Week 9 Questions - Theory of Computation

March 16, 2024

Note: $A \leq_m B$ denotes "A is mapping reducible to B".

1. (2 points) Consider the setup of the PCP problem with the unary alphabet $\Sigma = \{1\}$. Each domino is of the form

$$d_i = \begin{bmatrix} 1^{a_i} \\ \hline 1^{b_i} \end{bmatrix}$$

For positive integer valued sequences $\{a_i\}_{i=1}^{i=n}$ and $\{b_i\}_{i=1}^{i=n}$. Let the language where there is a match on the upper and lower tiles be called UPCP.

Pick all the correct options.

- ■ **If there exists i such that $a_i = b_i$, then $\langle P \rangle \in UPCP$.**
- ☐ $PCP \leq_m UPCP$
- ■ $UPCP \leq_m PCP$
- ■ **If there exists i, j such that $a_i < a_j$ and $b_j < b_i$ then $\langle P \rangle \in UPCP$**
- ■ **UPCP is decidable**

2. (2 points) Suppose that the binary alphabet ($\Sigma = \{0, 1\}$) is used in PCP. Let us call this language $BPCP$. Ram wishes to solve the PCP problem with a general alphabet $\Sigma' = \{a_1, a_2, \ldots, a_n\}$ using a decider for $BPCP$, i.e. he wishes to show that PCP is reducible to BPCP. In order to do this, he constructs a function that takes in an alphabet from PCP, and returns a binary string. Ram constructs

$$f : \Sigma' \to \{0, 1\}^*$$

Which of the following choices for f will show that $PCP \leq_m BPCP$? Select all the correct options.

- ■ $f(a_i) = 0^i 1$
- ☐ $f(a_i) = 0^i$
- ■ $f(a_i) = 10^i$
- ■ $f(a_i) = 01^i$

3. (2 points) Suppose that we modify the PCP problem in a way that we are only allowed to use each domino once, and we are to check whether the upper rows matches the lower one. Let us call this language OPCP. Select all the correct options.

- ☐ OPCP is undecidable when $\Sigma = \{1\}$
- ■ **OPCP is decidable when $\Sigma = \{1\}$**
- ■ **OPCP is decidable for any finite set Σ**
- ☐ OPCP is undecidable if $|\Sigma| > 1$

4. (2 points) Consider the Language A defined as follows:-

$$A = \{\langle G \rangle \mid G \text{ is a CFG that generates a string of the form } w\#w \text{ for some } w \in \{0,1\}^*\}$$

Pick all the correct options

- ■ $PCP \leq_m A$
- ■ $MPCP \leq_m A$
- □ $A \leq_m E_{CFG}$
- □ $\overline{A} \leq_m A_{TM}$

Solution: The correctness of the first two follow from This link.

The third option is incorrect since E_{CFG} is decidable.

The last option implies that A is co-turing recognizable and since A is recognizable, this implies that A is decidable, which is a contradiction.

5. (2 points) Let A be a decidable language, and M be a deterministic multi-tape TM that decides A in $O(n)$ time (n is the size of the input string). Let $t(n)$ denote the running time of the fastest deterministic single-tape TM S that decides A. Then which of the following is definitely true? Select all the correct options.

- ■ $t(n) = O(2^n)$
- ■ $t(n) = o(2^n)$
- ■ $t(n) = O(n^2)$
- □ $t(n) = O(n)$

6. (2 points) Let A be a decidable language, and N be a non-deterministic single-tape TM that decides A in $O(n)$ time (n is the size of the input string). Let $t(n)$ denote the running time of the fastest deterministic single-tape TM D that decides A. Then which of the following is definitely true? Select all the correct options.

- □ $t(n) = O(2^n)$
- ■ $t(n) = 2^{O(n)}$
- □ $t(n) = O(n^2)$
- □ $t(n) = O(n)$

7. (1 point) Any context free language is decidable in polynomial time, so all context free languages belong to P. True or False?

A. True

B. False

8. (2 points) Let $PRIMES$ denote the problem: Given a natural number N, output 1 if N is a prime, otherwise output 0. One straight-forward algorithm for $PRIMES$ is to check for divisibility by numbers between 2 until $\sqrt{N}$. With regards to this, select all the options that apply.

- ■ **This algorithm performs $O(\sqrt{N})$ division operations.**
- ■ **The algorithm correctly outputs if N is prime.**
- □ This algorithm has polynomial time complexity.
- □ This algorithm proves that $PRIMES$ is in P.

Solution: The running time is $O(\sqrt{N})$, but this does not mean that the algorithm is polynomial running time. Note that the running time should be expressed in terms of the number of input bits, and so this algorithm actually has an exponential running time.

PRIMES was later shown to be in P by Agrawal, Kayal and Saxena in 2002. They developed a polynomial time algorithm known as the AKS primality testing algorithm. You can read more about AKS algorithm here.

9. (2 points) Select all that are correct.

   - ■ **A language is decidable in polynomial time by a non-deterministic TM, iff it is in NP.**
   - ☐ If a language is verifiable by a deterministic TM iff it is in P
   - ■ **Let the running time of an algorithm be $f(n) = \lfloor \pi n \rfloor$. Then $f(n) = O(n)$**
   - ■ **There can exist functions $f(n)$ and $g(n)$ such that neither $f(n) = O(g(n))$ nor $g(n) = O(f(n))$ holds.**

Solution: All the statements except the last one are straightforward to see. For the last one, take $f(n) = n$ and $g(n) = n^{1+\sin n}$.

10. (2 points) The class P of languages is closed under which of the following operations? Select all that is/are correct.

    - ■ **Union**
    - ■ **Intersection**
    - ■ **Concatenation**
    - ■ **Complement**