

Linear Programming

Introduction

Linear programming, also known as linear optimization, is a method used to achieve the best outcome in a situation represented by linear relationships. Linear programming uses various linear inequalities and finds the “best” value obtainable under certain conditions or restrictions. One simple example of linear programming is a problem in which different inequalities are presented and requires the student to graph them to find the area bound by the graphs. In this case, you only need to check the sides and vertices of the shape in order to get the optimized solution.

Example 1 – Solving by Hand

Find the maximal and minimal values of $z = 3x + 4y$, subject to the following constraints

$$y - 4x \geq 0,$$

$$3x - y \geq -1,$$

$$2y + x \geq -12$$

Try solving the problem graphically. In order to ease this process, simplify the constraints so they are in slope intercept form.

$$y \geq 4x, \quad y \leq 3x + 1, \quad y \geq -\frac{1}{2}x - 6$$

Next, graph the three inequalities.

The next step in solving this problem is to find the corner points. Pair each line and solve the corresponding systems:

$$\begin{array}{lll}
 y = 4x, & y = 4x, & y = 3x + 1, \\
 y = 3x + 1 & y = \frac{-1}{2}x - 6 & y = -\frac{1}{2}x - 6 \\
 (1, 4) & (-\frac{4}{3}, -\frac{16}{3}) & (-2, -5)
 \end{array}$$

Now the last step is to substitute these values back into the original equation (z)

$$z = 3x + 4y$$

$$z(1, 4) = 19 \qquad z(-\frac{4}{3}, -\frac{16}{3}) = -\frac{76}{3} \qquad z(-2, -5) = -26$$

From this we can conclude:

max value of z = 19, @ (1,4) min value of z = -26 @ (-2,-5)

Example 2 – Solving using MATLAB

Note: You'll need MATLAB's Optimization toolbox installed to do this.

However, calculating by hand might not be a smart choice for timed competitions. MATLAB offers an easier way to solve linear programming problems.

For example, say that in a large scale problem, you've been able to reduce the prompt to a raw mathematics problem like in example 1, where you need to find the maximum and minimum values of some quantity, as limited by several factors.

Let $T = 6x + 3y + 4z$, **subject to the following conditions:**

$$x + y + z = 120,$$

$$x + 2z \leq 300,$$

$$3y + 4x \geq 100,$$

$$x \geq 30,$$

$$0 \leq y \leq 50,$$

$$z \geq 20$$

Find the maximum and minimum values of T .

First, you need to convert all the inequalities and equations into their standard form, which means that,

1. Matlab can only solve minimization problems; therefore, if the presented expression is a maximization problem, you will need to convert it to a minimization problem.

$$\max 6x + 3y + 4z \rightarrow \min -6x - 3y - 4z$$

2. If any of the inequalities/equations do not have one or more of the variables, you have to add them to the expressions (by adding the desired variable multiplied by 0) and rearrange every variable by alphabetical order.

$$x + 2z \leq 300 \rightarrow x + 0y + 2z \leq 300$$

$$3y + 4x \geq 100 \rightarrow 3y + 4x + 0z \geq 100,$$

3. As part of your conversion of the maximum problem to a minimum problem, reverse the inequality signs.

$$4x + 3y + 0z \geq 100 \rightarrow -4x - 3y + 0z \leq -100$$

In MATLAB, you need to either assign a *vlb* (variable lower bound) or *vub* (variable upper bound) to all or none of the variables.

A *vlb* is basically the lower endpoint of a compound inequality, and *vub* is the upper endpoint.

For instance, in this problem, y is limited between the *vlb* 0 and *vub* 50, as shown by the constraint $0 \leq y \leq 50$. In simpler terms, assigning a *vlb* and *vub* just means splitting up the compound inequality into two inequalities, as follows:

$$0 \leq y \leq 50 \text{ becomes...}$$

$$y \geq 0 \text{ and } 0x + y + 0z \leq 50$$

(Note the addition of the “zeroed” variables, since x and z are missing before)

After all this, we now have the following equations to use in Matlab.

