Data-driven Identification of the Breathing Mode governing equations

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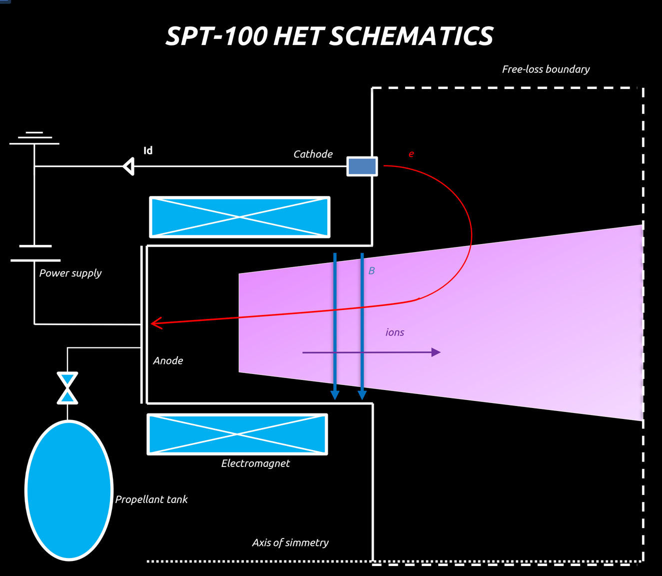
A framework developed to derive simple algebraic models from data is presented. When applied to the ion and neutral dynamics in the downstream channel region of a Hall Effect Thruster, it returns equations matching the ones coming from the theory. Parametric and pointwise analyses extend the 0D models to other operating points and thruster locations, providing further model interpretability and insight .

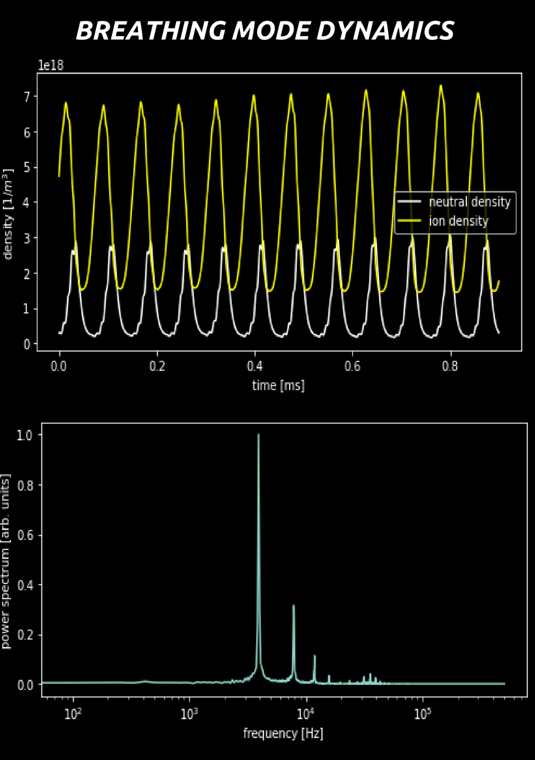
Breathing Mode

- Hall Effect Thrusters are one of the most mature electric space propulsion systems.
- They exhibit oscillations linked to plasma instabilities.
- The most prominent is the Breathing Mode, a low-frequency oscillation linked to an ionization instability.
- The simplest model that described it is the Lotka-Volterra predator-prey equations between ions and neutrals:

$$\begin{cases} \frac{\partial n_i}{\partial t} \equiv \dot{n}_i = \xi n_n n_i - \frac{u_i}{L} n_i \\ \frac{\partial n_n}{\partial t} \equiv \dot{n}_n = -\xi n_n n_i + \frac{u_n}{L} n_n \end{cases}$$

- More complex 0D models have been developed from analytical grounds [1].





