Spectral Properties of Hardy Kernel Operators and Application to Second Order Elliptic Boundary Value Problems

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Preliminaries

Let X be a Banach space and $T : X \longrightarrow X$ be a linear and bounded operator. The spectrum of T on X denoted by $\sigma(T, X)$ is given by:

$$\sigma(T, X) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible on } X\}.$$

For $K \in L^1(\mathbb{R})$ define the operator $K_* : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ by setting

$$(K_* u)(x) := \int_{-\infty}^{\infty} K(x-y)u(y)dy \qquad \forall\, u \in L^p(\mathbb{R})$$

It is straightforward that for each $K^1, K^2 \in L^1(\mathbb{R})$ there holds

$$K_*^1 \circ K_*^2 = \left(K_*^1(K^2)\right)_* \text{ and } \alpha K_*^1 + \beta K_*^2 = (\alpha K^1 + \beta K^2)_*,$$

for each α, β constants.

Preliminaries

Recall that if $K \in L^1(\mathbb{R})$ the Fourier transform of K (denoted by $\widehat{K}$) is given by:

$$\widehat{K}(\xi) := \int_{-\infty}^{\infty} K(x)e^{i\xi x}\,dx, \qquad \xi \in \mathbb{R}.$$

Let k be a measurable function on $(0, \infty)$. The Mellin transform of k, denoted by $\widetilde{k}$ is given by:

$$\widetilde{k}(z) = \int_0^{\infty} k(s)s^{z-1}ds$$

where this is defined on a subset of $\mathbb{C}$ on which the integral is absolutely convergent (typically a vertical strip).

Young′s Inequality for Convolutions : Let $p, q, r \in \mathbb{R}_{\geq 1}$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then for $f \in L^q(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ we have :

$$f * g \in L^r(\mathbb{R}^n)$$

$$\|f * g\|_r \leq \|g\|_p \cdot \|f\|_q$$

Wiener Lemma and Consequences

Lemma (Wiener)

Let $K \in L^1(\mathbb{R})$ *and suppose that* $\lambda \in \mathbb{C}$ *is such that:*

- $\lambda \neq 0$
- $\lambda \neq \widehat{K}(\xi) \ \forall \xi \in \mathbb{R}$

Then there exists $A_\lambda \in L^1(\mathbb{R})$ *such that*

$$(\lambda I - K_*)\, A_\lambda = K$$

Corollary (1)

Let $K \in L^1(\mathbb{R})$*, then the spectrum of* K_* *as an operator on* $L^p(\mathbb{R})$ *satisfies*

$$\sigma\big(K_*, L^p(\mathbb{R})\big) \subseteq \overline{\{\lambda \in \mathbb{C} \mid \lambda = \widehat{K}(\xi) \ \text{ for some } \ \xi \in \mathbb{R}\}}.$$

Proof of Corollary

Consider $\lambda \in \mathbb{C}$ such that $\lambda \notin \overline{\{\lambda \in \mathbb{C} \mid \lambda = \widehat{K}(\xi) \text{ for some } \xi \in \mathbb{R}\}}$. Our goal is to show that $\lambda \notin \sigma\big(K_*, L^p(\mathbb{R})\big)$.

The assumption on λ together with the Riemann-Lebesgue Lemma ensure that λ satisfies the conditions of Wiener's lemma. Hence:

$$\exists\ A_\lambda \in L^1(\mathbb{R}) \text{ such that } \lambda \cdot A_\lambda - K_* A_\lambda = K$$

Using the basic properties of the convolution operator and the fact that $K_* A_\lambda = -K + \lambda A_\lambda$ one obtains

$$\Big(\lambda^{-1}(I + A_{\lambda *})\Big) \circ (\lambda I - K_*) = I \text{ and } (\lambda I - K_*) \circ \Big(\lambda^{-1}(I + A_{\lambda *})\Big) = I$$

Thus $\lambda I - K_*$ is invertible on $L^p(\mathbb{R})$ so $\lambda \notin \sigma\big(K_*, L^p(\mathbb{R})\big)$, as desired.

