Parameterized Algorithms

Week 4 Assignment

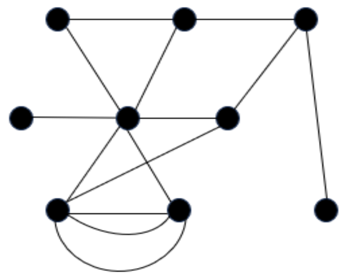


Figure 1: An instance of FVS

1. (1 point) Is the following statement True or False:

   An algorithm $\mathcal{A}$ is a Monte Carlo Algorithm with false negatives. When $\mathcal{A}$ outputs YES for an instance I, it can be a NO instance.

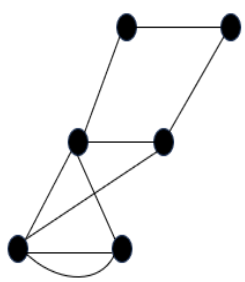
   A. True

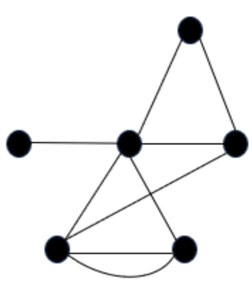
   B. False

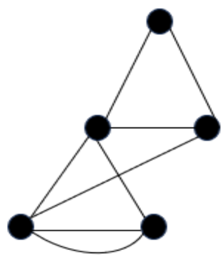
   Solution:

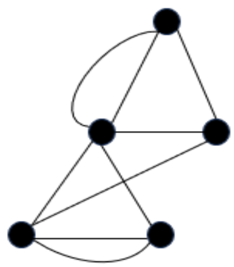
   If $\mathcal{A}$ has false negatives, then it means that it can output NO when the instance is a YES instance but it will definitely output NO when there is a NO instance. Therefore, it cannot output YES when there is a NO instance. Thus the statement is False.

2. (1 point) Which of the following correctly depicts the instance in Figure 1 after applying all the preprocessing rules for randomized FVS (such that the reduction rules cannot be applied any further)?

   
   A.

   
   B.

C. 

D. 

Solution:

In option A and C, a degree two vertex has been deleted without adding an edge between its end points. In option B, a degree one vertex has not been removed. (There may be other mistakes too). In Option D all the reduction rules have been correctly applied and no further application of each of the reduction rules is possible.

3. (1 point) Statement I: In a forest F, the number of leaves (degree 1 vertices) is always more than the number of vertices which are not leaves.

   Statement II: In a forest F with no degree 2 vertices and no isolated vertices, the number of leaves (degree 1 vertices) is always more than the number of vertices which are not leaves.

   Which of the following options is correct?

   A. Both Statement I and Statement II are correct.

   B. Statement I is wrong and Statement II is correct.

   C. Statement I is correct and Statement II is wrong.

   D. Both Statement I and Statement II are wrong.

Solution:

Consider a path on $n > 4$ vertices. It is a forest with $n - 2$ non-leaf vertices and 2 leaves. Thus Statement I is wrong.

Consider a forest with ℓ leaves and p non-leaf vertices such that it has no degree 2 vertex and no isolated vertex. Since it is a forest the number of edges is at most 1 less than the number of vertices, i.e., at most $\ell + p - 1$. Therefore sum of degrees of all vertices is at most $2\ell + 2p - 2$. Also, since there are no degree 2 or isolated vertices, each non-leaf vertex contributes at least 3 to the degree-sum. Each leaf contributes 1 to the degree-sum. Therefore the degree sum is at least $3p + \ell$ and hence we have, $3p + \ell \leq 2\ell + 2p - 2$, i.e., $p \leq \ell - 2$, i.e., $p < \ell$. Thus, Statement II is correct.

Therefore, option B is correct and the others are wrong.

