) $\int_{A_n} f d\mu \downarrow \int_X f d\mu$

   (d) $f\chi_{A_n} \downarrow f$

   Solutions : (a) and (b)
   (a) is trivially true and (b) follows from monotone convergence theorem.

6. Let $(X, \mathcal{F}, \mu)$ be a measure space and μ be a probability measure, that is $\mu(X) = 1$. Let $\{A_n\}$ be a sequence in $\mathcal{F}$. Which of the following are true?

   (a) $\mu(\limsup A_n) \geq \limsup \mu(A_n)$

   (b) $\mu(\limsup A_n) \leq \limsup \mu(A_n)$

   (c) $\mu(\liminf A_n) \geq \liminf \mu(A_n)$

   (d) $\mu(\liminf A_n) \leq \liminf \mu(A_n)$

   Solutions : (a) and (d)
   $\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$. Let $B_n = \cup_{k=n}^{\infty} A_k$. Then B_n is a decreasing sequence with $\mu(B_1)$ is finite. Hence $\mu(B_n) \downarrow \mu(\limsup A_n)$. Also $A_n \subset B_n$ for all n. So $\mu(A_n) \leq \mu(B_n)$ implies $\limsup \mu(A_n) \leq \lim \mu(B_n) = \mu(\limsup A_n)$.
   (d) also can be proved by similar arguments

7. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $f_n : X \to \mathbb{R}$ be measurable, $A = \{x \in X \mid \lim f_n(x) \text{ exists}\}$. Then,

   (a) $A \in \mathcal{F}$
   (b) $A = \phi$
   (c) $A = X$
   (d) $A^c \in \mathcal{F}$

   Solution : (a) and (d)

   $$\begin{aligned} A = &\ \{x \in X : \lim f_n(x) \text{ exists}\} \\ = &\ \{x \in X : \{f_n(x)\} \text{ is cauchy}\} \\ = &\ \cap_{k=1}^{\infty} \cup_{p=1}^{\infty} \cap_{m,n=p}^{\infty} \{x \in X : |f_n(x) - f_m(x)| < \tfrac{1}{k}\} \end{aligned}$$

   which is a countable union of measurable sets and hence measurable.

8. Let $(X, \mathcal{F}, \mu)$ be a measure space and $A_n \in \mathcal{F}$. Suppose $\mu(X) = 1, \sum \mu(A_n) < \infty$. Then which of the following are true?

   (a) $\mu(\limsup A_n) = 0$
   (b) $\mu(\liminf A_n) = 0$
   (c) $\mu(\limsup A_n) = 1$
   (d) $\mu(\liminf A_n) = 1$

   Solution : (a) and (b)
   Let $B_n = \cup_{k=n}^{\infty} A_k$. Then $\mu(B_n) \leq \sum_{k=n}^{\infty} \mu(A_n) \to 0$ as $n \to \infty$. Hence $\mu(\limsup A_n) = 0$. Then $\mu(\liminf A_n) = 0$ also.

9. $(X, \mathcal{F}, \mu)$ be a probability measure space. Suppose $f_n : X \to \mathbb{R}$ are measurable and $|f_n| \leq 1\ ae(\mu)\ \forall n$. Suppose $f_n \to 1\ ae(\mu)$. Then,

   (a) $\int_X f_n d\mu \to 1$
   (b) $\int_X f_n d\mu \to 0$
   (c) $\int_X f_n d\mu \to \infty$
   (d) $\int_X f_n d\mu$ does not converge

   Solution : (a)
   By Dominated convergence theorem

10. Let $(X, \mathcal{F}, \mu)$ be a measure space and $A_n \in \mathcal{F}$ such that $\mu(A_n) = 0\ \forall n$. Which of the following are true?

    (a) $\mu(\cup_{n=1}^{\infty} A_n) = 0$
    (b) $\mu(\cup_{n=1}^{\infty} A_n) = 1$
    (c) $\mu(\cup_{n=1}^{\infty} A_n) = \infty$
    (d) $\mu(\cup_{n=1}^{\infty} A_n) > 0$

    Solution : (a)
    By countable sub-additivity