Lemma 2

Lemma (2)

Let $1 < p < \infty$, *and consider* $K \in L^1(\mathbb{R})$ *satisfying:*

$$\int_{-\infty}^{\infty} |xK(x)|dx = M < \infty. \quad (\star)$$

Then for each $\xi \in \mathbb{R}$ *and* $\delta > 0$ $\exists\, u_{\delta,\xi} \in L^p(\mathbb{R})$ *such that:*

$$\|u_{\delta,\xi}\|_p = 1 \ \ \text{and} \ \ \|\left(\widehat{K}(\xi)I - K_*\right)u_{\delta,\xi}\|_p = \mathcal{O}(\delta),$$

uniformly in $\xi \in \mathbb{R}$.

Proof of Lemma

Given $\delta > 0$ and $\xi \in \mathbb{R}$ define the function $\nu_{\delta,\xi}$ by setting

$$\nu_{\delta,\xi}(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x} d\eta = 2e^{-i\xi x} \cdot \frac{\sin(\delta x)}{x}, \qquad x \in \mathbb{R}.$$

Then $\nu_{\delta,\xi}$ has the following properties:

$$\begin{cases} (i) & \nu_{\delta,\xi} \in L^p(\mathbb{R}) \quad \forall\, p \in (1, \infty) \\ (ii) & \|\nu_{\delta,\xi}\|_p = \delta^{1-\frac{1}{p}} \|\nu_{1,\xi}\|_p \end{cases}$$

where (i) is due to the fact that $\nu_{\delta,\xi}$ is bounded near zero, and at infinity

$$\left| e^{-i\xi x} \cdot \frac{\sin(\delta x)}{x} \right|^p \approx \left| \frac{1}{x^p} \right|.$$

(ii) follows from a change of variable (using the definition of the norm).

Proof of Lemma (continued)

Moving on, $\nu_{\delta,\xi}$ can be shown to satisfy:

$$K_*\nu_{\delta,\xi}(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x}\widehat{K}(\eta)d\eta.$$

Indeed, writing down the left hand side using the definition of K_* and then interchanging the order of integration yields the desired equality.

Next, for each $\delta > 0$, we consider the function $E_{\delta,\xi} : \mathbb{R} \longrightarrow \mathbb{C}$ given by

$$E_{\delta,\xi}(x) := \left(\widehat{K}(\xi)I - K_*\right)\nu_{\delta,\xi}(x) = \int_{\xi-\delta}^{\xi+\delta} e^{-i\eta x}\left(\widehat{K}(\xi) - \widehat{K}(\eta)\right)d\eta. \quad (\star\star)$$

Using $(\star)$, $\widehat{K}$ is differentiable and its derivative $(\widehat{K})'$ is bounded above by some constant M.

Proof of Lemma (continued)

This bound and the mean value theorem in ($\star\star$) yield on the one hand

$$|E_{\delta,\xi}(x)| < M\delta^2.$$

On the other hand integrating by parts in ($\star\star$) gives:

$$|E_{\delta,\xi}(x)| < 4M\delta x^{-1}$$

Thus:

$$\int_{-\infty}^{\infty} |E_{\delta,\xi}(x)|^p dx \leq \int_{|x|\leq \frac{4}{\delta}} (M\delta^2)^p dx + \int_{|x|>\frac{4}{\delta}} (4M\delta x^{-1})^p dx = \mathcal{O}(\delta^{2p-1})$$

and consequently $\|E_{\delta,\xi}\|_p = \mathcal{O}(\delta^{2-\frac{1}{p}})$.

Finally, for each $p \in (1, \infty)$, each $\xi \in \mathbb{R}$, and each $\delta > 0$ set:

$$u_{\delta,\xi} := \frac{\nu_{\delta,\xi}}{\|\nu_{\delta,\xi}\|_p}.$$

Using $\|\nu_{\delta,\xi}\|_p = \delta^{1-\frac{1}{p}} \|\nu_{1,\xi}\|_p$ we get the desired result.