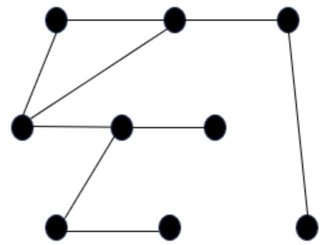
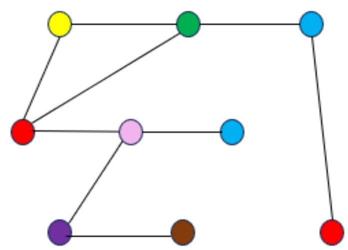
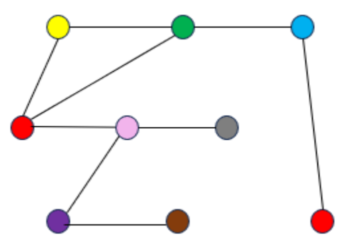
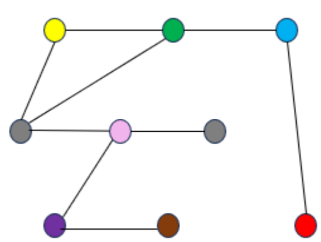


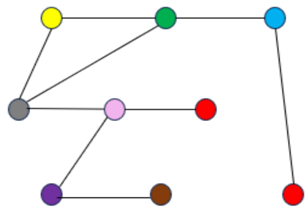
Figure 2: Longest Path Problem

4. (1 point) Consider the LONGEST PATH problem with parameter solution size (number of vertices) k. A coloring is said to be *good* when all the vertices of the $k-$path have distinct colors. Which of the following coloring(s) is/are good for the graph G given in Figure 2 with $k = 8$?

A. 

B. 

C. 



D.

Solution:

The figures in C and D have a path with 8 vertices and all of distinct colors i.e., red, blue, green, yellow, gray, pink, purple, brown. The figures in A and B do not have a path with 8 vertices and all of different colors. Thus, C and D are the correct options.

5. (1 point) Fill in the blank: In the LONGEST PATH problem with parameter k (solution size), we have a YES instance (with exactly one solution) and a coloring is given with each vertex in the solution colored with a different color. Given an ordering of the k colors for the "guess" in Week 4 Lecture 17, the probability that it is the correct ordering is $- - --$.

   A. $\frac{k!}{k^k}$

   B. $\frac{1}{k!}$

Solution:

The number of orderings of k colors is $k!$ out of which only 1 is correct hence the answer is B. No other ordering can be correct because it is given that we have a YES instance with exactly one solution.

6. (1 point) Is the following statement true or false: Suppose T is a tournament and X is a subset of arcs of T such that after deleting X, the resulting graph does not contain any directed cycle, then the tournament T' after reversing the arcs in X also does not contain any directed cycle.

   A. True

   B. False

Solution:

If X is the set of all arcs of T, then after deleting the edges in X the graph becomes a set of isolated vertices which does not contain any directed cycle. But if T contained a cycle, then after reversing all the edges the tournament T' will contain the same cycle in the reverse direction. Thus the statement is False.

7. (1 point) Is the following statement true or false: Every closed (directed) walk in a directed graph contains a directed cycle.

   A. True

   B. False

Solution:

Consider a closed (directed) walk $W = u_1u_2 \dots u_ku_1$ which is not a directed cycle. Let u_j be the lowest such index such that $u_i = u_j$ for some $i < j$. Then consider the walk $u_1u_2 \dots u_{i-1}u_ju_{j+1} \dots u_ku_1$. If we get a cycle, we are done. If not, repeat the same process. Since there are finitely many vertices and hence finitely many repetitions this process will eventually terminate and we will get a cycle which is contained in W. Thus the statement is True.

8. (1 point) Which of the following statement(s) is/are true?

   A. If two graphs are isomorphic, they have the same degree sequence (in non-increasing order).

   B. If two graphs are isomorphic, they may not have the same degree sequence (in non-increasing order).

   C. If two graphs have the same degree sequence (in non-increasing order), then they are isomorphic.

   D. If two graphs have the same degree sequence (in non-increasing order), then they may not be isomorphic.

Solution:

If two graphs G and H are isomorphic, then there exists a bijection f from G to H such that $uv \in E(G) \iff f(u)f(v) \in E(H)$. This implies that the degree of each vertex u in G is the same as the degree of the corresponding vertex $f(u)$ in H. This is because if the neighbours of u in G are $u_1, u_2, \ldots, u_d$, then the neighbours of $f(u)$ in H are exactly $f(u_1), f(u_2), \ldots, f(u_d)$. Thus the degree sequence of G and H (written in non-increasing order) must be the same. Hence A is correct and B is wrong.

