Inverting Double Layers on Lebesgue Spaces on the Boundary of Lipschitz Domains

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The Neumann Problem

Let $p \in (1, \infty)$. Recall the Neumann problem with data in $L^p(\partial\Omega)$:

$$(N) \quad \begin{cases} \triangle u = 0 \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega}^{n.t.} = f \in L^p(\partial\Omega) \quad \text{on} \quad \partial\Omega \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega) \end{cases}$$

Remarks:

- (N) is well-posed $\forall p \in (1, 2+\epsilon)$ and this is sharp in the class of Lipschitz domains
- Via the layer potential method (N) can be reduced to a BIE of the type $(-I + K^*)g = f$
- If Ω is a bounded C^1 domain then (N) is well-posed for each $p \in (1, \infty)$.Key: in this smooth setting K is compact on $L^p(\partial\Omega)$ for each $p \in (1, \infty)$

Definitions

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$:

- ν - the outward unit normal vector
- σ - the surface measure .

Recall:

$$K^* f(x) = p.v. \int_{\partial\Omega} \frac{\langle \nu(x), x-y \rangle}{|x-y|^n} f(y) d\sigma(y)$$

Goal :

$$\frac{1}{2} I + K^* : L^2(\partial\Omega) \to L^2(\partial\Omega)$$

is invertible.

Let $\overrightarrow{e}$ be an arbitrary vector in $\mathbb{R}^n$ define:

$$\overrightarrow{e}_{tan} = \overrightarrow{e} - \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle \overrightarrow{\nu}$$

Then the we have the following properties:

- $\langle \overrightarrow{e}_{tan}, \overrightarrow{\nu} \rangle = 0$
- $|\overrightarrow{e}|^2 = |\overrightarrow{e}_{tan}|^2 + \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle^2$
- $|\overrightarrow{e}_{tan}| \leq |\overrightarrow{e}|$

In particular for $\nabla_{tan} u = \nabla u - \langle \nabla u, \overrightarrow{\nu} \rangle \overrightarrow{\nu}$ we have:

$$\star : \begin{cases} |\nabla u|^2 = |\nabla_{tan} u|^2 + |\frac{\partial u}{\partial \nu}|^2 \\ \langle \nabla u, \overrightarrow{e} \rangle = \langle \nabla_{tan} u, \overrightarrow{e}_{tan} \rangle + \frac{\partial u}{\partial \nu} \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle \end{cases}$$

Rellich Identities / Estimates

Theorem

Let u be as follows:

$$(*)\begin{cases} u \in C^2(\Omega) \ \ s.t \ \ \Delta u = 0 \ \ in \ \Omega \\ \nabla u \Big|_{\partial\Omega}^{n.t.} \ exists \ \ and \ \ \mathcal{N}(\nabla u) \in L^2(\partial\Omega) \end{cases}$$

Then :

$$\left\| \nabla_{tan} u \right\|^2_{L^2(\partial\Omega)} \approx \left\| \frac{\partial u}{\partial \nu} \right\|^2_{L^2(\partial\Omega)} + \left\| comp(\nabla u) \right\|_X$$

$$comp : L^2(\partial\Omega) \longrightarrow X$$

Tools

Let u be defined on Ω such that u satisfies $(*)$, consider an arbitrary vector $\overrightarrow{e} \in \mathbb{R}^n$. Define:

$$\overrightarrow{F} = \left|\nabla u\right|^2 \overrightarrow{e} - 2\langle \nabla u, \overrightarrow{e} \rangle \nabla u \quad in \ \ \Omega.$$

Then $\overrightarrow{F}$ has the following properties:

- $\overrightarrow{F} \in \mathcal{C}^\infty(\Omega)$, thus $\overrightarrow{F} \in L^1_{loc}(\Omega)$.
- $\overrightarrow{F}\Big|_{\partial\Omega}^{n.t.}$ exists.
- $\overrightarrow{F}\Big|_{\partial\Omega}^{n.t.} \in L^1(\partial\Omega)$.
- $\overrightarrow{F}$ is divergence free.

