Difference Equations

Difference equations are a tool to model change. For example, we can use these equations to model change in a population or change of any other quantity. To understand difference equations, we shall look at some vocabulary first.

Sequences: A sequence of numbers is a function whose domain is 0 to infinity. An example would be the following:

$$\{0, 2, 4, 6, 8, 10, 12, \ldots\}$$

In this case, since 0 is the "zeroth term, " we denote it as a_0. The term 2 would be a_1, the term 4 would be a_2, and so forth. Note that in function notation:

$$f(0) = 0, f(1) = 2, f(3) = 3 .$$

Dynamical Systems: We will now formulate a rule between the terms of the sequence. This is called a dynamical system. A dynamical system is a rule between two adjacent terms of a sequence, namely, between a_{n+1} *and* a_n. We can use the following to equation to help us:

$$change = \Delta a_n = a_{n+1} - a_n$$

Another name for Δa_n is the *nth first difference.* Using the equation, we find the first couple *nth first differences* for the above sequence,

$$\Delta a_0 = a_1 - a_0 = 2$$

$$\Delta a_1 = 2$$

$$\Delta a_2 = 2$$

and so forth. To obtain the dynamical system for this sequence, we try to find a pattern between all the nth first differences and then find a general equation for a_{n+1}. This example is relatively straightforward. Since $\Delta a_n = a_{n+1} - a_n = 2$, the dynamical system would be:

$$a_{n+1} = a_n + 2$$

We shall take another, more challenging example. Suppose we were to find the dynamical system for the following sequence:

$$\{3, 9, 27, 81, 243, \ldots\}$$

Our first step would be to find the nth first differences:

$$\Delta a_0 = a_1 - a_0 = 6$$

$$\Delta a_1 = 18$$

$$\Delta a_2 = 54$$

$$\Delta a_3 = 162$$

From this pattern, we can see that the Δa_n is exactly $2a_n$. Thus, the change is dependent on the value itself. From here, we can say:

$$\Delta a_n = a_{n+1} - a_n = 2a_n$$

Solving for a_{n+1} we get

$$a_{n+1} = 3a_n$$

That is our dynamical system.

Solutions: Finally, we shall define the solution to a dynamical system. A dynamical system only gives you the n+1 term from the n term. If I have a_2, then The dynamical system would give me a_3. If I were to get a_{68}, then we have to **iterate** to find it. From the first term, we get the second, and then the third, all the way until term 68.

The solution of a dynamical equation allows us to find any term by just plugging it in an equation. For example, the solution to $a_{n+1} = a_n + 2$ is :

$$a_k = a_0 + 2k$$

Using this equation we can find the 68th term to be 136 (assuming ao to be 0).

But how exactly do we find the solution to the dynamical system? We shall take a look at that later.

TIP: During the rest of this tutorial, do not confuse a dynamical system with its solution.

Modeling with Δ Equations

Modeling Discrete Change: To model discrete change with difference equations is quite easy. Suppose we were modeling the interest accumulation of a deposit in a bank account. Let's say you have a principle of $1000 in the bank account and the bank pays you a monthly interest of 8%. Then, a_0 would be the principal amount, 1000; a_1 would be the total in the bank account after 1 month, a_2 would be the total in the bank account after 2 months, etc. To find the dynamical system, **we can calculate** $a_1, a_2, a_3, a_4, \ldots$ **for a couple months, find the nth first differences, and then deduce a pattern between them**. BUT, in this case, we are already given the change, $\Delta a_n = 0.08a_n$, because the change per month is just the interest. Solving for a_{n+1}, we get

$$a_{n+1} = a_n + \Delta a_n = 1.08a_n$$

Modeling Continuous Change: modeling continuous change is only slightly different. In the previous example, interest accumulation happens every month, so it was discrete. To model a continuous phenomenon such as growth of an organism, we should measure the growth at discrete intervals, such as every day. Thus, a_0 would be the initial state, a_1 would be the growth after 1 day, etc.

> **TIP**: When modelling with difference equations, *find the change* and *conjecture a pattern.*

Modeling Growth: Over the years, many scientists and mathematicians have made models pertaining to the growth of a population over time. One such model is the Malthusian Model, or the boundless-growth model. This model assumes that the growth rate is proportional to the size of the population. In other words,

$$\Delta a_n = ka_n$$

This makes sense because a population of size 2 would grow slower than a population of size 100. A bacterial colony of 2 members may divide to become 4 bacteria, but a colony of 100 bacteria would divide to become 200.