A Spectral Theorem

Theorem

Let $K \in L^1(\mathbb{R})$, $1 < p < \infty$, and recall the convolution operator K_*. Then

$$\sigma\big(K_*, L^p(\mathbb{R})\big) = \overline{\{\lambda \in \mathbb{C} \mid \lambda = \widehat{K}(\xi) \ \textit{ for some } \ \xi \in \mathbb{R}\}}$$

Remark: The inclusion

$$\sigma\big(K_*, L^p(\mathbb{R})\big) \subseteq \overline{\{\lambda \in \mathbb{C} \mid \lambda = \widehat{K}(\xi) \ \text{ for some } \ \xi \in \mathbb{R}\}}$$

was proved in Corollary 1. Moreover, since the spectrum is a closed set it suffices to show:

$$\{\lambda \in \mathbb{C} \mid \lambda = \widehat{K}(\xi) \ \text{ for some } \ \xi \in \mathbb{R}\} \subseteq \sigma\big(K_*, L^p(\mathbb{R})\big)$$

Proof of Theorem

Fix $\xi \in \mathbb{R}$. To show $\widehat{K}(\xi) \in \sigma\big(K_*, L^p(\mathbb{R})\big)$, start by considering for each $n \in \mathbb{N}$ the function $K_n \in L^1(\mathbb{R})$ given by

$$K_n(x) := \begin{cases} K(x) \text{ if } |x| \leq n \\ 0 \text{ if } |x| > n \end{cases} \qquad x \in \mathbb{R}.$$

Then K_n satisfies the hypotheses in Lemma 2. Consequently, applying this with $\delta := \frac{1}{n}$, there exists $u_{n,\xi}$ and $M > 0$ such that:

$$\begin{cases} \|u_{n,\xi}\|_p = 1 & (1) \\ \left\| \left((K_n)_* - \widehat{K_n}(\xi) I\right) u_{n,\xi} \right\|_p < \frac{M}{n} & (2). \end{cases}$$

Next, Young's convolution inequality (with $q = 1$) and LDCT yields:

$$\|(K_n)_* - K_*\|_p \leq \|K_n - K\|_1 = \int_{|x|>n} |K(x)| dx \underset{n\to\infty}{\longrightarrow} 0. \quad (3)$$

Continuation of Proof

Moreover, employing again LDCT,

$$|\widehat{K_n}(\xi) - \widehat{K}(\xi)| \leq \int_{-\infty}^{\infty} \left| \big(K_n(x) - K(x)\big) e^{i\xi x} \right| dx \xrightarrow[n\to\infty]{} 0. \qquad (4)$$

Moving on, we claim:

$$\|(K_* - \widehat{K}(\xi) I) u_{n,\xi}\|_p \xrightarrow[n\to\infty]{} 0. \qquad (\star)$$

Indeed, for each $n \in \mathbb{N}$ write

$$\begin{aligned}
&\|(K_* - \widehat{K}(\xi) I) u_{n,\xi}\|_p \\
&\leq \|(K_* - K_{n*}) u_{n,\xi}\|_p + \|(K_{n*} - \widehat{K_n}(\xi) I) u_{n,\xi}\|_p + \|(\widehat{K_n}(\xi) - \widehat{K}(\xi)) u_{n,\xi}\|_p \\
&\leq \underbrace{\|K_* - K_{n*}\|_p}_{\text{by } (3): \to 0} \underbrace{\|u_{n,\xi}\|_p}_{\text{by } (1): =1} + \underbrace{\|(K_{n*} - \widehat{K_n}(\xi) I) u_{n,\xi}\|_p}_{\text{by } (2): \leq \frac{1}{n}} + \underbrace{\left|\widehat{K_n}(\xi) - \widehat{K}(\xi)\right|}_{\text{by } (4): \to 0} \underbrace{\|u_{n,\xi}\|_p}_{\text{by } (1): =1} \\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Continuation of Proof