$\overrightarrow{F}$ Divergence Free

Let us prove that $\overrightarrow{F}$ mentioned above is indeed divergence free.

$$
\begin{aligned}
div\overrightarrow{F} &= \sum_{j=1}^{n} \partial_j F_j = \sum_{j=1}^{n} \partial_j \left(|\nabla u|^2 e_j - 2\langle \nabla u, \overrightarrow{e} \rangle \partial_j u \right) \\
&= \sum_{j=1}^{n} \partial_j (|\nabla u|^2) e_j - 2 \sum_{j=1}^{n} \left(\partial_j (\sum_{l=1}^{n} \partial_l u e_l) \partial_j u \right) - 2 \sum_{j=1}^{n} \left(\langle \nabla u, \overrightarrow{e} \rangle \partial_j^2 u \right) \\
&= \sum_{j=1}^{n} \partial_j (\sum_{k=1}^{n} (\partial_k u)^2) e_j - 2 \sum_{j=1}^{n} \left(\partial_j (\sum_{l=1}^{n} \partial_l u e_l) \partial_j u \right) - 2 \left(\langle \nabla u, \overrightarrow{e} \rangle \Delta u \right) \\
&= 2 \sum_{j,k=1}^{n} (\partial_j \partial_k u)(\partial_k u) e_j - 2 \sum_{j,l=1}^{n} (\partial_j \partial_l u)(\partial_j u) e_l = 0
\end{aligned}
$$

So $div\overrightarrow{F} = 0$ in Ω.

Divergence Theorem

Apply the divergence theorem to $\overrightarrow{F}$:

$$0 = \int_{\Omega} div\, \overrightarrow{F}\, dV = \int_{\partial\Omega} \overrightarrow{\nu} \cdot \overrightarrow{F}\Big|_{\partial\Omega}^{n.t.} d\sigma.$$

So :

$$\int_{\partial\Omega} |\nabla u|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma = 2 \int_{\partial\Omega} \langle \nabla u, \overrightarrow{e} \rangle \frac{\partial u}{\partial \nu} d\sigma$$

Using ★ we get:

$$\int_{\partial\Omega} |\nabla_{tan} u|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma - \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} \right)^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma = \qquad (*)$$

$$2 \int_{\partial\Omega} \langle \nabla_{tan} u, \overrightarrow{e}_{tan} \rangle \frac{\partial u}{\partial \nu} d\sigma$$

Let $a \cdot b = 1$ and $a, b > 0$, then:

$$\pm 2\langle \nabla_{tan} u, \vec{e}_{tan}\rangle \frac{\partial u}{\partial \nu} = \pm 2a \cdot b \langle \nabla_{tan} u, \vec{e}_{tan}\rangle \frac{\partial u}{\partial \nu}$$
$$\leq a^2 |\langle \nabla_{tan} u, \vec{e}_{tan}\rangle|^2 + b^2 \left(\frac{\partial u}{\partial \nu}\right)^2$$
$$\leq a^2 |\nabla_{tan} u|^2 |\vec{e}|^2 + b^2 \left(\frac{\partial u}{\partial \nu}\right)^2 .$$

use this in $(*)$ to get

$$(**) : \int_{\partial\Omega} |\nabla_{tan} u|^2 \langle \vec{e}, \vec{\nu}\rangle d\sigma \leq \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \langle \vec{e}, \vec{\nu}\rangle d\sigma + a^2 \|\nabla_{tan} u\|^2_{L^2(\partial\Omega)} |\vec{e}|^2 + b^2 \left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\partial\Omega)}$$

$$(***) : \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \langle \vec{e}, \vec{\nu}\rangle d\sigma \leq \int_{\partial\Omega} |\nabla_{tan} u|^2 \langle \vec{e}, \vec{\nu}\rangle d\sigma + a^2 \|\nabla_{tan} u\|^2_{L^2(\partial\Omega)} |\vec{e}|^2 + b^2 \left\|\frac{\partial u}{\partial \nu}\right\|^2_{L^2(\partial\Omega)}$$

Transversality

Consider Ω the upper graph of $\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$,

i.e $\Omega = \{(x_1, ..., x_n) \mid x_n > \phi(x_1, ..., x_{n-1})\}$, and $x = (x', \phi(x')) \in \partial\Omega$ for $x' \in \mathbb{R}^{n-1}$.