The dynamical system for the above difference equation is:

$$a_{n+1} = a_n + ka_n = (k+1)a_n = ra_n$$

$$a_{n+1} = ra_n$$

where r is another constant.

How do we know when the Malthusian Model applies to a particular population? If we are given data on the growth of the population, we can find the nth first differences. Then we can graph $a_n \textit{ with } \Delta a_n$ for all n. Since you know that the two quantities are directly proportional, all we have to see is if the said graph is a straight line. Then we know that we can model the growth of a population with the Malthusian Model.

Note that although the dynamical system of this growth model is linear, the solution to the dynamical system is not. In fact, it has an exponential growth. The solution to the Malthusian growth model is actually given as:

$$a_k = a_0 r^k$$

The implications of the model is profound. Theoretically at a distant point in time, because the growth is exponential, the population size should be infinity. The Malthusian model only works best for populations with abundant resources. In fact, this model RARELY applies in nature. This is because normally, a population is constrained by many factors. Organisms are constantly competing over resources such as land, food, water, etc. If the number of organisms exceeds a certain value, the environment would not be able to support them all, and many animals would start dying.

A new model can be formulated taking these into account. This is called the logistic growth model. The maximum number of organisms an environment can support due to constrained resources is called the carrying capacity. The dynamical system for this model is:

$$\Delta a_n = ka_n(C - a_n)$$

where C is the carrying capacity and k is a constant. Note the beauty of this equation: when a_n is small, the change is large, but as a_n approaches C, the change becomes less and less. Finally, when a_n exceeds C, the growth rate becomes negative. The solution of the dynamical system looks like this:

How do we find out if the logistical growth is appropriate, and how do we find k? Note that $\Delta a_n \text{ and } a_n(C - a_n)$ are directly proportional. Graphing the two quantities should yield a straight line, if the logistical growth applies. From there, finding k is as easy as eating pie, by using the slope formula.

There are many other growth models out there. The Malthus model factors in very few variables. The logistical growth is more complex. The important thing in modeling is to see where to draw the line, it shouldn't be too complex, nor should it be too simple.

Solving with Δ Equations

Finding Solutions: Solving difference equations is a little more tricky. To help us solve these equations, we turn to something called the **method of conjecture.** The method of conjecture is a process that uses patterns to find the solution. The steps are:

1. Observe a pattern
2. Conjecture a solution
3. Test the solution by substitution
4. Based on the testing, either accept or reject the solution

The method of conjecture can be slightly difficult, especially the first step. Observing a pattern is quite difficult. We shall go through an example. We stated that the solution to the dynamical system $a_{n+1} = ra_n$ is $a_k = a_0 r^k$. How did we arrive to this? We shall first find a pattern between the first couple terms of the sequence. Using the dynamical system, we get:

$$a_0 = a_0$$
$$a_1 = ra_0$$
$$a_2 = r(ra_0) = r^2 a_0$$
$$a_3 = r(r^2 a_0) = r^3 a_0$$

Observing the pattern, we can extend this to say that our solution is:

$$a_k = a_0 r^k$$

To test this solution, we can plug in some values and see if it checks out.

Long-Term Behavior: Examining the long term behavior of a dynamical system is often useful. Does it grow without bound? Does it decay? Does it remain at a constant value?

The long-term behavior of solutions of linear dynamical systems ($a_{n+1} = ra_n,\ \textit{whose solution is}\ a_k = r^k a_0$) varies on the constant r. The behavior is outlined in the table below.

LONG-TERM BEHAVIOR TABLE

Value of R	Behavior of solution	Reason
r = 0	Constant at 0	The solution, $a_k = r^k a_0$ will be 0 because r = 0.
r = 1	Constant at the initial value	The solution, $a_k = r^k a_0$ will be a_0 because r = 1.
r < 0	Oscillation	Since r is being raised to a power, even exponents will give positive values and odd exponents will give negative values
|r|<1	Decay towards 0	Since r is being raised to a fractional exponent, it will curve down towards 0.
|r|>1	Boundless growth	The pattern is exponential growth.

Another pattern to look for in dynamical systems is the **equilibrium point**. The equilibrium point *e* is such that if *e* is the initial value of the dynamical system, then all other following values will be *e*. That is, if $a_0 = e$, then $a_n = e$ for all n.

A common technique to find the equilibrium point is to graph the dynamical system for different initial values. If, say, the equilibrium point of a certain dynamical system is 12, then graphing the dynamical system whose initial value is 12 will yield a horizontal line.

