Mellin Transform Techniques for the Mixed Problem in Two Dimensions

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The Dirichlet Problem

Let Ω be a bounded domain in $\mathbb{R}^n$

The Dirichlet problem:

$$(D) \quad \begin{cases} \triangle u = 0 \quad \text{in} \quad \Omega \\ u\Big|_{\partial\Omega}^{n.t.} = f \quad \text{on} \quad \partial\Omega \\ \mathcal{N}(u) \in L^p(\partial\Omega) \end{cases}$$

Remarks:

- (D) is well posed $\forall p \in (2-\epsilon, +\infty)$ and this result is sharp in the class of Lipschitz domains
- Via the layer potential method (D) is reduced to a BIE of the type $(I+K)g = f$, where K is a singular integral operator of Calderón-Zygmund type

Non-tangential condition

Definition (Non-tangential approach region)

Fix $a > 1$ and for $x \in \partial\Omega$, consider the non-tangential approach region.

$$\Gamma_a + x := \{y \in \Omega : |x - y| < a \cdot dist(y, \partial\Omega)\}$$

Remarks:

- When $\Omega = \mathbb{R}^n_+$, $\Gamma_a + x$ is an infinite upright cone with vertex at x.
- a determines the aperture of the cone/non-tangential region.

Fix $a > 1$ and $u : \Omega \longrightarrow \mathbb{R}$.

Definition (Non-tangential Maximal Function and Limits)

The non-tangential maximal function of u at $x \in \partial\Omega$ and the non-tangential limit of u at $x \in \partial\Omega$ are:

$$\mathscr{N}(u)(x) := \sup_{y \in \Gamma_a + x} |u(y)| \quad and \quad u\Big|_{\partial\Omega}^{n.t.}(x) := \lim_{\substack{y \to x \\ y \in \Gamma_a + x}} u(y)$$

On the condition $\mathcal{N}(u) \in L^p(\partial\Omega)$

The condition $\mathcal{N}(u) \in L^p(\partial\Omega)$ is necessary and natural even in the case of the Dirichlet problem in the upper half space $\mathbb{R}^n_+$

Indeed, consider $n = 2$ and $p \in (1, \infty)$ and

$$\begin{cases} \triangle u = 0 \quad \text{in} \quad \mathbb{R}^2_+ \\ u\Big|^{n.t.}_{\partial\mathbb{R}^2_+} = 0 \in L^p(\partial\mathbb{R}^2_+) \quad \text{on} \quad \partial\mathbb{R}^2_+ \end{cases}$$

Then

1. $u_1(x, y) \equiv 0$ is obviously one solution,
2. However $u_2(x, y) = \dfrac{y}{x^2 + y^2}$ is a solution as well, violating uniqueness.

This happens because $\mathcal{N}(\frac{y}{x^2 + y^2}) \notin L^p(\partial\mathbb{R}^2_+)$, for any $p \in (1, \infty)$

The Neumann Problem

Consider next $p \in (1, \infty)$ and the Neumann problem with data in $L^p(\partial\Omega)$:

$$(N) \quad \begin{cases} \triangle u = 0 \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega}^{n.t.} = f \in L^p(\partial\Omega) \quad \text{on} \quad \partial\Omega \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega) \end{cases}$$

Remarks:

- (N) is well-posed $\forall p \in (1, 2+\epsilon)$ and this is sharp in the class of Lipschitz domains
- Via the layer potential method (N) can be reduced to a BIE of the type $(-I + K^*)g = f$
- If Ω is a bounded C^1 domain then (D) and (N) are well-posed for each $p \in (1, \infty)$. Key: in this smooth setting K is compact on $L^p(\partial\Omega)$ for each $p \in (1, \infty)$

Functional Analysis Digression

Let X, Y be Banach spaces

Recall:

- $\mathcal{L}(X, Y)$ stands for the class of bounded linear operators $T : X \longrightarrow Y$.
- $T \in \mathcal{L}(X, Y)$ is said to be upper Fredholm if it has finite dimensional null space (i.e. $\alpha := \dim(ker(T)\langle \infty)$ and a closed range.
- $T \in \mathcal{L}(X, Y)$ is said to be lower Fredholm if its range is closed and it has a finite co-dimension (i.e. $\beta := \dim(Y/T(X)) < \infty$).
- $T \in \mathcal{L}(X, Y)$ is Fredholm if it is upper and lower Fredholm and

$$index(T) := \alpha - \beta \in \mathbb{Z}.$$

- When Ω is a bounded C^1 domain in $\mathbb{R}^n$, $K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega)$ is compact. As such $I + K$ is Fredholm with index zero.

The Mixed Boundary Value Problem

Consider next the mixed boundary value problem (MBVP), also known as the Zaremba problem. Let Ω be a bounded domain in $\mathbb{R}^n$ with the splitting :

$$\partial\Omega = \bar{D} \cup \bar{N} \qquad \bar{D} \cap \bar{N} = \partial D = \partial N$$

Then the MBVP with Dirichlet and Neumann type boundary conditions with L^p data, $p \in (1, \infty)$, is:

$$(MBVP) \quad \begin{cases} \triangle u = 0 \quad \text{in} \quad \Omega \\ u\Big|_D^{n.t.} = f \in L_1^p(D) \quad \text{on} \quad D \\ \dfrac{\partial u}{\partial \nu}\Big|_N^{n.t.} = g \in L^p(N) \quad \text{on} \quad N \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega) \end{cases}$$

A few remarks on the (*MBVP*)

- If $D = \emptyset$ then, (*MBVP*) becomes (*N*).
- If $N = \emptyset$ then, (*MBVP*) becomes (*D*) (with data in $L^p_1(\partial\Omega)$ - also known as the Regularity problem).
- This is an open problem in C. Kenig's 1994 *CBMS Book*: *characterize the smoothness of the gradient of the solution of (MBVP)*.
- The study of (*MBVP*) has applications in engineering and mathematical physics. Indeed, this has connections with
  1. conductivity
  2. heat transfer
  3. wave phenomena
  4. electrostatics
  5. metallurgical melting
  6. stamp problems in elasticity and hydrodynamics

A Brief History

Classical work by I. N. Sneddon (1966) E. Shamir (1968). Additional contributions by J. Lagnese (1983), S.R. Simanca (1987), G. Savaré (1997)...

- (*MBVP*) in the class of *domains with isolated singularities*
  1. S.Rempel and B.-W. Schulze (1989)
  2. M. Dauge (1992),
  3. V. Maz'ya and J.Rossmann (Stokes -2004)
- (*MBVP*) in the class of *Lipschitz creased domains*
  1. R. Brown (1994)
  2. R. Brown and J. Sykes (2001)
  3. I. Mitrea & M. Mitrea (2007)
  4. T. Jakab, I. Mitrea & M. Mitrea(2009)
  5. I. Mitrea & R. Brown (2009)
  6. R. Brown, I. Mitrea, M. Mitrea & M. Wright (2010)

A Brief History

- *(MBVP) in more general domains*
  1. R.Brown, L.Capogna and L. Lanzani (2 dimensions - 2005)
  2. R. Brown, and K. Ott
     - the MBVP for the Laplacian in Lipschitz domains (without the creased condition) in $\mathbb{R}^n$ for $n \geq 3$ (2013)
     - the MBVP for the Lame system in Lipschitz domains in two dimensions (2013)
  3. R. Brown, K. Ott, and J. Taylor (2013) studied the (MBVP)for the Laplacian in Lipschitz domains with very general (geometrically) decompositions of the boundary
- *(MBVP) in connection with numerical analysis*
  1. T. von Petersdorff and E.P. Stephan - BEM (1990)
  2. V.A. Kozlov, V.G. Maz'ya and A.V. Fomin - Cauchy problem (1991)
  3. J. Elschner, Y. Jeon, I.H. Sloan and E.P. Stephan - collocation methods (1997)