Radamacher's theorem yields $\left\|\nabla\phi\right\|_{L^\infty(\mathbb{R}^{n-1})} \leq M < \infty$.

It's elementary that for $x = (x', \phi(x')) \in \partial\Omega$ we have $\nu(x) = \frac{(\nabla\phi, -1)}{\sqrt{1+|\nabla\phi|^2}}$.

Now for a special $\overrightarrow{e} = -\overrightarrow{e}_n = (0, ..., 0, -1)$ we get:

1. $0 \leq \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle = \frac{1}{\sqrt{1+|\nabla\phi|^2}} \leq 1.$
2. $\langle \overrightarrow{e}, \overrightarrow{\nu} \rangle \geq \frac{1}{\sqrt{1+\left\|\nabla\phi\right\|^2_{L^\infty(\mathbb{R}^{n-1})}}} \geq \frac{1}{\sqrt{1+M^2}} = c > 0.$

This makes :

$$\langle \overrightarrow{e}, \overrightarrow{\nu} \rangle \approx c$$

Let us recall what we had so far :

$$(**): \int_{\partial\Omega} |\nabla_{tan} u|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma \leq \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} \right)^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma$$
$$+ a^2 \|\nabla_{tan} u\|^2_{L^2(\partial\Omega)} |\overrightarrow{e}|^2 + b^2 \|\tfrac{\partial u}{\partial \nu}\|^2_{L^2(\partial\Omega)}$$

$$(***): \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} \right)^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma \leq \int_{\partial\Omega} |\nabla_{tan} u|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma$$
$$+ a^2 \|\nabla_{tan} u\|^2_{L^2(\partial\Omega)} |\overrightarrow{e}|^2 + b^2 \|\tfrac{\partial u}{\partial \nu}\|^2_{L^2(\partial\Omega)}$$

$(**)$ becomes:

$$(**): \quad \begin{aligned} c\|\nabla_{tan}u\|^2_{L^2(\partial\Omega)} &\leq L.H.S((**)) \\ &\leq \|\tfrac{\partial u}{\partial \nu}\|^2_{L^2(\partial\Omega)} + a^2\|\nabla_{tan}u\|^2_{L^2(\partial\Omega)} + b^2\|\tfrac{\partial u}{\partial \nu}\|^2_{L^2(\partial\Omega)} \end{aligned}$$

Let $a = \frac{\sqrt{c}}{2}$ in $(**)$, then :

$$\|\nabla_{tan}u\|^2_{L^2(\partial\Omega)} \leq C_1\|\frac{\partial u}{\partial \nu}\|^2_{L^2(\partial\Omega)}$$

$(***)$ becomes:

$$(***): \quad \begin{aligned} c\|\tfrac{\partial u}{\partial \nu}\|^2_{L^2(\partial\Omega)} &\leq L.H.S((***)) \\ &\leq \|\nabla_{tan}u\|^2_{L^2(\partial\Omega)} + a^2\|\nabla_{tan}u\|^2_{L^2(\partial\Omega)} + b^2\|\tfrac{\partial u}{\partial \nu}\|^2_{L^2(\partial\Omega)} \end{aligned}$$

Let $b = \frac{\sqrt{c}}{2}$ in $(***)$, then :

$$\|\frac{\partial u}{\partial \nu}\|^2_{L^2(\partial\Omega)} \leq C_2\|\nabla_{tan}u\|^2_{L^2(\partial\Omega)}$$

$\left\|\nabla_{tan} u\right\|_{L^2(\partial\Omega)} \approx \left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial\Omega)}$

For $f \in L^2(\partial\Omega)$ let u be be defined as follows:

$$u(x) = \mathcal{S}f(x) = c_n \int_{\partial\Omega} \frac{1}{|x-y|^{n-1}} f(y) d\sigma(y)$$

Then u satisfies the following properties:

1. $u \in \mathcal{C}^\infty(\Omega)$, and $\Delta u = 0$ in Ω.
2. $\mathcal{N}(\nabla u) \in L^2(\partial\Omega)$.
3. $\nabla u\Big|_{\partial\Omega}^{n.t.}$ exists.