This brings us to stable equilibrium points. A stable equilibrium point is such that if the initial value is a slightly different , the dynamical system will still converge on the equilibrium point. Continuing with our previous example, assuming our eq. point- 12- is stable, then if we enter an initial value of 11.9 or 12.1, it will still converge to 12. If our equilibrium point was unstable, than the initial values of 11.9 or 12.1 would lead to divergent series which would not converge on 12. With unstable equilibrium points, we would need a PRECISE initial value for the rest of the series to converge on that point, slight deviations would lead to divergence.

Another, less haphazard way of finding equilibrium points exist. We shall take the example of the dynamical system $a_{n+1} = ra_n + b$. Since at the equilibrium points, all terms of the series are same, we get

$$a = ra + b$$

Solving for a, we get

$$a = \frac{b}{1-r}$$

However, if a is 0/0, all numbers are equilibrium points. If a is n/0, then no equilibrium points exist. Another fun fact is that for the dynamical system $a_{n+1} = ra_n + b$, if |r| > 1, the equilibrium is unstable, and if |r|<1, the equilibrium is stable. Finally, we can say that the solution to the dynamical system $a_{n+1} = ra_n + b$ is

$$a_k = r^k c + \frac{b}{1-r}$$

where c is some constant. The above equation can be proven using substitution.

Finally, we shall look at chaotic systems. Dynamical systems can be classified as chaotic or non-chaotic by changing the value of constant parameters. Take the dynamical system for the logistic growth model. Using algebraic manipulation, we can simplify the dynamical system to the form:

$$a_{n+1} = r(1 - a_n)a_n.$$

Keeping the initial value constant, if we substitute different values of r, iterate to find the resulting numerical solution, and graphing, we see that each instance is wildly different from each other. Since we changed a constant parameter, r, and obtained wildly different results each time, we can say that this is a chaotic system.

Systems of Δ Equations

Systems of Difference Equations are multiple difference equations modelling the same system. As we proceed, we should keep in mind to determine long-term behavior, including sensitivity and chaos.

We shall first take an example. There are two coastal cities, A and B. Each city has a port full of ships and boats. Thirty percent of all boats in city A go to city B, and there rest return to the city A dock. Forty percent of all boats in city B go to city A, and there rest return to the city B dock. We will make a system of equations to model the boats in each dock.

$$A_{n+1} = .7A_n + .4B_n$$
$$B_{n+1} = .6B_n + .3A_n$$

Now, we will try to find the equilibrium point. Set $A = A_{n+1} = A_n$ and $B = B_{n+1} - B_n$. Solving, we get

$$B = .75A$$

This means that if we had a total of 70 boats in both docks, we would expect a long-term 3:4 ratio between the two docks, even if one city started with no boats at all.

Applications of Difference Equations

Competitive Hunter Model: We will now turn our attention to specific applications of difference equations. One well-known model is the competitive hunter model. Let us assume, in an aquatic environment, there are small fish and two types of shark, lemon sharks and tiger sharks. Each species competes for the same food source. Now, we shall try to model the growth of each population. Remember, each population has a birth and death rate. We shall try to find a model for the population growth of the lemon shark.

Birth	**Death**	
Birth	Natural death	Mutual decrement
$k_a L$	$-k_b L$	$-k_c LT$

Obviously, the birth factor models the birth rate and the natural death factor models the natural death rate. We can combine the two terms and get our familiar exponential growth dynamical system:

$$\Delta L_n = k_1 L_n$$

So far, none of this is new. We have just deduced the exponential growth model.However, in this example, tiger sharks can also decrease the lemon shark population. How? Because both species compete for the same food, the tiger shark removes the food source for the lemon shark, thus decreasing survival rate. We model this decrement by counting the possible interactions between the two species (i.e. Sharks Bob and Fred are fighting for the same little fishy and only one gets it.) The mutual decrement factor is given as

$$m.d. = -k_2 L_n T_n$$

Thus, we get a revised population growth model(difference equation) for the lemon shark:

$$\Delta L_n = k_1 L_n - k_2 L_n T_n$$

The model for the tiger shark is the same, with a different variable and different constants of proportionality:

$$\Delta T_n = k_3 T_n - k_4 L_n T_n$$

Now, we shall find the dynamical systems. Since *next term = old term + change*, we get

$$L_{n+1} = (1+k_1)L_n - k_2 L_n T_n$$
$$T_{n+1} = (1+k_3)T_n - k_4 L_n T_n$$

Now, each constant has a specific value. For now, I will just replace them with arbitrary values:

$$L_{n+1} = aL_n - bL_n T_n$$
$$T_{n+1} = cT_n - dL_n T_n$$

Let's now find the equilibrium values substituting all L_k *for* L *and all* T_k *for* T:

$$L = aL - bLT$$
$$T = cT - dLT$$

Solving, we get:

$$0 = L(a - bT - 1)$$
$$0 = T(c - dL - 1)$$

Both equations are satisfied when (L,T) = (0,0) or (L,T)= $(\frac{c-1}{d}, \frac{a-1}{b})$. Those are our equilibrium values. This means that if Lo=L and To=T, then all the rest of the terms will be constant.