Objectives

Data-driven equation discovery allows the description of complex physical phenomena where prior knowledge is lacking. In this work, **sparse regression** is used to recover the Breathing Mode equations with few initial assumptions. Some notable achievements include:

- A model selection procedure for large parametric searches **without researcher bias**.
- Several extensions to the original equation discovery algorithm SINDy [2] to facilitate **real-world applications.**
- Pointwise analysis to study the change in the **Dominant Physics Balance** along the thruster

EP2-SINDy Framework

SINDy

- Expressing the system dynamics as a library of functions of the state variables (features) multiplied by a set of coefficients

$$\dot{x}_i(t) = f_i(\vec{x}(t), t) = \beta_{ij} \Theta_j(\vec{x}(t), t)$$

- The coefficients β_{ij} can be obtained by minimizing the Least Square error:

$$\varepsilon_{ik}^{s} = \left\| \hat{\dot{x}}_{ik} - \beta_{ij} \Theta_j(\hat{\vec{x}}(t_k), t_k) \right\|^2$$

where $\hat{x}_{ik}$ denotes the data-point of variable i at time k.

- By adding a **sparsity promoting term** to the minimization we can expand Θ_j without overfitting. We use the Adaptive LASSO penalty [3]:

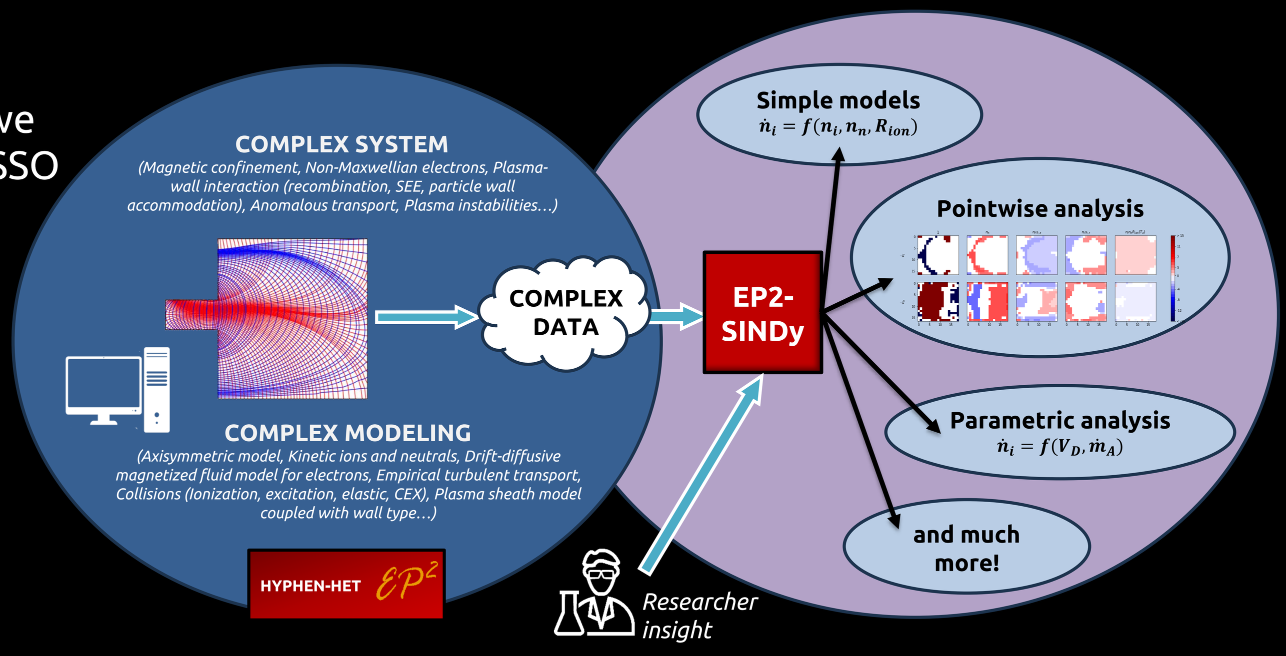
$$\varepsilon_{ij}^{\lambda} = |a_{ij}\beta_{ij}| \quad where \;\; a_{ij} = \left(\operatorname{argmin}\left\{ \sum_k \varepsilon_{ik}^{s} \right\} \right)^{-1}$$