We also have the following relations at the boundary: (jump relations):

- $$\frac{\partial \mathcal{S}f}{\partial \nu}\Big|_{\partial\Omega\pm}^{n.t.} = \left(\pm\frac{1}{2}I - K^*\right) f \qquad \sigma \text{ a.e on } \partial\Omega$$
- $\nabla_{tan}\mathcal{S}f$ does not jump on the boundary.

Fredholm

Let $f \in L^2(\partial\Omega)$, then:

$$f = \left(\tfrac{1}{2}f - K^* f\right) - \left(-\tfrac{1}{2}f - k^* f\right)$$

$$\left\|f\right\|_{L^2(\partial\Omega)} \leq \left\|\left(\tfrac{1}{2}I - K^*\right) f\right\|_{L^2(\partial\Omega)} + \left\|\left(-\tfrac{1}{2}I - K^*\right) f\right\|_{L^2(\partial\Omega)}$$

But :

$$\left\|\frac{\partial \mathcal{S}}{\partial \nu} f\Big|_{\partial\Omega_+}^{n.t.}\right\|_{L^2(\partial\Omega)} \underset{\text{Rellich}}{\approx} \left\|\nabla_{tan}\mathcal{S} f\Big|_{\partial\Omega_\pm}^{n.t.}\right\|_{L^2(\partial\Omega)} \underset{\text{Rellich}}{\approx} \left\|\frac{\partial \mathcal{S}}{\partial \nu} f\Big|_{\partial\Omega_-}^{n.t.}\right\|_{L^2(\partial\Omega)}$$

So we get:

$$\left\|f\right\|_{L^2(\partial\Omega)} \leq C \left\|\left(\frac{1}{2}I + K^*\right) f\right\|_{L^2(\partial\Omega)}$$

Similarly one can get:

$$\left\|f\right\|_{L^2(\partial\Omega)} \leq C \left\|\left(\frac{1}{2}I - K^*\right) f\right\|_{L^2(\partial\Omega)}$$

whence :$\frac{1}{2}I \pm K^*$ is Semi-Fredholm and injective if Ω is upper graph.

Let $\Omega = \Omega_+$ be a bounded Lipschitz domain in $\mathbb{R}^n$ and recall

$$I_+ : \quad \int_{\partial\Omega} |\nabla_{tan} u|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma - \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma = 2 \int_{\partial\Omega} \langle \nabla_{tan} u, \overrightarrow{e}_{tan} \rangle \frac{\partial u}{\partial \nu} d\sigma$$

Set $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$, and let $f \in L^2(\partial\Omega)$we define :

$$\begin{cases} u^+ = \mathcal{S}f \ \ in \ \ \Omega_+ \\ u^- = \mathcal{S}f \ \ in \ \ \Omega_- \end{cases}$$

let $\nu_- = -\nu$ one can get the following equality for u^-

$$I_- : \quad \int_{\partial\Omega} |\nabla_{tan} u^-|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma - \int_{\partial\Omega} \left(\frac{\partial u^-}{\partial \nu_-}\right)^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma = 2 \int_{\partial\Omega} \langle \nabla_{tan} u^-, \overrightarrow{e}_{tan} \rangle \frac{\partial u^-}{\partial \nu_-} d\sigma$$

Let us recall the jump relations:

$$\begin{cases} \frac{\partial u^+}{\partial \nu} = \frac{\partial \mathcal{S} f}{\partial \nu}\Big|^{n.t.}_{\partial\Omega+} = \left(\frac{1}{2} I - K^*\right) f & \sigma \text{ a.e on } \partial\Omega \\ \frac{\partial u^-}{\partial \nu} = \frac{\partial \mathcal{S} f}{\partial \nu}\Big|^{n.t.}_{\partial\Omega-} = \left(-\frac{1}{2} I - K^*\right) f & \sigma \text{ a.e on } \partial\Omega \end{cases}$$