Now that we have our equilibrium values, we can test for sensitivity by entering initial values slightly different from L and T and still see if our same equilibrium values are reached.

Predator Prey Models: A similar model to the one above is the predator-prey models. Let us consider the populations of red-tailed hawks and shrews. Obviously, here, the hawks are predators and the shrews are prey. Three major assumptions we will make is that prey die mostly because they are eaten. We also assume that the predator's sole food source is the single prey population, on which the predator's survival rate is dependent on most. Finally we assume that the number of interactions between the populations is SH. We will try to model the growth of each population:

Prey Shrew	$k_1 S$	**Prey Growth**: in the absence of predators, the growth rate is proportional to the population size
	$-k_2 SH$	**Prey Death**: in the presence of predators, the death rate is proportional to the number of possible interactions between the two populations
Predator Hawk	$-k_3 H$	**Predator Death**: in the absence of prey, the death rate is proportional to the population size
	$k_4 SH$	**Predator Growth**: in the presence of prey, the death rate is

		proportional to the number of possible interactions between the two populations

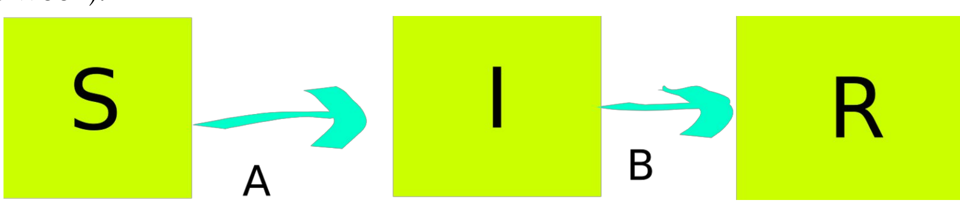
Now, we can make our models:

$$\Delta S = k_1 S - k_2 SH$$
$$\Delta H = -k_3 H + k_4 SH$$

With our difference equations, we can proceed to find dynamical systems, equilibrium points, and sensitivity using techniques covered in *Systems of Difference Equations* and in *The Competitive Hunter Model*.

Discrete Epidemic Models: We shall now turn to the potential of difference equations to model the spread of disease. One common model is called the SIR model, or the susceptible-infected-removed model. Before we get into the model, we shall take a look at the assumptions.

- No immigration/emigration occurs with the population.
- Each person is either susceptible, infected, or removed(unable to get it again, either immune or dead)
- Initially, everyone is either susceptible or infected, but once someone gets the infection, they are immune.
- The average time someone is infected is k weeks.
- Our model will increment by a week (The time difference between n and n+1 is one week).



We shall make the models of S(t), I(t), and R(t) now.

Infect Time	We shall assume an infection time of θ. The removal rate, is therefore 1/ θ, which we will denote γ.
S-I relation	The number of people who go from susceptible to removed is proportional to the number of interactions(SxI) between the two populations. This quantity is β SI/N. β is the proportionality constant. The fraction is divided by N because it is actually a percent.
I-R relation	The number of people who go from infected to removed is γ I(x). This is true because the number of people transferred from the infected to removed segments is equal to the removal rate times the

	infected people.

Now, we can formulate the difference equations:

$$\Delta S_n = -\frac{\beta S(n)I(n)}{N}$$
$$\Delta I_n = \frac{\beta S(n)I(n)}{N} - \gamma I(n)$$
$$\Delta R_n = \gamma I(n)$$

From here, we shall find the dynamical systems for each of three difference equations. Then, with the initial values of S(0), I(0), and R(0), and then by iterating, we can model the spread of disease in a population.

Note, the SIR model is only the basic model. There are many more epidemic models, which take into many other factors overlooked in the SIR model.