Finally, by contradiction we assume that $\widehat{K}(\xi) \notin \sigma(K_*, L^p(\mathbb{R}))$. This implies $\widehat{K}(\xi)I - K_*$ is invertible on $L^p(\mathbb{R})$, and hence, there exists T_ξ a linear and bounded operator on $L^p(\mathbb{R})$, such that:

$$T_\xi \circ \left(\widehat{K}(\xi)I - K_*\right) = I.$$

Thus, on the one hand

$$1 = \|u_{n,\xi}\|_p = \left\|\left(T_\xi \circ \left(\widehat{K}(\xi)I - K_*\right)\right) u_{n,\xi}\right\|_p.$$

On the other hand, using $(\star)$ and the boundedness of T_ξ, we obtain:

$$\left\|\left(T_\xi \circ \left(\widehat{K}(\xi)I - K_*\right)\right) u_{n,\xi}\right\|_p \leq \|T_\xi\|_p \cdot \|(\widehat{K}(\xi)I - K_*)u_{n,\xi}\|_p \xrightarrow[n\to\infty]{} 0. \quad ※$$

Corollary 2

Corollary (2)

Let $p \in (1, \infty)$ and k be a measurable function on $\mathbb{R}_+$, such that:

$$\int_0^\infty |k(s)| s^{-1/p} ds < \infty,$$

and consider the operator T defined by:

$$Tf(t) := \int_0^\infty k(s) f(ts) ds, \qquad \forall f \in L^p(\mathbb{R}_+).$$

Then T is a bounded operator from $L^p(\mathbb{R}_+)$ into itself and

$$\sigma(T, L^p(\mathbb{R}_+)) = \overline{\left\{ \widetilde{k}(\tfrac{1}{q} + i\xi) : \xi \in \mathbb{R} \right\}},$$

where q is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof of Corollary

Let $\mathcal{K}(x) = k(e^{-x}) \cdot e^{-x/q}$ for $x \in \mathbb{R}$. A change of variable ($s = e^{-x}$), together with the hypothesis $\int_0^\infty |k(s)| s^{-1/p} ds < \infty$, imply $\mathcal{K} \in L^1(\mathbb{R})$.

Next, define the following operator $\mathcal{Q} : L^p(\mathbb{R}_+) \longrightarrow L^p(\mathbb{R})$ given by

$$\mathcal{Q}(f)(x) = f(e^x) e^{x/p}, \qquad x \in \mathbb{R}.$$

The change of variable ($t = e^x$) shows that $\mathcal{Q}$ is well defined, and

$$\|\mathcal{Q}(f)\|_{L^p(\mathbb{R})} = \|f\|_{L^p(\mathbb{R}_+)}$$

Moreover $\mathcal{Q}$ has the inverse $\mathcal{Q}^{-1} : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R}_+)$ where

$$\mathcal{Q}^{-1}(u)(x) = \frac{u(\ln x)}{x^{1/p}}, \qquad x \in \mathbb{R}_+.$$

Hence $\mathcal{Q}$ is an isometry.

Proof (continued)

In addition, we claim that $\mathcal{Q}$ satisfies

$$\left(\mathcal{Q} \circ T \circ \mathcal{Q}^{-1}\right) u = \mathcal{K}_* u \quad \forall\, u \in L^p(\mathbb{R}).$$

Indeed, if $u \in L^p(\mathbb{R})$ and $x \in \mathbb{R}$, then

$$\left(\mathcal{Q} \circ T \circ \mathcal{Q}^{-1}\right) u(x) = T(\mathcal{Q}^{-1} u)(e^x) \cdot e^{x/p} = \int_0^\infty k(s)(\mathcal{Q}^{-1} u)(e^x s)\, ds \cdot e^{x/p}$$

$$= \int_0^\infty k(s) \frac{u(\ln(e^x s))}{(e^x s)^{1/p}}\, ds \cdot e^{x/p}$$

$$= \int_0^\infty k(s) \frac{u(\ln(e^x s))}{s^{1/p}}\, ds.$$

Letting $y = \ln(e^x s)$ (thus $s = e^{y-x}$ and $ds = e^{y-x} dy$), we obtain

$$\left(\mathcal{Q} \circ T \circ \mathcal{Q}^{-1}\right) u(x) = \int_{-\infty}^\infty k(e^{y-x}) \frac{u(y)}{(e^{y-x})^{1/p}} e^{y-x} dy$$