- In this way, the coefficients are obtained from solving

$$\beta_{ij} = \operatorname{argmin}\left\{ \sum_k \varepsilon_{ik}^{s} + \lambda_i \sum_j \varepsilon_{ij}^{\lambda} \right\}$$

Limitations

1. The correct sparsity level is not known a-priori
2. Regressing on the derivatives amplifies noise and does not account for the sequential nature of time-series data
3. It does not benefit from the knowledge of conservation laws and symmetries of the physical system.
4. Feature selection can be inconsistent for different samplings of $\hat{\dot{x}}_{ik}$ and $\Theta_j(\hat{\vec{x}}(t_k), t_k)$.



Extensions

In our work we address all these issues by expanding upon standard SINDy:

1. For a wide range of λ_i, plot the resulting models in a *Model Error* vs *Model Complexity* plot. This will form a L-shaped Pareto front where the "knee" denotes **the simplest model able to describe the data**.
2. We define weak form and integral error variants

$$\varepsilon_{in}^{w} = \left\| \left(\hat{x}_{ik} - \beta_{ij} \Theta_j(\hat{\vec{x}}(t_k), t_k) \right) \phi_{in} w_{ik} \right\|^2$$

$$\varepsilon_{in}^{I} = \left\| \hat{x}_{ik}^{(n)} - \tilde{x}_{ik}^{(n)} \right\|^2 \quad where \;\; \tilde{x}_i^{(n)}(t) \begin{cases} \dot{x}_i = \beta_{ij}\Theta_j(\vec{x}(t), t) \;\; \forall i \\ x_i(t_0^n) = \hat{x}_i(t_0^n) \end{cases}$$

Leading to the minimization of $\beta_{ij}^{w} = \operatorname{argmin}\{\sum_n \varepsilon_{ik}^{w} + \lambda_i \sum_j \varepsilon_{ij}^{\lambda}\}$ and $\beta_{ij}^{I} = \operatorname{argmin}\{\sum_i \sum_n \varepsilon_{in}^{I} + \lambda_i \sum_i \sum_j \varepsilon_{ij}^{\lambda}\}$, respectively.

3. We introduce linear constraints through a matrix C_{ijl}

$$\varepsilon_{ik}^{C} = \left\| \hat{\dot{x}}_{ik} - C_{ijl} b_l \Theta_j(\hat{\vec{x}}(t_k), t_k) \right\|^2 \qquad \varepsilon_{ij}^{\lambda c} = |a_{ij} C_{ijl} b_l|$$

leading to $\beta_{ij}^{c} = C_{ijl} \operatorname{argmin}\{\sum_i \sum_k \varepsilon_{ik}^{c} + \lambda \sum_i \sum_j \varepsilon_{ij}^{\lambda c}\}$.

4. We repeat the sparse regression procedure for several bootstraps obtained from random subsampling with replacement of the data.