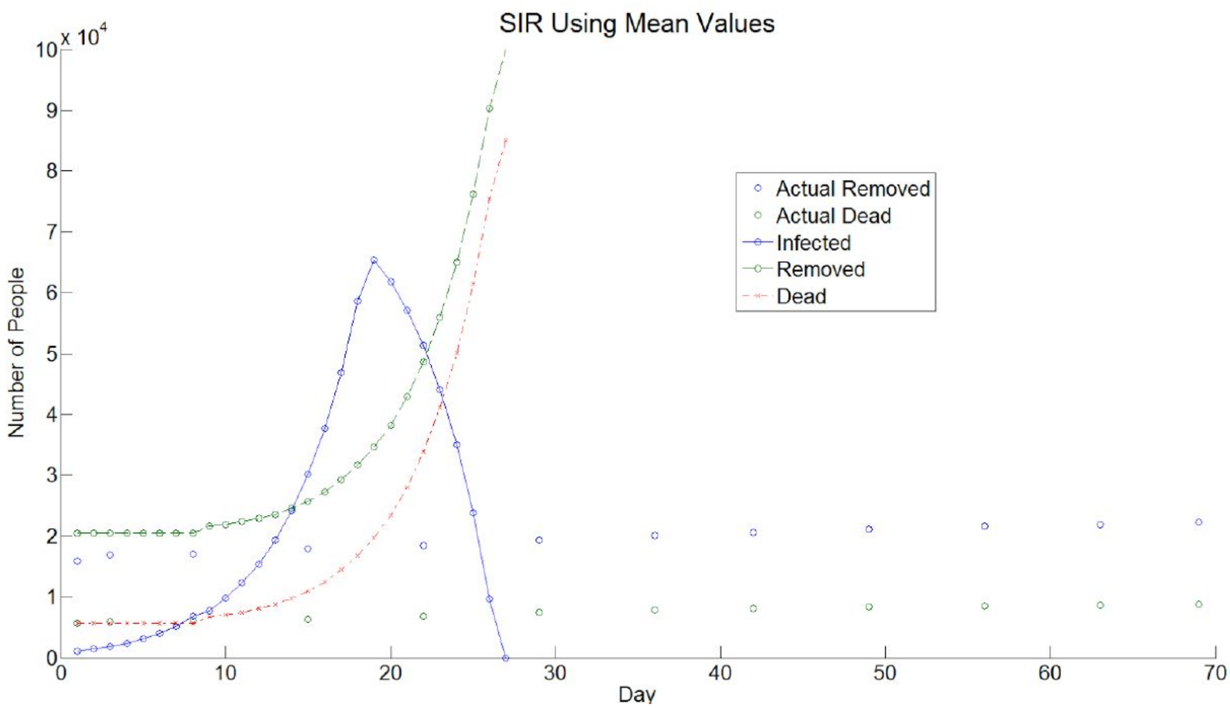
TIP: In many of the models we saw so far, we overlooked many factors. The real world is much too complex. When constructing models, it is important to define our bounds and assumptions.

SIR Model: Difference Equations are extremely useful in simulating the development of a disease, if you do not know much about advanced Calculus. A model that can simulate the development of a disease overtime is required to solve this type of problem. Let's take a look at a simple application of Difference Equation in SIR (Susceptible – Infected – Removed) model.

Function	Description	Equation
S(t)	Susceptible Population	S(t - 1) - I(t - 1)
I(t)	Infected Population	I(t - 1) + S(t - 1) - S(t) - I(t - μ) + I(t - μ - 1) + R(t - μ - 1) - R(t - μ)
R(t)	Removed Population	R(t - 1) + I(t - μ) - I(t - μ - 1) - R(t - μ - 1) + R(t - μ)
C(t)	Cured Population	C(t - 1) + δ [R(t) - R(t - 1)]
D(t)	Dead Population	D(t - 1) + γ [R(t) - R(t - 1)]

To calculate S(t), we will need to compute both S(t–1) and I(t–1). For I(t), we even need to compute the value for all the other variables at different stages! Rather than finding the general form, we can just use our programming skills to let the computers do all the work for you (If you do not know much about coding in Matlab, try to use Java or

C++ and make them print out all the values for Matlab to graph). Since those calculations will still take computers considerable amount of time to run it, it is really important to keep your code efficient. Personally, I recommend vectors rather than 2D arrays, because they take up much less memory and we only need 2 spots to store our values. The code is really long but not hard to program so we will not put it here. The result is shown below.



Of course, SIR model might be too simple for a complicated real-world scenario like this. Standard distribution might be introduced so that values such as transmission rate are no longer fixed numbers, which enables us to figure out best and worst cases.