Proof (continued)

Consequently

$$\left(\mathcal{Q} \circ T \circ \mathcal{Q}^{-1}\right) u(x) = \int_{-\infty}^{\infty} \underbrace{k(e^{-(x-y)})(e^{-(x-y)})^{1/q}}_{\mathcal{K}(x-y)} u(y)\, dy$$

$$= \int_{-\infty}^{\infty} \mathcal{K}(x-y) u(y) dy = \mathcal{K}_* u(x)$$

Since $\mathcal{Q}$ is an isometry this implies

$$\sigma\big(T, L^p(\mathbb{R}_+)\big) = \sigma(\mathcal{K}_*, L^p(\mathbb{R})). \quad (\star)$$

However, using the earlier theorem:

$$\sigma(\mathcal{K}_*, L^p(\mathbb{R})) = \overline{\{\lambda \in \mathbb{C} \mid \lambda = \widehat{\mathcal{K}}(\xi), \ \text{where } \xi \in \mathbb{R}\}}. \quad (\star\star)$$

Proof (continued)

Finally observe that $\widehat{\mathcal{K}}(\xi) = \widetilde{k}(\frac{1}{q} - i\xi)$. Indeed,

$$\widehat{\mathcal{K}}(\xi) = \int_{-\infty}^{\infty} \mathcal{K}(x) \cdot e^{i\xi x} dx = \int_{-\infty}^{\infty} k(e^{-x}) \cdot e^{-x/q} \cdot e^{i\xi x} dx \qquad \boxed{\underbrace{y = e^{-x}}_{dy = -e^{-x}dx}}$$

$$= \int_{0}^{\infty} k(y) \cdot y^{(1/q) - i\xi} \frac{dy}{y} = \widetilde{k}(\frac{1}{q} - i\xi).$$

The conclusion of the corollary immediately follows now from this and $(\star) - (\star\star)$

Hardy Kernels

Definition

Let $k(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a Lebesgue measurable function s.t.

($\star$) k is homogeneous of degree -1

$\left(\text{i.e. for any } t \in \mathbb{R}_+ \ k(tx, ty) = \frac{1}{t} k(x,y) \text{ for all } x, y \in \mathbb{R}_+\right)$

($\star$) $\int_0^\infty |k(1,t)| t^{-1/p} dt = \int_0^\infty |k(s,1)| s^{(1/p)-1} ds < \infty.$

Define the operator $T : L^p(\mathbb{R}_+) \longrightarrow L^p(\mathbb{R}_+)$:

$$Tf(s) := \int_0^\infty k(s,t) f(t) dt \quad \text{for a.e. } s \in \mathbb{R}_+.$$

Then k is called a Hardy kernel and T is called a Hardy-kernel operator for $L^p(\mathbb{R}_+)$.

Corollaries

Corollary (3)

Let $1 < p < \infty$ *and* T *be a Hardy-kernel operator for* $L^p(\mathbb{R}_+)$*. Then*

$$\sigma(T, L^p(\mathbb{R}_+)) = \overline{\left\{ \left(\widetilde{k}(\cdot, 1)\right)\left(\tfrac{1}{p} + i\xi\right) : \xi \in \mathbb{R} \right\}}.$$

In the sequel we shall work in the matrix setting, i.e. when $k = \left(k_{ij}\right)_{ij}$ with the entries k_{ij} being Hardy kernels.