On the other hand, $\nabla_{tan} u^+ = \nabla_{tan} u^-$ does not jump on the boundary. Now we consider the following equality:

$$\left(\lambda - \frac{1}{2}\right) I_+ - \left(\lambda + \frac{1}{2}\right) I_- \qquad (\bigstar)$$

The LHS($\bigstar$):

$$-\int_{\partial\Omega} \left|\nabla_{tan} u\right|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma - \int_{\partial\Omega} \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle A_\lambda d\sigma$$

Where $A_\lambda = \left[\left(\lambda - \frac{1}{2}\right)\left(\frac{1}{2} f - K^* f\right)^2 - \left(\lambda - \frac{1}{2}\right)\left(\frac{1}{2} f + K^* f\right)^2\right]$

And the RHS($\bigstar$):

$$2\int_{\partial\Omega} \langle \nabla_{tan} u, \overrightarrow{e_{tan}} \rangle \left[\left(\lambda - \frac{1}{2}\right)\left(\frac{1}{2} f - K^* f\right) - \left(\lambda - \frac{1}{2}\right)\left(\frac{1}{2} f + K^* f\right)\right]$$

$$A_\lambda = \left[\left(\lambda - \tfrac{1}{2}\right) \left(\tfrac{1}{2} f - K^* f\right)^2 - \left(\lambda - \tfrac{1}{2}\right) \left(\tfrac{1}{2} f + K^* f\right)^2 \right]$$
$$= -(\tfrac{1}{4}) f^2 - 2\lambda K^* f - (K^* f)^2 = (\lambda^2 - \tfrac{1}{4}) f^2 - [(\lambda I + K^*) f]^2$$

So: LHS($\star$):

$$-\int_{\partial\Omega} |\nabla_{tan} u|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma - \int_{\partial\Omega} \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle \left[(\lambda^2 - \frac{1}{4}) f^2 - ((\lambda I + K^*) f)^2 \right] d\sigma$$

And the RHS ($\star$):

$$2 \int_{\partial\Omega} \langle \nabla_{tan} u, \overrightarrow{e_{tan}} \rangle \left[\lambda f + K^* f \right] d\sigma$$

Putting everything together we get $(* * **)$:

$$\int_{\partial\Omega} |\nabla_{tan} u|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma + (\lambda^2 - \frac{1}{4}) \int_{\partial\Omega} \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle |f|^2 d\sigma$$
$$= \int_{\partial\Omega} \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle ((\lambda I + K^*) f)^2 d\sigma - 2 \int_{\partial\Omega} \langle \nabla_{tan} u, \overrightarrow{e_{tan}} \rangle \left[\lambda f + K^* f \right] d\sigma$$

Recall :

$$\langle \overrightarrow{e}, \overrightarrow{\nu} \rangle \approx c$$

Then using this in $(****)$ we get the following:

$$c\left\|\nabla_{tan}u\right\|^2_{L^2(\partial\Omega)} + c(\lambda^2 - \tfrac{1}{4})\left\|f\right\|^2_{L^2(\partial\Omega)} \leq$$

$$\int_{\partial\Omega} |\nabla_{tan}u|^2 \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle d\sigma + (\lambda^2 - \frac{1}{4}) \int_{\partial\Omega} \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle |f|^2 d\sigma$$

$$= \int_{\partial\Omega} \langle \overrightarrow{e}, \overrightarrow{\nu} \rangle ((\lambda I + k^*)f)^2 d\sigma - 2\int_{\partial\Omega} \langle \nabla_{tan}u, \overrightarrow{e_{tan}} \rangle \left[\lambda f + k^* f\right] d\sigma \leq$$

$$\left\|(\lambda I + K^*)f\right\|^2_{L^2(\partial\Omega)} + a^2\left\|\nabla_{tan}u\right\|^2_{L^2(\partial\Omega)} + b^2\left\|(\lambda I + K^*)f\right\|^2_{L^2(\partial\Omega)}$$

where $a \cdot b = 1$ and $a, b > 0$.
So for the appropriate a and b we will get:

$$\left\|f\right\|^2_{L^2(\partial\Omega)} \leq c_\lambda \left\|(\lambda I + K^*)f\right\|^2_{L^2(\partial\Omega)}$$