Smooth domains for $(MBVP)$

Consider: $\Omega = \{z \in \mathbb{C} : \ 0 < \arg z < \alpha\} \cap \{|z| < 1\}$
with the splitting : $N = \{z = r\, e^{i\alpha}; 0 < r \ < 1\}, \quad D = \partial\Omega \setminus \overline{N}$

Let $(x, y) \equiv z = re^{i\theta}$ and $u(x, y) = r^{\frac{\pi}{2\alpha}} \sin\left(\dfrac{\pi\theta}{2\alpha}\right)$

1. u is harmonic in Ω $(u = \mathrm{Im}\left(z^{\pi/2\alpha}\right))$
2. $u\Big|_D^{n.t} \in L^2_1(D)$
3. $\frac{\partial u}{\partial \nu}\Big|_N^{n.t.} \in L^2(N)$
4. if $\alpha \geq \pi$ then $\nabla u \notin L^2(\partial\Omega)$. In particular if $\alpha = \pi$ (i.e. in the smooth case, one does not have an L^2 theory)

Indeed $\nabla u\Big|_{\partial\Omega}^{n.t} \approx r^{\frac{\pi}{2\alpha}-1}$. To integrate this around the origin one $\partial\Omega$ requires $\dfrac{\pi}{\alpha} - 2 > -1$, i.e.

$$\alpha < \pi$$

Layer Potentials

Let Γ be such that $\Delta\Gamma = 2\delta$. Introduce the single layer potential:

$$\mathcal{S}g(X) := \int_{\partial\Omega} \Gamma(X - Q)g(Q)\, d\sigma(Q) \qquad X \in \mathbb{R}^n \setminus \partial\Omega$$

Boundary behavior of $\mathcal{S}$ and $\frac{\partial \mathcal{S}}{\partial \nu}$:

- $\frac{\partial \mathcal{S}\, g}{\partial \nu}\Big|_{\partial\Omega}^{n.t.} = (-I + K^*)\, g$
- $\mathcal{S}\, g\Big|_{\partial\Omega}^{n.t.} = S\, g$

Here, for $P \in \partial\Omega$

$$Sg(P) := \int_{\partial\Omega} \Gamma(P - Q)g(Q)\, d\sigma(Q),$$

and

$$K^*g(P) := p.v. \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu(P)}(Q - P)g(Q)\, d\sigma(Q)$$

The Case of a Sector

Ω -sector of aperture $\theta \in (0, 2\pi)$ and D/N be the left/right rays of $\partial\Omega$

Seek a solution of the *(MBVP)* as $u = \mathcal{S}h$ for some $h : \partial\Omega \to \mathbb{R}$. Then

- $\Delta u \equiv 0$ in Ω
- $u\Big|_D^{n.t.} = Sh\Big|_D = f \in L^p_1(D)$. Consequently $\partial_\tau Sh\Big|_D = \partial_\tau f \in L^p(D)$
- $\dfrac{\partial u}{\partial \nu}\Big|_N^{n.t.} = (-I + K^*)h\Big|_N$

This leads us to consider $\widetilde{T} : L^p(\partial\Omega) \to L^p(D) \oplus L^p(N)$ with

$$\widetilde{T}(h) = \left(\partial_\tau Sh|_D, (-I + K^*)h\Big|_N\right)$$

The BIE satisfied by h is:

$$\left(\partial_\tau Sh|_D, (-I + K^*)h\Big|_N\right) = (\partial_\tau f, g) \quad \text{i.e.} \quad \widetilde{T}(h) = (\partial_\tau f, g)$$

Main Result

Theorem

Let Ω as before. Then the operator $\widetilde{T}$ is an isomorphism for all $p \in (1, \infty)$ s.t.

$$p \neq \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \text{if } \theta \in (0, \pi/2) \\ \frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi/2, \pi) \\ \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi, 3\pi/2) \\ \frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi} & \text{if } \theta \in (3\pi/2, 2\pi) \\ 3 & \text{if } \theta = \pi/2 \\ 3/2, 3 & \text{if } \theta = 3\pi/2 \\ 2 & \text{if } \theta = \pi \end{cases}$$

Tools for Proof - Mellin Transform

Idea: use Mellin Transform to study the invertibility of $\tilde{T}$ on L^p.

