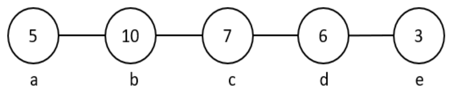
Week 5 Assignment

Vertex v	a	b	c	d	e
$A(v)$	5	10	x	16	z
$B(v)$	0	5	10	12	y

Table 1: DP Table for Maximum Independent Set on Paths for Questions $1, 2$ and 3.

1. (1 point) Consider the DP table shown in Table 1 for the MAX WEIGHT INDEPENDENT SET problem on a path P given below. Let $A(v)$ denote the weight of the maximum independent set of the path ending at v (from left to right) and $B(v)$ denote the weight of the maximum independent set of the path ending at v (from left to right) which does not contain v. What is the value of x?



   A. 10
   B. 11
   C. 12
   D. 13

   Solution:

   The path ending at a is the single vertex a and hence the maximum independent set of the path ending at a is $\{a\}$. The maximum independent set of the path ending at a which does not contain a is the empty set. Therefore, $A(a) = 5$ and $B(a) = 0$. The maximum independent set of the path ending at b which does not contain b is also the maximum independent set of the path ending at a i.e. $B(b) = A(a) = 5$. The weight of maximum independent set of the path ending at b is the maximum of the weight of the maximum independent set of the path ending at a and the sum of the weight of b and the weight of the maximum independent set of the path ending at a which does not contain a. Therefore, $A(b) = \max(B(b), b + B(a)) = \max(5, 0 + 10) = 10$. Again, the maximum independent set of the path ending at c which does not contain c is also the maximum independent set of the path ending at b i.e. $B(c) = A(b) = 10$. The weight of the maximum independent set of the path ending at c is the maximum of the weight of the maximum independent set of the path ending at b and the sum of the weight of c and the weight of maximum independent set of the path ending at b which does not contain b. Therefore, $A(c) = \max(B(c), c + B(b)) = \max(10, 7 + 5) = 12$. Thus $x = 12$ and option C is the only correct choice.

2. (1 point) Consider the same problem on the same path in Question 1. What is the value of y in Table 1 ?

   A. 15
   B. 16
   C. 17
   D. 18

   Solution:

   The maximum independent set of the path ending at d which does not contain d is also the maximum independent set of the path ending at c i.e. $B(d) = A(c) = 12$. The weight of the maximum independent set of the path ending at d is the maximum of the weight of the maximum independent set of the path ending at c and the sum of the weight of d and the weight of the maximum independent set of the path ending at c which does not contain c. Therefore, $A(d) = \max(B(d), d + B(c)) = \max(12, 6 + 10) = 16$. The maximum independent set of the path ending at e which does not contain e is also the maximum independent set of the path ending at d i.e. $B(e) = A(d) = 16$.Thus $y = 16$ and option B is the only correct choice.

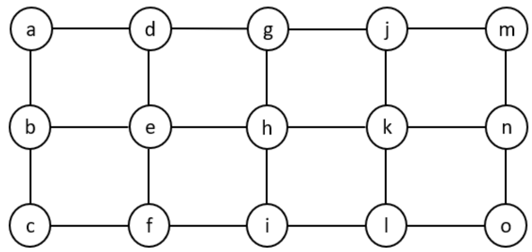


Figure 1: A 3×5 grid

3. (1 point) What is the weight of the maximum weight independent set of the path in Question 1 ?
   A. 15
   B. 16
   C. 17
   D. 18

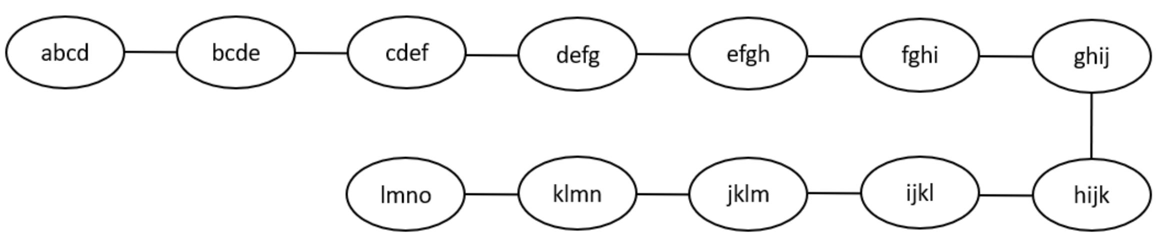
   Solution:

   The path in Question 1 ends at e. The weight of the maximum independent set of the path ending at e is the maximum of the weight of the maximum independent set of the path ending at d and the sum of the weight of e and the weight of the maximum independent set of the path ending at d which does not contain d. Therefore, $A(e) = \max(B(e), e + B(d)) = \max(16, 3 + 12) = 16$. Therefore the value of z, i.e., the weight of the maximum weight independent set of the path P is 16. Thus option B is the only correct choice.

4. (1 point) Is the following statement True or False: Every subgraph of the $k \times n$ grid where $k < n$ and $k \geq 2$ has treewidth exactly k.
   A. True
   B. False

   Solution:

   The $k \times n$ grid has a tree decomposition of width k. The figure below shows a decomposition of width 3 of the 3×5 grid in Figure 1.



   But a subgraph of the $k \times n$ grid may have a decomposition of smaller width. For example, a path is also a subgraph of the the $k \times n$ grid but it has treewidth 1.

5. (1 point) The treewidth of the graph G shown in Figure 2 is —— .

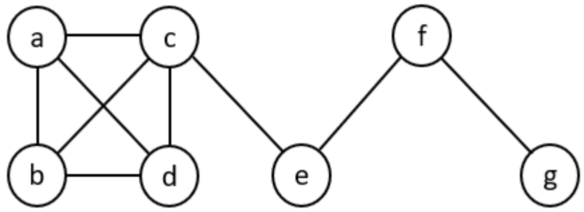


Figure 2: Graph G

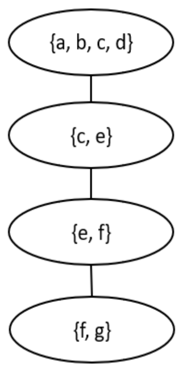
A. 2

B. 3

C. 4

D. 5

Solution: The graph G has a clique of size 4 hence its treewidth cannot be less than 3. A tree decomposition of width 3 of graph G is given below:



Hence option B is the only correct answer.

6. (1 point) Statement I: The treewidth of a graph must decrease after removing (deleting) a vertex.

   Statement II: The treewidth of a graph may decrease after removing (deleting) a vertex.

   Statement III: It is possible that the treewidth of a graph does not decrease after removing (deleting) a vertex.

   Which of the above statement(s) is/are correct?

   A. I

   B. II

   C. III

   Solution:

   Consider the case where graph G is a path on $n > 2$ vertices. After deleting a degree one vertex, it still remains a path and thus its treewidth remains unchanged. Thus Statement I is not correct and Statement III is correct.

   If the graph G is a cycle, then it becomes a path after deleting any one vertex and its treewidth decreases from 2 to 1.