Results

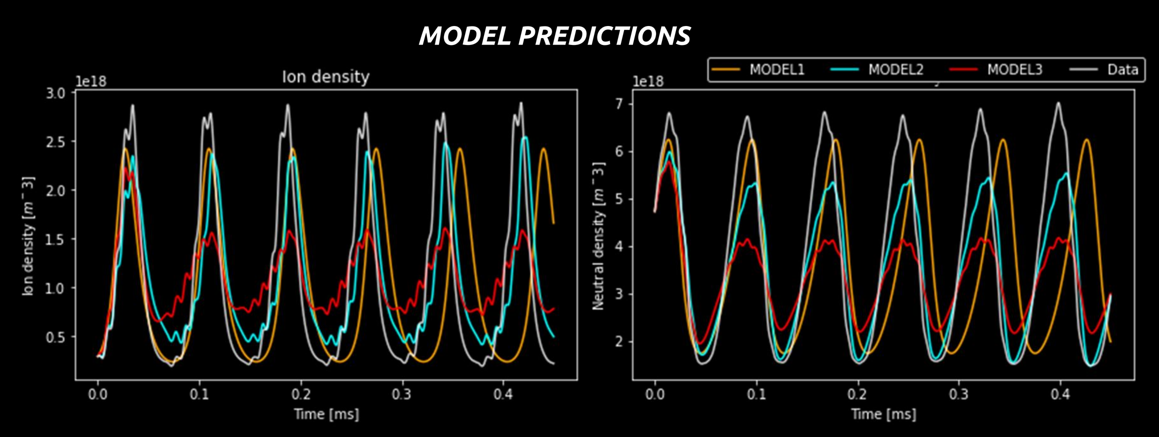
We use data coming from simulations of the SPT-100 HET using the in-house code HYPHEN [4]. We build Θ_j with all polynomial combinations of the $\vec{x}(t)$ state variables up to degree 3. Running the EP2-SINDy framework we obtain the following Pareto optimal models:

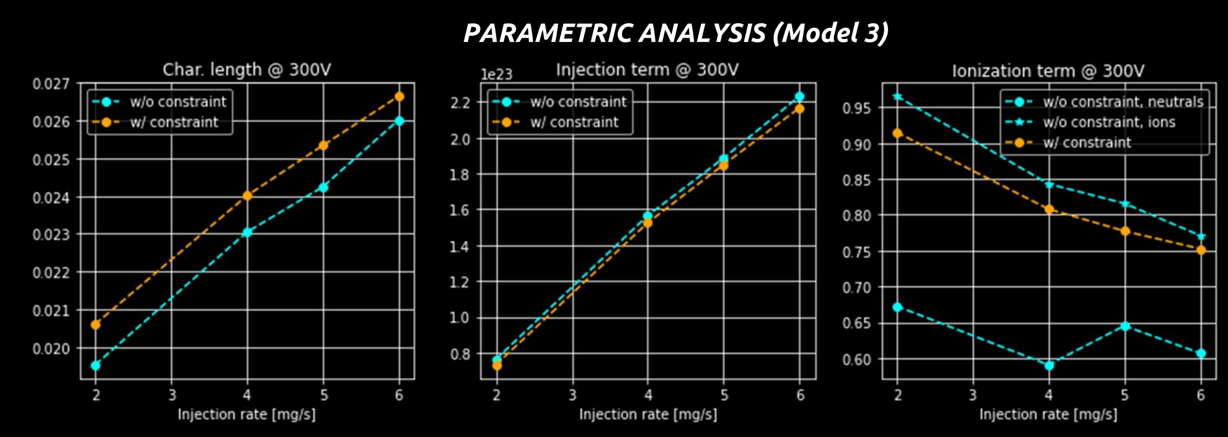
$$\text{Model 1} \begin{cases} \dot{n}_i = 4.12 \cdot 10^{-14}\, n_n n_i - 1.45 \cdot 10^{5}\, n_i \\ \dot{n}_n = -4.89 \cdot 10^{-14}\, n_n n_i + 4.58 \cdot 10^{4}\, n_n \end{cases} \quad \text{for } \vec{x}(t) = [n_i, n_n]$$