$$x + 2y + 3z = 120$$

$$x + 0y + 2z \leq 300$$

$$-4x - 3y + 0z \leq -100$$

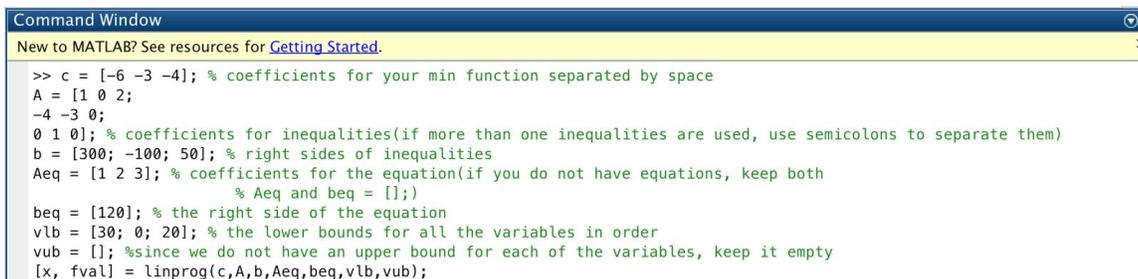
$$0x + y + 0z \leq 50$$

$$x \geq 30$$

$$y \geq 0$$

$$z \geq 20$$

Now let's initialize all the matrices in Matlab.



```
Command Window
New to MATLAB? See resources for Getting Started.
>> c = [-6 -3 -4]; % coefficients for your min function separated by space
A = [1 0 2;
     -4 -3 0;
     0 1 0]; % coefficients for inequalities(if more than one inequalities are used, use semicolons to separate them)
b = [300; -100; 50]; % right sides of inequalities
Aeq = [1 2 3]; % coefficients for the equation(if you do not have equations, keep both
         % Aeq and beq = []);
beq = [120]; % the right side of the equation
vlb = [30; 0; 20]; % the lower bounds for all the variables in order
vub = []; %since we do not have an upper bound for each of the variables, keep it empty
[x, fval] = linprog(c,A,b,Aeq,beq,vlb,vub);
```

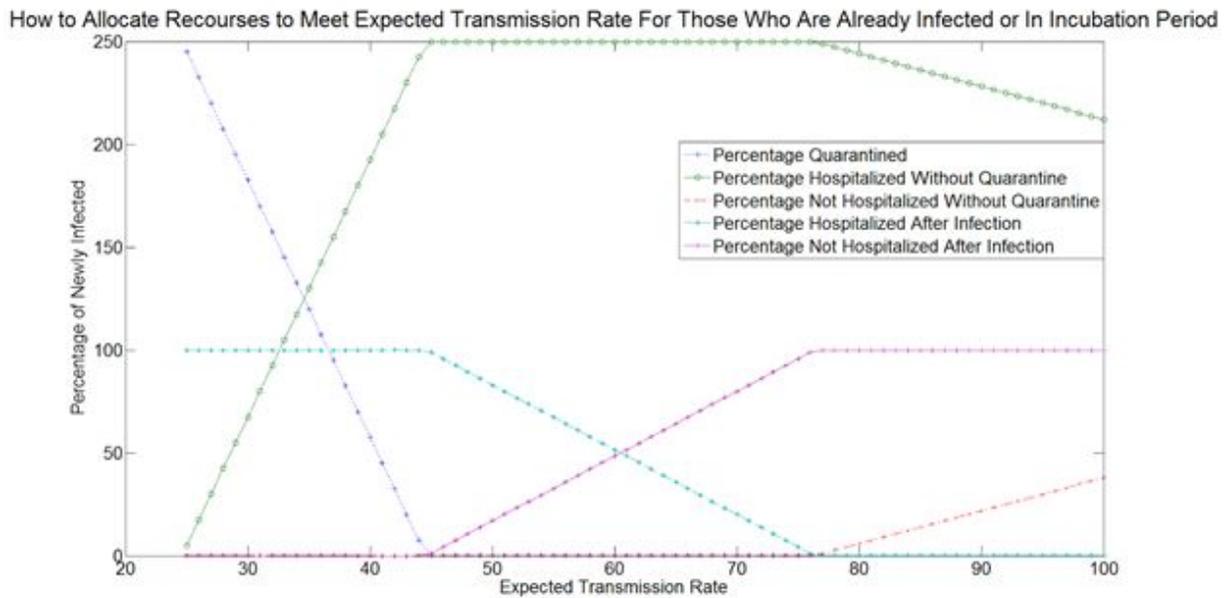
Run it and Matlab will give you the optimum values for x, y, z in the matrix named “ x ”, and the minimum value in the matrix called “ $fval$ ”.

Applications

Linear programming can be a useful technique in solving math modeling problems.

For instance, if we would like to model how to keep a wide-spreading disease under control, linear programming would be essential in minimizing costs while keeping transmission rate under the expected value. In addition, by using a *for-loop* to run through the cost it takes to maintain different

transmission values, we can determine the optimum balance between cost and number of death/infection. Sample graph is shown below.



Another typical example is maximizing the profit of a company that is selling different types of products. Such application is really straightforward and is not very likely to appear by itself. Normally, a mathematical modeling problem requires participants to combine different algorithms together to arrive at a solution. Linear programming is usually a really useful tool for most of the problems.