Corollary (4)

Consider k *a Hardy kernel and* T *be a Hardy-kernel operator for* $[L^p(\mathbb{R}_+)]^2$, $1 < p < \infty$, *with kernel* k *as above. Then, for each* $\lambda \in \mathbb{C}$ *the operator* $\lambda I - T$ *is invertible on* $L^p(\mathbb{R}_+)$*, if and only if the following holds:*

$$det\left(\lambda I - \left(\widetilde{k}(\cdot, 1)\right)\left(\tfrac{1}{p} + i\xi\right)\right) \neq 0 \quad \forall \xi \in \overline{\mathbb{R}}.$$

Applications to PDE

Let Ω be a reasonable domain in $\mathbb{R}^2$. The Dirichlet problem with L^p data:

$$(D) \quad \begin{cases} \triangle u = 0 \quad \text{in} \quad \Omega \\ u\Big|_{\partial\Omega}^{n.t.} = f \in L^p(\partial\Omega) \qquad\qquad 1 < p < \infty. \\ \mathcal{N}(u) \in L^p(\partial\Omega) \end{cases}$$

Via the layer potential method (D) is reduced to a BIE of the type $(\frac{1}{2}I + K)g = f$, where K is a SIO of Calderón-Zygmund type. Indeed, let Γ be such that $\Delta\Gamma = \delta$ as distributions in $\mathbb{R}^2$,

$$\Gamma(X) = \tfrac{1}{2\pi} \ln|X|, \qquad \forall\, X \in \mathbb{R}^2 \setminus \{0\}.$$

Introduce the double layer potential:

$$\mathcal{D}g(X) := \int_{\partial\Omega} \frac{\partial}{\partial\nu(Q)}[\Gamma(X-Q)]g(Q)\, d\sigma(Q), \qquad X \in \mathbb{R}^2 \setminus \partial\Omega.$$

Applications to PDE - continued

The principal value harmonic double layer potential operator is given by

$$Kg(P) := p.v. \int_{\partial\Omega} \frac{\partial}{\partial\nu(Q)}[\Gamma(P-Q)]g(Q)\,d\sigma(Q), \qquad \text{for } \sigma\text{-a.e. } \; P \in \partial\Omega,$$

and it satisfies the jump relations (for $g \in L^p(\partial\Omega)$):

$$\mathcal{D}g\big|^{\text{n.t.}}_{\partial\Omega}(P) = \left(\tfrac{1}{2}I + K\right)g(P), \quad \sigma - \text{a.e. } \; P \in \partial\Omega.$$

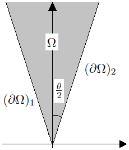
Thus, the solvability of (D) can be recast as a spectral problem, namely matters reduce to showing

$$-\frac{1}{2} \notin \sigma(K, L^p(\partial\Omega)).$$

Key Observation: *When Ω is an infinite sector in $\mathbb{R}^2$, then K can be naturally identified with a Hardy kernel operator- thus the earlier technology applies*.

Application: Ω infinite sector

The problem is rotation invariant , so Wlog assume Ω is an upright sector symmetric w.r.t. y-axis. Denote by $(\partial\Omega)_1$ and $(\partial\Omega)_2$ the left and the right side of $\partial\Omega$, resp. . Concretely, one can write:



$$(\partial\Omega)_1 := \left\{(-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}) : \ s \in \mathbb{R}_+\right\}$$

$$(\partial\Omega)_2 := \left\{(s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}) : \ s \in \mathbb{R}_+\right\}.$$

Hence, via the mapping:

$$(\partial\Omega)_j \ni P \mapsto |P| \in \mathbb{R}_+,$$

for $j = 1, 2$ and for all $p \in [1, \infty)$ one can identify:

$$(\partial\Omega)_j \ \text{ with } \ \mathbb{R}_+ \ \text{ and } \ L^p(\partial\Omega) \ \text{ with } \ L^p(\mathbb{R}_+) \oplus L^p(\mathbb{R}_+).$$

Moreover,

$$\mathcal{K}(P, Q) = \frac{1}{2\pi} \cdot \frac{\langle Q - P, \nu(Q)\rangle}{|Q - P|^2} \quad \forall P, Q \in \partial\Omega, P \neq Q$$