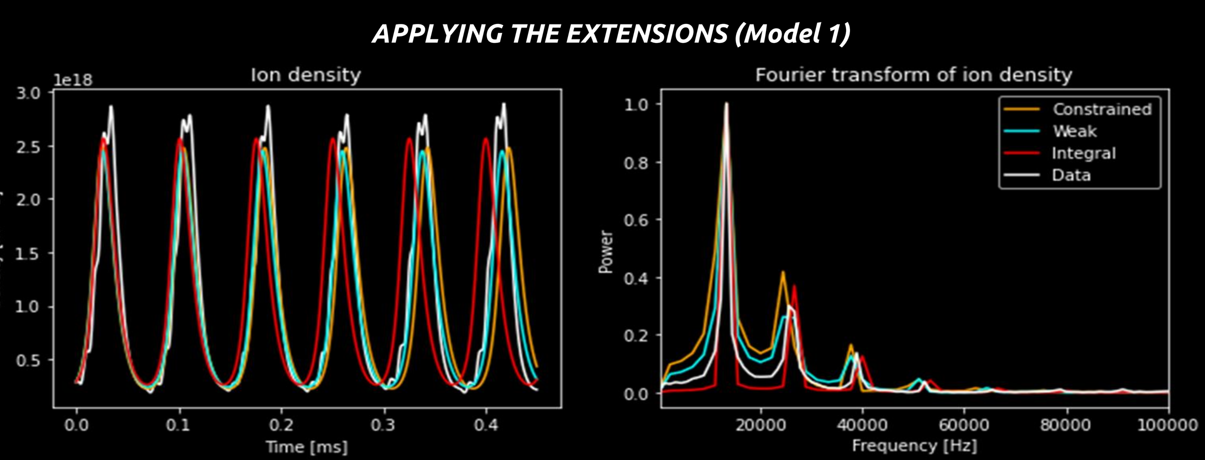
$$\text{Model 2} \begin{cases} \dot{n}_i = 1.16\, n_n n_i R_{ion} - 2.66 \cdot 10^{5}\, n_i \\ \dot{n}_n = -0.77\, n_n n_i R_{ion} + 1.83 \cdot 10^{23} \end{cases} \quad \text{for } \vec{x}(t) = [n_i, n_n, R_{ion}(T_e)]$$

$$\text{Model 3} \begin{cases} \dot{n}_i = 0.82\, n_n n_i R_{ion} - 41.2\, n_i u_{i,z} \\ \dot{n}_n = -0.77\, n_n n_i R_{ion} + 1.83 \cdot 10^{23} \end{cases} \quad \text{for } \vec{x}(t) = [n_i, n_n, R_{ion}(T_e), u_{i,z}, u_{i,r}]$$

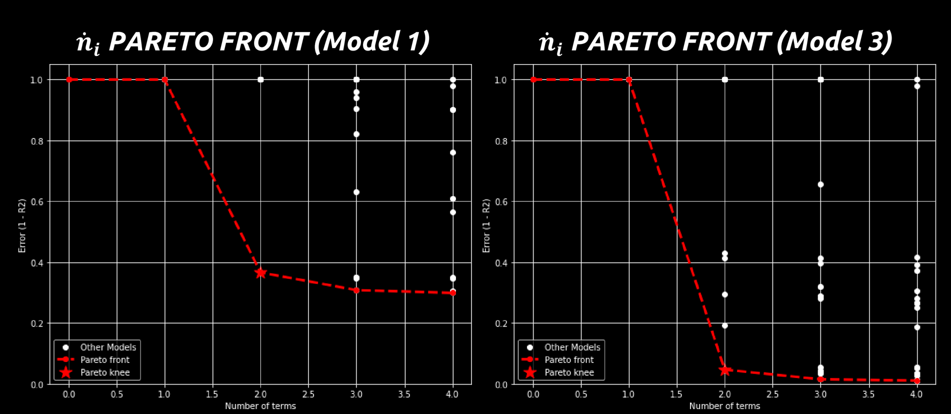
We have the possibility of forcing the ionization terms to be equal.

