Week 8 Questions - Theory of Computation

March 11, 2024

Note the following languages:-

$$A_{TM} = \{\langle M, w\rangle \text{ Where } M \text{ is a Turing machine that accepts } w\}$$

$$EQ_{TM} = \{\langle M_1, M_2\rangle \text{ Where } M_1, M_2 \text{ are Turing machines such that } L(M_1) = L(M_2)\}$$

1. (2 points) Let A, B be two languages such that $A \leq_m B$. Which of the following is/are correct?
   - ☐ If A is Turing recognisable then B is Turing recognisable.
   - ☐ If A is co-Turing recognisable then B is co-Turing recognisable.
   - ■ **If B is Turing recognisable then A is Turing recognisable.**
   - ■ **If B is co-Turing recognisable then A is co-Turing recognisable.**

2. (2 points) Which of the following is/are true about languages A and B?
   - ☐ If $A \leq_m B$, then $B \leq_m A$
   - ■ **If $A \leq_m \bar{A}$, then $\bar{A} \leq_m A$**
   - ■ **If $A \leq_m \bar{A}$ and A is recognizable, then A is decidable**
   - ☐ If A is decidable, $A \leq_m \bar{A}$

   Solution: The last option is incorrect since it does not hold when $A = \Sigma^*$

3. (2 points) For a property $\mathcal{P}$, recall the language $\mathcal{P}_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \in \mathcal{P}\}$. For which of the following properties $\mathcal{P}$ is $\mathcal{P}_{TM}$ undecidable?
   - ■ **$\mathcal{P} = \{A \in \Sigma^* \mid |w| \text{ is even for all } w \in A\}$**
   - ■ **$\mathcal{P} = \{A \in \{a, b, c, d\}^* \mid w \text{ contains at most 2 distinct symbols for all } w \in A\}$**
   - ☐ $\mathcal{P} = \{A \in \Sigma^* \mid A \text{ is countable}\}$
   - ☐ $\mathcal{P} = \{A \in \Sigma^* \mid A \text{ is uncountably infinite}\}$

4. (2 points) Consider a Turing Machine M and a string w. Define $ACH_{M,w}$ as the language of all strings that are accepting configuration histories when M is run on input w.
   Which of the following options is/are correct? Select all the correct options.
   - ■ **$\overline{ACH_{M,w}}$ can be generated by a context-free language**
   - ☐ $\langle M, w\rangle \in A_{TM} \Leftrightarrow$ The grammar that generates $ACH_{M,w} \in ALL_{CFG}$
   - ■ **$\langle M, w\rangle \in A_{TM} \Leftrightarrow$ The grammar that generates $\overline{ACH_{M,w}} \in \overline{ALL_{CFG}}$**
   - ☐ ALL_{CFG} is decidable

5. (2 points) Choose the true statements
   Let B be such that $EQ_{TM} \leq_m B$ and $\overline{A_{TM}} \leq_m B$

- ■ $A_{TM} \leq_m B$
- ☐ B is decidable
- ■ **B is not recognizable**
- ☐ $\bar{B}$ is decidable

Solution: Theorem 5.30 in Sipser shows that $A_{TM} \leq_m EQ_{TM}$. Hence option 1 correct.

B being decidable implies that EQ_{TM} is decidable which is not true. Hence option 2 is incorrect.

It is known EQ_{TM} is not recognizable. Hence it follows that option 3 is correct.

Option 4 is incorrect since B is undecidable.

6. (2 points) Consider the following reduction function f, that takes as input strings of the form $\langle M, w\rangle$ and returns strings of the format $\langle M_1, M_2\rangle$, where M, M_1, M_2 are Turing machines and w is a string, i.e. consider a function $f : \Sigma^* \to \Sigma^*$

$$f(\langle M, w\rangle) = \langle M_1, M_2\rangle$$

Where M_1 is a Turing machine that rejects all strings $x \in \Sigma^*$. The TM M_2 accepts all strings $x \in \Sigma^*$ if M accepts w otherwise it does not accept any string $x \in \Sigma^*$.

Ram claims that this reduction is not valid, because this is not a **halting (computable)** reduction, since it checks whether w is accepted by M, and this check may lead to a loop in the computation of f.

Pick all the correct statements about this reduction.

- ☐ Ram is False, because A_{TM} is recognizable, and therefore the reduction is halting (computable)
- ■ **The reduction is valid and shows that $A_{TM} \leq_m \overline{EQ_{TM}}$**
- ■ **The reduction is valid and therefore $\overline{A_{TM}} \leq_m EQ_{TM}$**
- ■ **The reduction shows that EQ_{TM} is not Turing recognizable**

Solution: f needs to only create a Turing machine M_2 from M. For this, we can simulate M on w, and accept the input of M_2 if M accepts w. If M rejects/loops on w, M_2 also does the same. Hence $L(M_2) = \Sigma^*$ if M accepts w and $L(M_2) = \emptyset$ if M does not accept (rejects/loops) w.

The last option is correct because A_{TM} is not co-Turing recognizable, and by option 3.

7. (2 points) Pick all the statements that are true

- ■ **If $A \leq_m 0^*1^*$ then $\bar{A}$ is recognizable**
- ■ **If $A \leq_m 0^*1^*$, then we cannot have $EQ_{TM} \leq_m A$**
- ■ **If $A \leq_m A_{TM}$ and $A \leq_m \bar{A}$ then $A \leq_m 0^*1^*$**
- ■ **If A is decidable then $A \leq_m 0^*1^*$**

8. (2 points) For any language L and a string $w \in \Sigma^*$, define L_w as $L_w = L \cup \{w\}$. Consider the statements.

Statement P: L is Turing-recognizable $\Longrightarrow$ L_w is Turing-recognizable.

Statement Q: L_w is Turing-recognizable $\Longrightarrow$ L is Turing-recognizable.

Select the correct option.

A. Statements P and Q are both true.

B. Statement P is true, but statement Q is not.

C. Statement Q is true, but statement P is not.

D. Neither statement P nor statement Q are true.

9. (1 point) Given a language A, define A_{TWICE} as $A_{TWICE} = \{ww \mid w \in A\}$.

   A is decidable $\Longrightarrow$ A_{TWICE} is decidable. True or false?

   A. True

   B. False

Solution: We construct a decider for A_{TWICE} as follows. Given input x, it can be checked if it is of the form ww. If not, it can be rejected. If yes, we can use the decider for A and accept/reject as per this decider.

10. (1 point) Given a language A, define A_{TWICE} as $A_{TWICE} = \{ww \mid w \in A\}$.

    A_{TWICE} is decidable $\Longrightarrow$ A is decidable. True or false?

    A. True

    B. False

Solution: We construct a decider for A as follows. Given any string w, we give ww as input to the decider of A_{TWICE}. We accept or reject as per the decider.

11. (2 points) Out of the given choices, select all the properties that are nontrivial properties of a language L over the alphabet $\{0, 1\}$.

    - ■ **L contains the string 001.**
    - ■ **L is a finite-sized language.**
    - ☐ L is uncountably infinite.
    - ☐ $L \subset \{0, 1\}^*$.

Solution: All languages are finite or countably infinite. Hence the third option is an empty property, and hence trivial.

A language is defined to be the subset of Σ^*. Hence the last option is satisfied by all languages, and hence is trivial.