Continuation

Going further, $\mathcal{K}$ can be regarded as a kernel on $\mathbb{R}_+ \times \mathbb{R}_+$. Specifically the function $\mathcal{K}(\cdot,\cdot)$ on $\partial\Omega \times \partial\Omega$ shall be identified with the following 2×2 kernel matrix $\mathscr{K} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow M_{2\times 2}(\mathbb{R})$ given by

$$\mathscr{K}(s,t) := \frac{1}{2\pi}\begin{pmatrix} 0 & \frac{s\sin(\theta)}{s^2+t^2-2st\cos(\theta)} \\ \frac{s\sin(\theta)}{s^2+t^2-2st\cos(\theta)} & 0 \end{pmatrix},$$

The Mellin transform of $\mathscr{K}(\cdot,1)$ is given by:

$$\widetilde{\mathscr{K}}(\cdot,1)(z) := \frac{1}{2\pi}\begin{pmatrix} 0 & \frac{\pi\sin((\pi-\theta)z)}{\sin(\pi z)} \\ \frac{\pi\sin((\pi-\theta)z)}{\sin(\pi z)} & 0 \end{pmatrix}.$$

Hence $\frac{1}{2}I + K$ is invertible on $L^p(\partial\Omega)$ iff $\forall\xi \in \mathbb{R}$ and for $z = \frac{1}{p} + i\xi$

$$\det\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\frac{\sin((\pi-\theta)z)}{\sin(\pi z)} \\ \frac{1}{2}\frac{\sin((\pi-\theta)z)}{\sin(\pi z)} & \frac{1}{2} \end{pmatrix} \neq 0 \iff p \neq \begin{cases} \frac{2\pi-\theta}{\pi} & \text{for } \theta \in (0,\pi) \\ \frac{\theta}{\pi} & \text{for } \theta \in (\pi,2\pi) \end{cases}$$

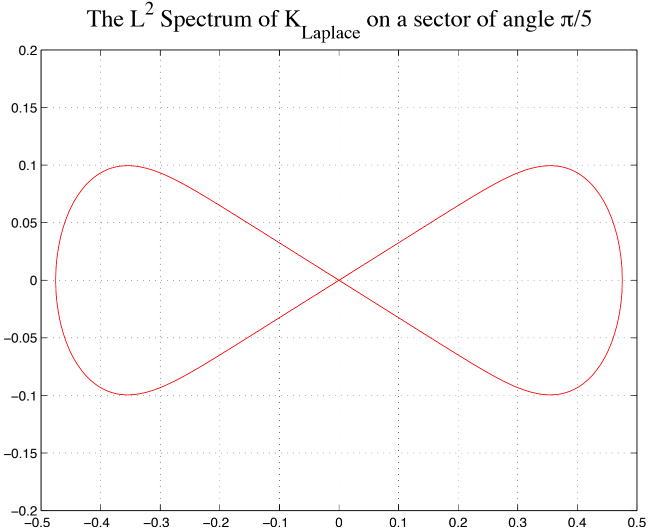
Spectrum for K

Consequently, based on the earlier discussion, given $p \in (1, \infty)$ the spectrum $\sigma(K, L^p(\partial\Omega))$ is explicitly characterized as the set

$$\overline{\left\{ \lambda \in \mathbb{C} : \det \begin{pmatrix} \lambda & A(\theta, z) \\ A(\theta, z) & \lambda \end{pmatrix} = 0 \text{ for some } z \in \mathbb{C} \ \ \mathrm{Re}\, z = \frac{1}{p} \right\}}.$$