Fredholm Index

Thus $\forall \lambda \in \mathbb{R}$ such that $|\lambda| > \frac{1}{2}$:

$$\lambda I + K^* : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \quad \mathit{is\ semi - Fredholm}$$

Then for every λ defin the operator:

$$\lambda \longrightarrow T_\lambda = \lambda I + K^* \in \mathcal{L}(L^2(\partial\Omega), L^2(\partial\Omega))$$

Notice the following:

1. The operator defined above is *continuous* in λ.
2. Functional analysis will give us that all the operators T_λ *will have the same index*.
3. For λ *large enough* one has that $T_\lambda = \lambda I + K^* = \lambda(1 + \frac{1}{\lambda}K^*)$, and so you can have $\|\frac{K^*}{\lambda}\| < 1$, which implies that T_λ is *invertible*, and hence of *index zero*.
4. $\forall \lambda \in \mathbb{R}$ such that $|\lambda| > \frac{1}{2}$ $index(T_\lambda) = zero$, inparticular $index(\frac{1}{2}I + K^*) = zero$.

Injectivity (Ω *bounded*)

Let $f \in L^2(\partial\Omega)$ be such that $(\frac{1}{2}I + K^*)f = 0$
Our goal is to show that $f = 0$ this will imply that $\frac{1}{2}I + K^*$ is an injective operator on $L^2(\partial\Omega)$.

$$\int_\Omega |\nabla u|^2 dV = \int_\Omega \langle \nabla u, \nabla u \rangle dV = -\int_\Omega \Delta u.u dV + \int_{\partial\Omega} u.\frac{\partial u}{\partial \nu} d\sigma.$$

Then :

$$\int_{\Omega_-} |\nabla u|^2 dV = -\int_{\partial\Omega} u.\frac{\partial u}{\partial \nu} d\sigma = 0$$

So : $|\nabla u| = 0$ which implies that $u \equiv c$ on Ω_-.
Hence we have:

$$\int_\Omega |\nabla u|^2 dV = c\int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma = c\int_\Omega \Delta u dV = 0$$

And so we get that $u \equiv c$ in Ω_+ , so $(\frac{1}{2}I - K^*)f = 0$
But $f = (\frac{1}{2}I - K^*)f + (\frac{1}{2}I + K^*)f$

Definitions

Again we define Ω as the upper graph of $\phi : \mathbb{R} \longrightarrow \mathbb{R}$.
Then the unit outward normal at a point $X = (x, \phi(x))$ is given by

$$\nu(X) = \frac{(\phi^{'}(x), -1)}{\sqrt{1 + |\phi'(x)|^2}}$$

If $x \in \mathbb{R}$ define $z = x + i\phi(x)$ and write

$$dz = (1 + i\phi^{'}(x))dx = \underbrace{\frac{1 + i\phi^{'}(x)}{\sqrt{1 + |\phi'(x)|^2}}}_{i\nu} \underbrace{\sqrt{1 + |\phi'(x)|^2} dx}_{d\sigma}$$

Recall: if $F = u + iv$ holomorphic in Ω then

- $\overline{\partial} F = (\partial_x + i\partial_y)F \equiv 0$
- $\displaystyle\int_{\partial\Omega} F\, dz = 0$

Note that $\Re(i\nu(X)) = \frac{1}{\sqrt{1+|\phi'(x)|^2}} \geq \frac{1}{\sqrt{1+\left\|\phi'\right\|_{L^\infty}^2}} = c > 0$ Then :

$$c\int_{\partial\Omega}|F|^{2m}d\sigma \leq \int_{\partial\Omega}|F|^{2m}\ \Re(i\nu(X))d\sigma$$

$$= \Re\left(\int_{\partial\Omega}|F|^{2m}i\nu(X)d\sigma\right) = \Re\left(\int_{\partial\Omega}|F|^{2m}dz\right)$$

$$= \Re\left(\int_{\partial\Omega}F^m(\bar{F}^m \pm F^m\right)dz = \Re\left(\int_{\partial\Omega}F^m\underbrace{(\Re(F^m))}_{(\Im(F^m))}dz\right)$$