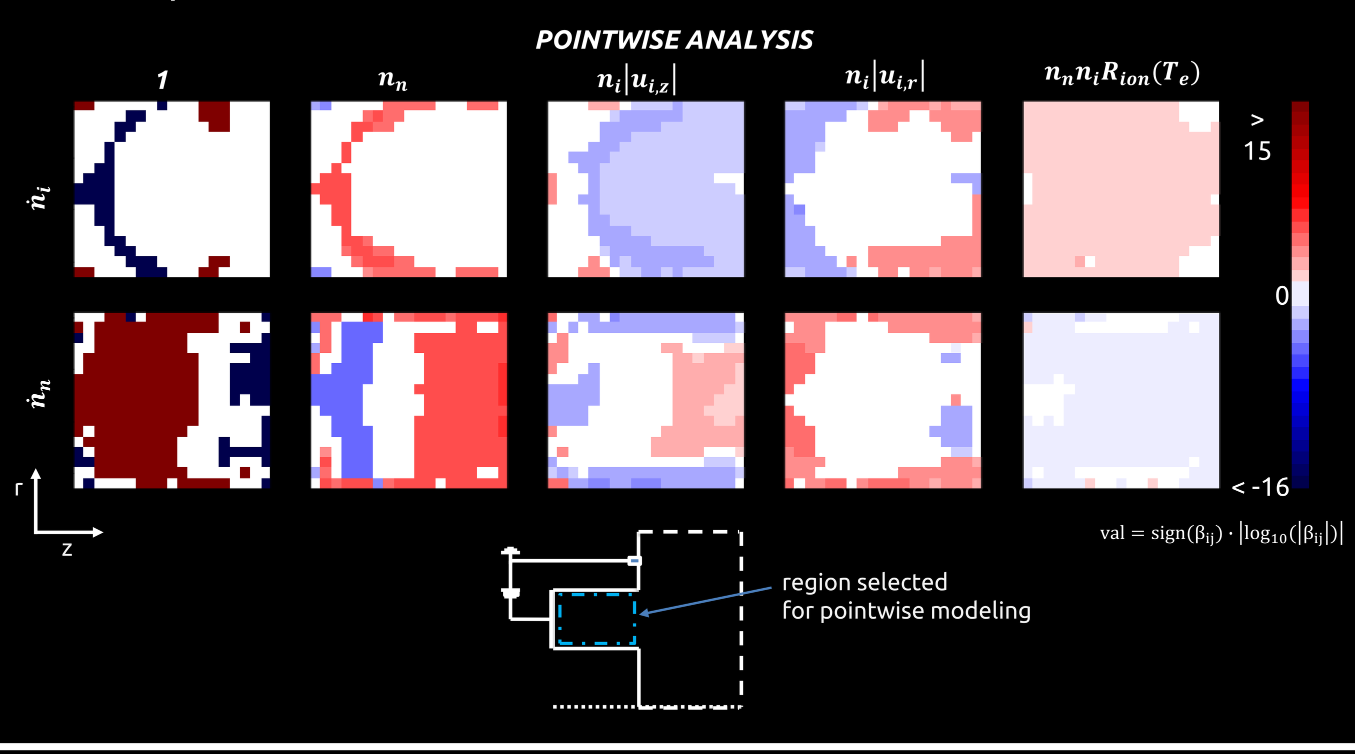






- Parametric analysis obtained by re-fitting Model 3 for data coming from several operating points.
- Pointwise analysis is carried out by obtaining the Pareto-optimal model for every point in the thruster channel using the library of Model 3 and plotting the coefficient distribution.





Discussion

- The EP2-SINDy was able to retrieve models resembling those derived from analytical grounds, both in shape and magnitude, in a completely data-driven way.
- The fine-tuned Model 1 outperforms the rest when integrated
- Adding the ionization rate leads from a proportional injection to a constant injection neutral influx term.
- Parametric analysis reveals that the ionization term also accounts for other phenomena (possibly recombination + others)
- Pointwise analysis reveals the spatial dependence of the neutral influx term depending on the distance to the anode, the relevance of axial ion transport close to walls and the overall global ion dynamics.

Acknowledgments

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