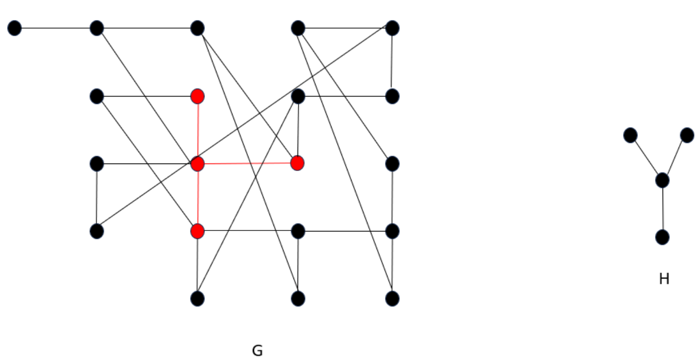
Consider a graph G such that $V(G) = \{a, b, c, d, e, f\}$ and $E(G) = \{ab, bc, cd, de, cf\}$. Now consider a graph H such that $V(H) = \{u, v, w, x, y, z\}$ and $E(G) = \{uv, vw, wx, xy, vz\}$. The degree sequence of both these vertices (written in non-increasing order) will be $3, 2, 2, 2, 1, 1$. Now any isomorphism from G to H must map c to v as these are the only degree 3 vertices. But c has two degree-2 neighbours and one degree-1 neighbour whereas v has two degree-1 neighbours and one degree-2 neighbour. Hence there cannot be a bijection between the same degree neighbours of c and the same degree neighbours of v. Hence G and H are non-isomorphic. Thus C is wrong and D is correct.

9. (1 point) Consider the instance of the subgraph isomorphism problem in Figure 3. It is a —- instance. (Fill in the blank.)

   A. YES

   B. NO

Solution: One possible solution is given by:



Here the red colored graph is an isomorphic copy of H. Therefore, it is a YES instance.

10. (1 point) Consider the 14 colored vertices in the figure given below and the set of colors: {Red, Pink, Yellow, Blue, Green}. Fill in the blank: There are —- sets of size 5 which are subsets of these 14 vertices which are colored injectively (as described in Week 4 Lecture 20).

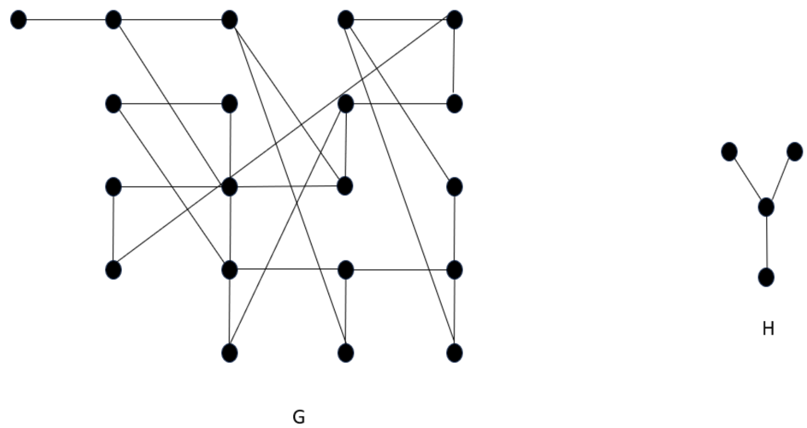


Figure 3: An instance of subgraph isomorphism problem with host graph G and pattern graph H

A. 144

B. 288

C. 48

D. 72

Solution: If we pick any one red vertex out of 3, one pink vertex out of 4, one blue vertex out of 2, one yellow vertex out of 2 and one green vertex out of 3 then we will get a set of size 5 which is injectively colored and any other set of size 5 will contain two vertices of the same color. Therefore the number of subsets of size 5 which are injectively colored will be: $3 \times 4 \times 2 \times 2 \times 3 = 144$. Hence A is correct and others are wrong.