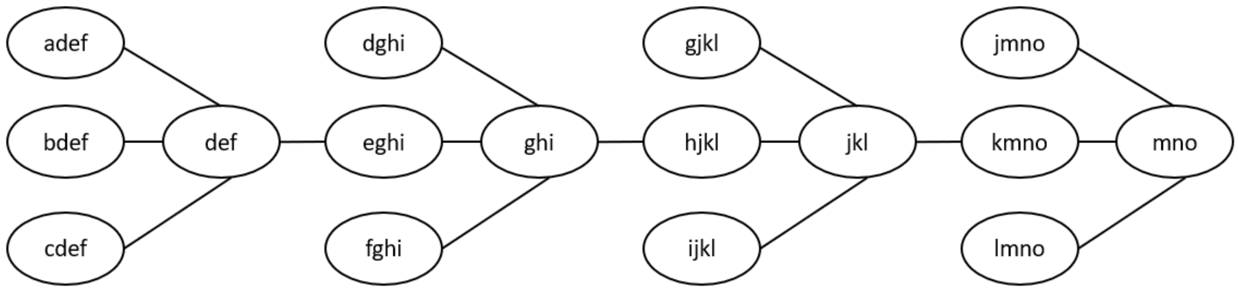


Figure 3: A possible tree decomposition of the 3×5 grid

7. (1 point) Consider the 3×5 grid in Figure 1. Is the following statement True or False: The graph in Figure 3 shows a possible tree decomposition of the 3×5 grid in Figure 1.

   A. True

   B. False

   Solution:

   The set of all the bags containing the vertex d does not induce a connected subgraph. Hence it is not a correct tree decomposition. There may also be other mistakes.

8. (1 point) Statement I: The only graphs of treewidth 1 are trees.

   Statement II: Graphs of treewidth 5 cannot contain a clique on 7 vertices.

   Statement III: A graph of treewidth k must contain a clique of $k+1$ vertices.

   Which of the following statement(s) is/are correct?

   A. I

   B. II

   C. III

   Solution:

   Statement I is incorrect because forests also have treewidth 1. Let C be a clique in a graph G. Consider any tree decomposition $\mathcal{T}$ of G. Either there exists a bag which contains all the vertices in C or there exists an edge t_1t_2 of $\mathcal{T}$ such that after deleting $X_{t_1} \cap X_{t_2}$, the graph induced by the remaining vertices have at least two components whose intersection with the vertices in C is non-empty. Suppose there does not exist a bag which contains all the vertices of C; and u and v are vertices of C which belong to different components of G (after deleting $X_{t_1} \cap X_{t_2}$), then there cannot be an edge between u and v. But this is not possible as C is a clique. Therefore, there must be a bag containing all the vertices of C. Thus treewidth of a graph containing a clique on 7 vertices must be at least 6. Thus Statement II is correct. Consider a cycle $a_1a_2 \dots a_na_1$ and its tree decomposition given by a path $v_2 \dots v_{n-1}$ where $n > 3$ and each $v_i = \{a_1, a_i, a_{i+1}\}$ for $i \in [2, n]$. Thus treewidth of this cycle is 2 (it cannot be less than 2 because the only graphs of treewidth 1 are forests). Thus Statement III is incorrect.

9. (1 point) A chordal graph is a graph which does not contain any induced cycle of length 4 or more. Is the following statement true or false: A minimal cut S of a connected chordal graph G induces a clique of G.

   A. True

   B. False

Solution: Suppose G is a chordal graph and S is a cut of G. If S is not a clique then there exist u and v in S such that u and v are non-neighbours. Let a and b be vertices which belong to different components of $G \setminus S$. Since S is a minimal seperator there exists at least two disjoint paths in G from a to b one of which passes through u and the other through v. Consider the union of the shortest path from a to b passing through u and the shortest path from a to b passing through v. This induces a cycle of length at least 4 in G which is not possible. Therefore, S must contain a clique. Thus the statement is true.

10. (1 point) Which of the following statement(s) is/are correct?

    A. If G has a vertex cover of size k, the treewidth of G is less than $2k + 1$.

    B. If G has a vertex cover of size k, the treewidth of G is less than $k + 1$.

    C. The treewidth of a subgraph of G cannot be more than the treewidth of G.

Solution:

Let V be a vertex cover of G of size k. Let $I = V(G) \setminus V$ be the independent set in G after deleting V. Consider the tree decomposition such that the root of the tree is a bag containing all the vertices of V and each leaf of the tree is a bag which contains all the vertices of V and one vertex of I. The root is adjacent to all the leaves of the tree. This is a tree decomposition of G of width k. Thus both option A and option B are correct.

Any tree decomposition of G can be used to get a tree decomposition of a subgraph H of G by replacing each bag with its intersection with $V(H)$. Thus the treewidth of H cannot be more than that of G. Thus option C is correct.