Week 7 Questions - Theory of Computation

March 2, 2024

Consider the language

$$\text{ALL-DFA} = \{\langle M\rangle |\ M \text{ is a DFA that accepts all strings in } \Sigma^*\}$$

$$E_{DFA} = \{\langle M\rangle |\ L(M) = \phi\}$$

1. (1 point) Pick all the statements that are true
   - ☐ The language ALL-DFA is Recognizable but not Co-turing recognizable
   - ☐ The language ALL-DFA is co-turing recognizable but not recognizable
   - ■ **To build a recognizer R for the complement of ALL-DFA, we list down all the strings s in Σ^* in order and then make R simulate the processing of M on s. If any string s is rejected, then we accept, else we continue.**
   - ■ **We can also build a decider for D by first constructing the DFA that accepts the complement of $L(M)$ which is $\langle \bar{M}\rangle$ and then passing it to the decider D' for E_{DFA}**

2. (2 points) Which of the following statements is/are true? Select all that is/are correct.
   - ☐ We can always construct a bijective mapping between two countable sets.
   - ■ **We can never construct a bijective mapping between a countable set and an uncountable set.**
   - ■ **We can always construct a bijective mapping between two sets that are countably infinite**
   - ☐ We can never construct a bijective mapping between any two uncountable sets

3. (2 points) Consider the parametrized language $L_{R,A}$, where R is a regular expression and A is an arbitrary subset of Σ^*. The language is defined as follows:-

$$L_{R,A} = \{\langle G\rangle \mid G \text{ is a CFG over } \{0,1\} \text{ and } R \cap L(G) = A\}$$

   - ■ **$L_{0^*1^*,\emptyset}$ is recognizable**
   - ■ **$L_{0^*1^*,\emptyset}$ is decidable**
   - ☐ $L_{0^*1^*,\Sigma^*}$ is recognizable because it is the complement of $L_{0^*1^*,\emptyset}$
   - ■ **$L_{R,A}$ such that $|A| < \infty$ is co-Turing recognizable**

Solution: The intersection of a context free language and a regular language is context free. To construct a decider D for $L_{R,A}$, the TM D will first construct a DFA X that generates the regular expression R, and then perform a "product of states construction" with the PDA $\langle G\rangle$. The resultant PDA K would recognize the language $L(G) \cap R$.

The first 2 follow from the construction and the decider for E_{CFG}. The third option is wrong because those two languages are not complements of each other, and the fourth is co-turing recognizable, because we can iterate over every string in A and check if the CFG K accepts it (using CYK/Chomsky construction), if it does not accept some string, then accept, else continue to check every other string $e \in \Sigma^* - A$, if any string accepts, then accept, else reject

4. (2 points) (Infinity) Consider the languages defined below , here $\mathbb{S}$ is a positive integer valued function that takes as input the description of a DFA $\langle M \rangle$

$$\text{INF-DFA} = \{\langle M \rangle \mid M \text{ is a DFA and } L(M) \text{ is an infinite language } \}$$
$$\text{ARB-DFA}_{\mathbb{S}} = \{\langle M \rangle \mid M \text{ is a DFA and } L(M) \text{ contains a string of length } k > \mathbb{S}(\langle M \rangle) \}$$

■ **INF-DFA is decidable**

☐ INF-DFA is the complement of E_{DFA}, and is therefore, recognizable

☐ ARB-DFA$_{\mathbb{S}}$ = INF-DFA if $\mathbb{S}(\langle M \rangle)$ = Number of States in the DFA $\langle M \rangle$

■ **The complement of INF-DFA is recognizable**

Solution: The pumping lemma and the pigeonhole principle

5. (2 points) Tony is trying to prove that every context free language is decidable. Tony constructs the following TM M. The Turing Machine M knows the PDA P that recognizes this CFL A.

TM M: On an input w:-

1. Convert the PDA P to an NTM N that recognizes the same language A. The NTM N will use another tape to simulate the stack of PDA P. N will faithfully replicate the moves of P using the second tape to mirror the stack.
2. Convert the NTM N to a DTM D using the standard construction
3. Run the input w on the DTM D, accept if D accepts, else reject.

Out of the given choices, select all the correct options.

Hint: Does a PDA necessarily accept or reject on all branches of computations?

■ **This TM M recognizes the CFL A**

☐ This TM M decides the CFL A

☐ Every branch of computation taken by N on input w will halt.

☐ It is possible that there is no such NTM N that can simulate this PDA P

6. (2 points) Let L be a decidable language. Let M be a decider of L. Now construct the complement of L, $L^c = \{w | w \text{ is not accepted by } M\}$. Select all the correct options

■ **L^c is Turing-recognizable**

☐ L^c is not Turing-recognizable

■ **L^c is decidable**

☐ L^c is not decidable

7. (2 points) Let L^c denote the complement of a language L. Select all the correct options

■ **If L and L^c are both decidable, then they are both Turing-recognizable**

■ **If L and L^c are both Turing-recognizable, then they are both decidable**

☐ If L is Turing-recognizable, but not decidable, then L^c is decidable

☐ If L is not Turing-recognizable, then L^c is decidable

8. (2 points) How many of the following sets are countably infinite? ____**3**____

   (In the below options, the notation [10] stands for the set $[10] = \{1, 2, 3, \dots 9, 10\}$ and $\mathbb{N}$ stands for the set of all natural numbers)

   1. set of all rational numbers that have an even denominator
   2. set of all functions $f : [10] \to \mathbb{N}$
   3. set of all functions $f : \mathbb{N} \to [10]$
   4. set of all functions $f : \mathbb{N} \to \mathbb{N}$
   5. $[10]^*$ (i.e., the set of all finite length strings formed from the alphabet [10])

   Solution: Set of all rational numbers is countable, hence option 1 is countable. We have seen that Σ^* is countable for any finite alphabet Σ, hence option 5 is countable as well.

   We can show that option 2 (set of all functions $f : [10] \to \mathbb{N}$) is countable by ordering the functions by $\sum_{i=1}^{10} f(i)$.

   However options 3 and 4 are not countable. The proof is similar to the proof that all the set of all infinitely long binary strings are uncountable.

9. (2 points) Let

$$A = \{\langle R, S\rangle | R, S \text{ are regular expressions and } L(R) \subseteq L(S)\}$$

$$B = \{\langle R, S\rangle | R, S \text{ are regular expressions and } L(S) \subseteq L(R)\}$$

   Select all the correct options

   ☐ Both A and B are Turing-recognizable but not decidable

   ■ **Both A and B are decidable**

   ■ $A \cap B$ **is decidable**

   ☐ $A \cup B$ is Turing-recognizable, but not decidable

10. (2 points) Select all the decidable languages.

   ■ $A = \{ww^R | w \in \{0, 1\}^*\}$

   ■ $B = \{a^i b^j c^k | i + j = k$ **and** $i, j, k \geq 1\}$

   ■ $C = \{0^{2^n} | n \geq 0\}$

   ■ $D_n = \{\#x_1\#x_2\# \dots \#x_n|$ **each** $x_i \in \{0, 1\}^*$ **and** $x_i \neq x_j \; \forall \; i \neq j\}$