Brief History in Connection with SIO: Mellin analysis – fundamental tool in the study of the spectra of certain SIO's. In particular:

- Lewis & Lewis and Parenti (1980)
  - Calculus of pseudodifferential operators of Mellin type
  - Explicit descriptions of the L^p spectra of these operators
  - Determine compactness on L^p of Mellin operator via computation of principal symbols
  - Applied this for the study of BVP arising in elasticity in curvilinear polygons
- Fabes and Jodeit (1976)
  - Give invertibility conditions of $\frac{1}{2}I - K$ on L^p where K is the double layer potential operator and Ω is a polygon
  - Further studied the L^p spectrum of this operator

Hardy Kernels

Will need the notion of Hardy Kernel:

Definition (Hardy Kernel)

Let $k(\cdot,\cdot)$ be a measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$. Then k is a Hardy kernel for $L^p(\mathbb{R}_+)$ provided that

1. *$k(\cdot,\cdot)$ is homogenous of degree -1, and*
2. $\int_0^\infty |k(1,y)| y^{-1/p} dy < \infty$

If $k(x,y)$ is a Hardy Kernel, define the Hardy Kernel operator T as

$$Tf(x) = \int_0^\infty k(x,y) f(y) dy$$

Theorem [Fabes, Jodeit, Lewis, Boyd]

Let h be a Hardy kernel on $(L^p(\mathbb{R}_+))^m$, for some $p \in (1, \infty)$, and let A, B be $m \times m$ matrices with real entries, and $c_1, c_2, c_3 \in \mathbb{R}$ given constants. Consider the operator $T : (L^p(\mathbb{R}_+))^m \longrightarrow (L^p(\mathbb{R}_+))^m$ given by

$$\boxed{Tf(s) := c_3 \cdot Af + \int_0^\infty \mathfrak{K}(s,t) f(t)\, dt} \quad \text{for a.e.} \quad s \in \mathbb{R}_+ \ \& \ \forall\, f \in (L^p(\mathbb{R}_+))^m$$

$$\text{with} \quad \boxed{\mathfrak{K}(s,t) := c_1 \cdot h(s,t) + \frac{c_2}{s-t} \cdot B}, \qquad \forall\, s, t \in \mathbb{R}_+ \ \text{ with } \ s \neq t,$$

Then T is a linear bounded operator and its spectrum ($\sigma(T; L^p(\mathbb{R}_+))$) satisfies:

$$\boxed{\sigma(T, L^p(\mathbb{R}_+)) = \overline{\{w \in \mathbb{C} : (wI - c_3 A - \mathcal{M}\mathfrak{K}(\cdot, 1)(1/p + i\xi)) = 0, \xi \in \mathbb{R}\}}}$$

where I is the identity operator, $\mathcal{M}$ denotes the Mellin Transform, and $\overline{E}$ denotes the closure of the set $E \subseteq \mathbb{C}$ in $\mathbb{R}^2$.