Apply Holder :

$$c\int_{\partial\Omega}|F|^{2m}d\sigma \leq \left(\int_{\partial\Omega}|F|^{2m}d\sigma\right)^{\frac{1}{2}}\left(\int_{\partial\Omega}\underbrace{\left|\Re(F^m)\right|^2}_{\left|\Im(F^m)\right|^2}d\sigma\right)^{\frac{1}{2}}$$

Therfore:

$$\int_{\partial\Omega} |F|^{2m} d\sigma \leq c \begin{cases} \int_{\partial\Omega} |\Re(F^m)|^2 d\sigma \\ \int_{\partial\Omega} |\Im(F^m)|^2 d\sigma \end{cases}$$

Then :

$$* \qquad F^m = (u+iv)^m = \sum_{k=0}^{m} i^k \begin{pmatrix} m \\ k \end{pmatrix} u^{m-k} v^k$$

$$* \qquad \Re(F^m) = \sum_{\substack{k=0 \\ even}}^{m} \pm \begin{pmatrix} m \\ k \end{pmatrix} u^{m-k} v^k$$

Claim :

m *odd* $\Longrightarrow \forall \epsilon > 0 \ \ \exists c_\epsilon > 0$ such that :

$$\sum_{\substack{k=0 \\ even}}^{m} |u|^{m-k} v^k \leq c_\epsilon |u|^m + \epsilon |v|^m.$$

Then for m *odd*, one can deduce the following:

$$\int_{\partial\Omega} |u|^{2m} d\sigma \approx \int_{\partial\Omega} |v|^{2m} d\sigma$$

Now let $f \in L^{2m}(\partial\Omega)$ real valued:

$$F(z) = \mathcal{C}f(z) := \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi$$

Then :

- $u = \Re(F) = \mathcal{D}f$
- $v = \Im(F) = -\mathcal{S}(\partial_T f)$

and we have for all $f \in L^{2m}(\partial\Omega)$:

$$v\big|_{\partial\Omega_+} = v\big|_{\partial\Omega_-}$$

Then we have the following inequality:

$$\int_{\partial\Omega} |f|^{2m} d\sigma \leq c \int_{\partial\Omega} \left|(\tfrac{1}{2} I + K) f\right|^{2m} d\sigma + \int_{\partial\Omega} \left|(-\tfrac{1}{2} I + K) f\right|^{2m} d\sigma$$

In conclusion for *m odd* we have:

$$\left\|f\right\|_{L^{2m}(\partial\Omega)} \leq \left\|(\pm\tfrac{1}{2} I + K) f\right\|_{L^{2m}(\partial\Omega)}$$

Notice this also proves that the operators: $\pm\frac{1}{2} I + K$ are injective.
So for *m odd* we have:

$$\pm\tfrac{1}{2} I + K : L^{2m}(\partial\Omega) \longrightarrow L^{2m}(\partial\Omega)$$

- One-to-One
- Closed range

Zero Index

Define the following operator:

$$t \in [0,1] \quad \longrightarrow \quad \pm\mathcal{L}_t = \pm\frac{1}{2}I + K_{t\phi}$$

Then notice the following:

- For $t = 0$ we get $\pm\mathcal{L}_0 = \pm\frac{1}{2}I$ which is invertible and hence of *index zero*.
- For $t = 1$ we get $\pm\mathcal{L}_1 = \pm\frac{1}{2}I + K$.
- The operator defined above is continuous with respect to t.
- Hence $\forall t \in [0,1]\ \pm\mathcal{L}_t$ is of *index zero*.

In conclusion :

$$\frac{1}{2}I + K : L^{2m}(\partial\Omega) \longrightarrow L^{2m}(\partial\Omega) \quad \text{invertible} \quad \forall\ m \text{ odd}$$

Thank You