Assignment 11 - Theory of Computation

Jan 2023

1. (1 point) We define the language

$$SUB\text{-}GRAPH = \{\langle G, H\rangle \mid G, H \text{ are graphs and } H \text{ appears as a subgraph of } G\} \quad (1)$$

   Then $SUB\text{-}GRAPH \in$ NP.

   A. True

   B. False

2. (2 points) Pick all the statements that are true.

   ■ $ALL_{DFA} \in P$

   ■ $\overline{ALL_{DFA}} \in NP$

   ■ **If $\overline{ALL_{DFA}}$ is NP-Complete then $P = NP$**

   ■ **If $3SAT \leq_P ALL_{DFA}$ then $P = NP$**

   Solution: It can be shown that ALL_{DFA} is in P, since we can construct the complement of a DFA $D \in ALL_{DFA}$ and check whether the complement is in E_{DFA}, both of which can be done in Polynomial time (since E_{DFA} requires only a path-check to the final state)

3. (2 points) In the below, consider the below languages:

$$3\text{-}CLIQUE = \{\langle G\rangle \mid G \text{ is an undirected graph that contains a clique of size } 3\}.$$

$$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph that contains a clique of size } k\}.$$

   Pick all the statements that are true.

   ■ $3 - CLIQUE \in P$

   ■ $CLIQUE \in NP$

   ☐ If $CLIQUE$ is NP complete then $P = NP$

   Solution: 3-CLIQUE is in P because we can check all the 3-tuples of vertices of the graph and check if they form a 3-clique. This can be done in $O(n^3)$ time.

4. (2 points) Pick all the true statements

   ■ **$\{\langle G\rangle \mid G$ is an undirected connected graph $\} \in P$**

   ■ $EQ_{DFA} \in P$

   ☐ $EQ_{NFA} \in P$ since $EQ_{NFA} \leq_P EQ_{DFA}$, using the NFA to DFA construction

   ■ **$\{\langle G\rangle \mid G$ is an undirected graph containing a cycle of length 4 $\} \in P$**

Solution: The first option is true because we can perform a traversal like DFS. The second option is true by closure properties of regular languages and reduction to E_{DFA}. All the involved operations can be performed in polynomial time.

The third option is not correct because converting NFA to DFA involves an exponential blow-up, and hence the resulting algorithm may not be in P. The last option is true by checking all 4-tuples of vertices of the graph.

5. (2 points) Consider the language $HALF{-}CLIQUE = \{\langle G = (V, E)\rangle \mid \text{G contains a clique of size } |V|/2\}$. Pick all the statements that are known to be true.

   - ■ $HALF - CLIQUE \in$ **NP**
   - ☐ $HALF - CLIQUE$ is in P
   - ■ $3SAT \leq_P HALF - CLIQUE$
   - ■ $CLIQUE \leq_P HALF - CLIQUE$

Solution: HALF-CLIQUE is NPC because we can reduce CLIQUE to HALF-CLIQUE. The idea is to add an arbitrary set of extra nodes to a Graph with a half-clique so that it becomes a larger graph with a $k - clique$ for arbitrary k. For the exact solution refer to Page 329 in Sipser 3rd edition or this link. Since it is already established that $3SAT \leq_P CLIQUE$, and that $SAT \leq_P 3SAT$, option 3 follows and HALF-CLIQUE is NPC.

6. (1 point) P is closed under the * operation - (Kleene closure). True or False?

   A. True

   B. False

Solution: We can tilize dynamic programming as described in this link.

7. (1 point) NP is closed under the * operation - (Kleene closure). True or False?

   A. True

   B. False

Solution: Non deterministically break the input string of the language into pieces and then non-deterministically guess the certificates for each piece.

8. (2 points) Consider the following variant of SUBSET-SUM: Given a target integer t, and a set of positive integers $\{x_1, x_2, \ldots, x_k\}$, determine if there exists a set of integer (positive or negative) weights $w_1, w_2, \ldots, w_k$ such that $\sum_{i=1}^{k} w_i x_i = t$. Which of the following statements is/are correct about this problem? Select all the correct options.

   - ■ **This problem is in P**
   - ☐ This problem is not known to be in P
   - ■ **This problem is in NP**
   - ☐ This problem is NP-complete

Solution: This language is in P. This is because there exists a solution if and only if t is a multiple of the gcd of the integers $x_1, x_2, \ldots, x_k$. The gcd can be computed in polynomial time by Euclid's gcd algorithm.

9. (2 points) HAM-PATH is in P when the given graph G is a tree.

   A. True

   B. False

10. (2 points) Consider a generalized version of Sudoku played on a $n^2 \times n^2$ grid ($n = 3$ for normal Sudoku). Each row, column, and $n \times n$ sub-grid should contain the numbers $1, 2, \ldots n^2$ exactly once. Define the language

$$SUDOKU = \{\langle G \rangle \mid G \text{ is a partially filled generalized sudoku grid and has a valid solution}\}$$

Which of the following statements is/are true? Select all that are correct.

   A. $SUDOKU \leq_P$ SAT

   B. $SUDOKU$ over a 9×9 input grid $\leq_P$ 9-colorability

Solution: Option 1 can be shown by assigning n^2 variables to each square, with variable $x^k_{(i,j)}$ signifying whether the square with coordinates (i, j) has value k. The corresponding equations for validity and correctness can then be computed.

Option 2 can be shown by assigning one vertex to each square, and connecting two vertices if their corresponding squares lie in the same row/column/sub-grid.

To represent the initially filled positions, we add extra vertices labelled $s_1, s_2, \ldots s_9$, and connect them all to each other, making them all 9 different colors. Now if we are given that a certain vertex has value k, we connect it to every s_ℓ, $\ell \neq k$. This makes this vertex the same color as s_k.

11. (2 points) We also define the Latin Square, as a $n \times n$ grind where each row and column contains each of the numbers $1, 2, \ldots n$ exactly once. Note that this is similar to sudoku but without the sub-grid condition.

We define the language LATIN SQUARE COMPLETION or LSC as

$$LSC = \{\langle G \rangle \mid G \text{ is a partially filled Latin Square and has a valid solution}\}$$

Which of the following statements is/are true? Select all that are correct.

   A. $LSC \leq_P SUDOKU$

   B. LSC is NP-complete $\implies$ $SUDOKU$ is NP-complete

Solution: See http://www.cs.ox.ac.uk/people/paul.goldberg/FCS/sudoku.html for a good explanation for **Option 1**.

Option 2 clearly follows, SUDOKU is in NP by the previous problem, and the NP hardness follows by Option 1.