Corollary

Corollary (*)

Consider the operator T as in the previous Theorem, such that

$$\det(c_3 \cdot A - \pi i c_2 \cdot B) \neq 0.$$

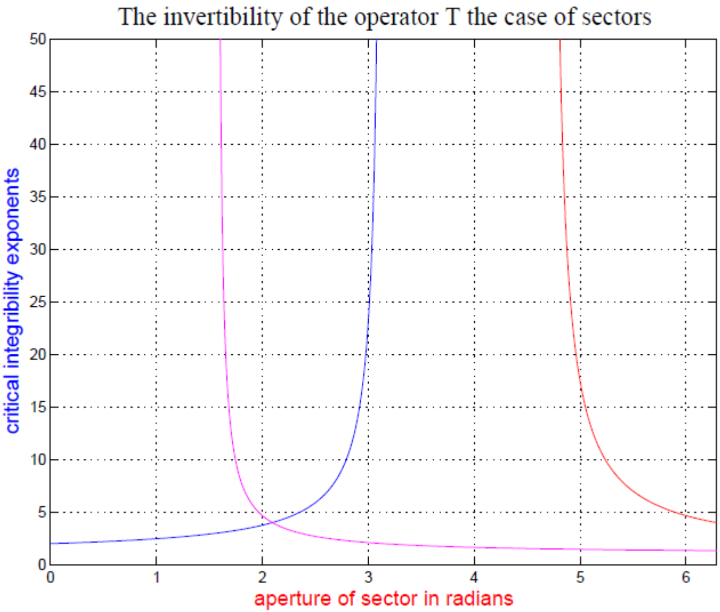
Then T is invertible on $(L^p(\mathbb{R}_+))^m$, $1 < p < \infty$, if and only if the following holds

$$\boxed{\det\,(c_3 \cdot A + \mathcal{M}(\mathfrak{K}(\cdot,1))(1/p + i\xi)) \neq 0 \quad \forall\, \xi \in \mathbb{R}.}$$

When $\Omega \subseteq \mathbb{R}^2$ is a sector we let $\partial\Omega_1$ and $\partial\Omega_2$ denote the left and right rays of $\partial\Omega$. The idea is to identify $\partial\Omega_j, j = 1, 2$ with $\mathbb{R}_+$ via

$$\partial\Omega_j \ni P \mapsto |P| \in \mathbb{R}_+$$

Then, $\widetilde{T}$ becomes a Hardy Kernel operator and (*) provides a mechanism to compute the critical values p as stated in the Theorem.


The invertibility of the operator T the case of sectors
critical integribility exponents
aperture of sector in radians
0
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1
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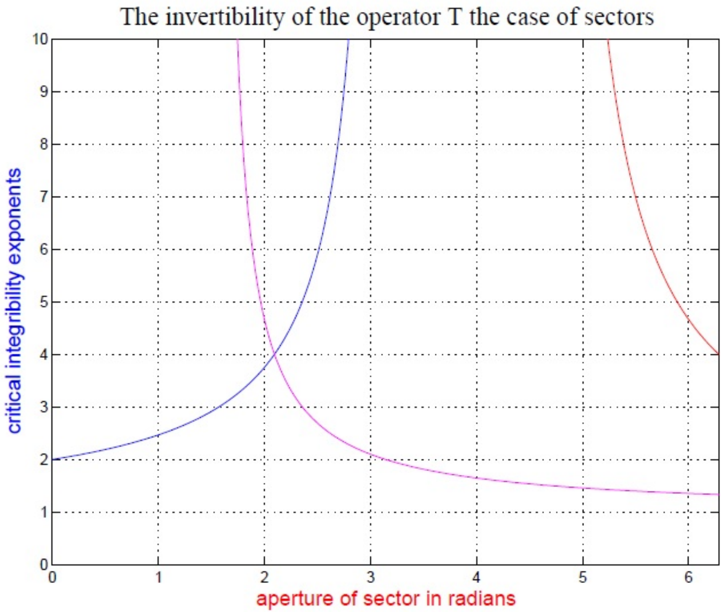
5
6


The invertibility of the operator T the case of sectors
critical integribility exponents
aperture of sector in radians
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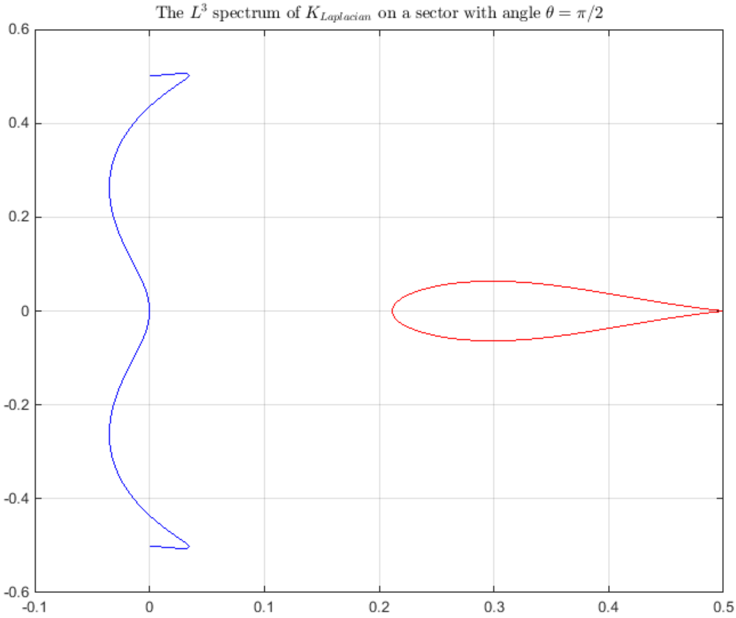
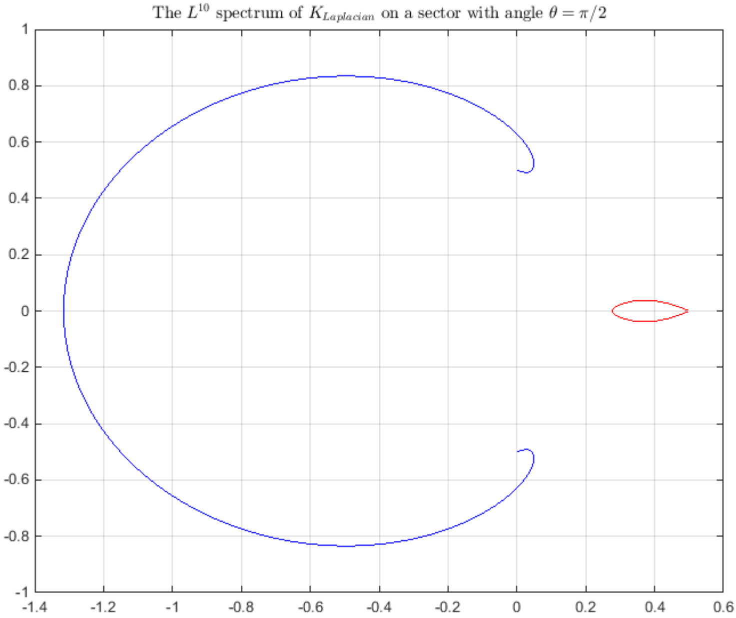
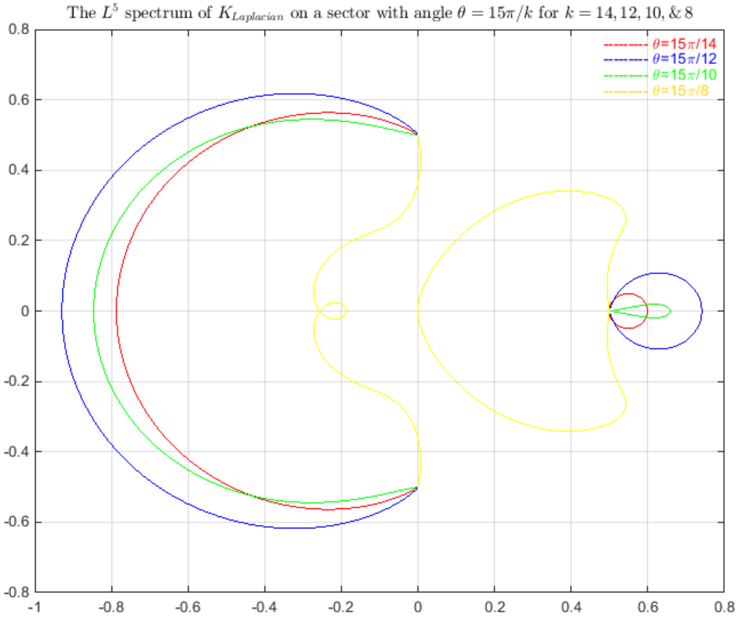

The L^3 spectrum of $K_{Laplacian}$ on a sector with angle $\theta = \pi/2$
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.1
0
0.1
0.2
0.3
0.4
0.5


The L^{10} spectrum of $K_{Laplacian}$ on a sector with angle $\theta = \pi/2$
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.4
-1.2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6


The L^5 spectrum of $K_{Laplacian}$ on a sector with angle $\theta = 15\pi/k$ for $k = 14, 12, 10, \& 8$
θ=15π/14
θ=15π/12
θ=15π/10
θ=15π/8
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

Mixed BVPs for 2nd Order Elliptic Systems of PDEs

Consider an $m \times m$ elliptic system in $\mathbb{R}^n$, given by

$$\left(\mathcal{L}\vec{u}\right)^{\alpha} := a_{ij}^{\alpha\beta}\partial_i\partial_j u^{\beta}, \text{ with } a_{ij}^{\alpha\beta} \in \mathbb{R}.$$

where the coefficient tensor $A := \left(a_{ij}^{\alpha\beta}\right)_{\alpha,\beta,i,j}$ satisfies $\exists c > 0$ s.t.
$a_{ij}^{\alpha\beta}\xi_i\xi_j\eta^{\alpha}\eta^{\beta} \geq c|\xi|^2|\eta|^2 \quad \forall\, \xi, \eta$.
Mixed boundary value problem:

$$(MBVP) \begin{cases} \mathcal{L}\vec{u} = \vec{0} \quad \text{in} \quad \Omega, \\ \vec{u}\Big|_{D}^{\text{n.t.}} = \vec{f} \in \left(L_1^p(D)\right)^m \\ \frac{\partial \vec{u}}{\partial \nu_A}\Big|_{N}^{\text{n.t.}} = \vec{g} \in \left(L^p(N)\right)^m \\ \mathcal{N}(\nabla\vec{u}) \in L^p(\partial\Omega), \end{cases} \qquad 1 < p < \infty.$$

The Lamé System of Elastostatics in 2D

Here $\frac{\partial}{\partial \nu_A}$ stands for the conormal derivative

$$\left(\frac{\partial \vec{u}}{\partial \nu_A(Q)}\right)^{\alpha} = \nu_i(Q) a_{ij}^{\alpha\beta} \partial_j u^{\beta}.$$

Consider next the Lamé differential operator ($\mu > 0$ and $\lambda + \mu \geq 0$):

$$\mathcal{L}\vec{u} = \mu \triangle \vec{u} + (\lambda + \mu)\nabla \mathrm{div}\, \vec{u} = \vec{0},$$

Componentwise $\left(\mathcal{L}\vec{u}\right)^{\alpha} := a_{ij}^{\alpha\beta} \partial_i \partial_j u^{\beta}$, $\alpha \in \{1, 2\}$ for the (*infinitely many*) choices

$$a_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(r) = \mu\, \delta_{ij}\delta_{\alpha\beta} + (\mu + \lambda - r)\delta_{i\alpha}\delta_{j\beta} + r\delta_{i\beta}\delta_{j\alpha}$$

For each $r \in \mathbb{R}$, this coefficient tensor is elliptic (in the earlier sense).

Special choices $r_{\psi} := \dfrac{\mu(\mu + \lambda)}{(3\mu + \lambda)} \in (-\mu, \mu)$ and $r_{\text{traction}} = \mu$.

THANK YOU