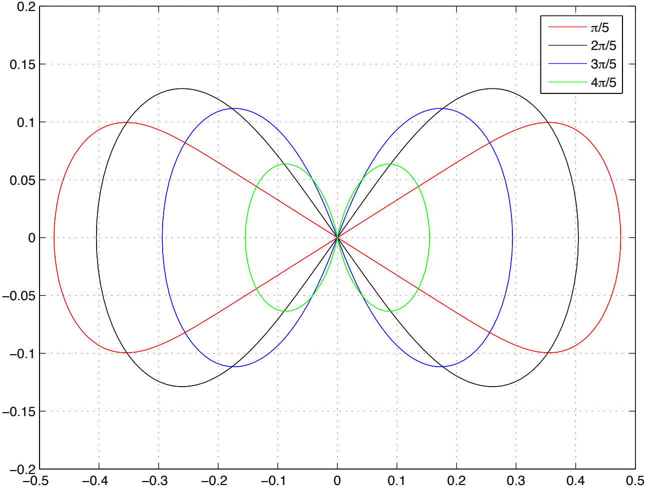
where $A(\theta, z) := -\frac{1}{2} \frac{\sin((\pi - \theta) z)}{\sin(\pi z)}$. This, in turn implies $\lambda = \pm A(\theta, z)$.

Spectrum for K

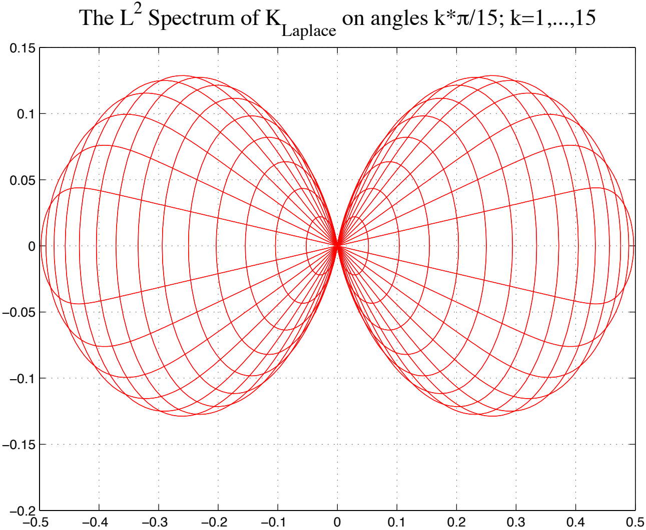


Spectrum for *K*

The L^2 Spectrum of $K_{Laplace}$ on sectors of angles $k*\pi/5$, k=1,...,4



Spectrum for *D*



Application: Mixed Boundary Value Problem

Let Ω be as above. The boundary value problem with mixed Dirichlet and Neumann type boundary conditions with L^p data, $p \in (1, \infty)$, is formulated as follows.

$$
(MBVP_p) \quad \begin{cases} \triangle u = 0 \quad \text{in} \quad \Omega \\ u\Big|_D^{n.t.} = f \in L_1^p(D) \quad \text{on} \quad D \\ \dfrac{\partial u}{\partial \nu}\Big|_N^{n.t.} = g \in L^p(N) \quad \text{on} \quad N \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega) \end{cases}
$$

The same kind of analysis applied above can be carried again here.

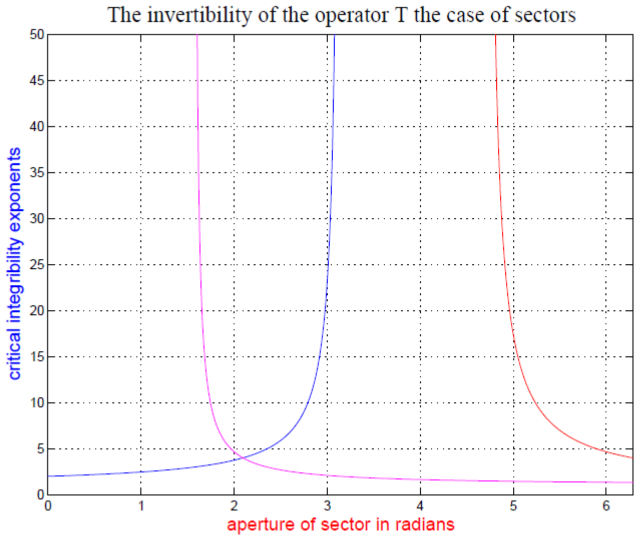
Mixed Boundary Value Problem Result

Theorem

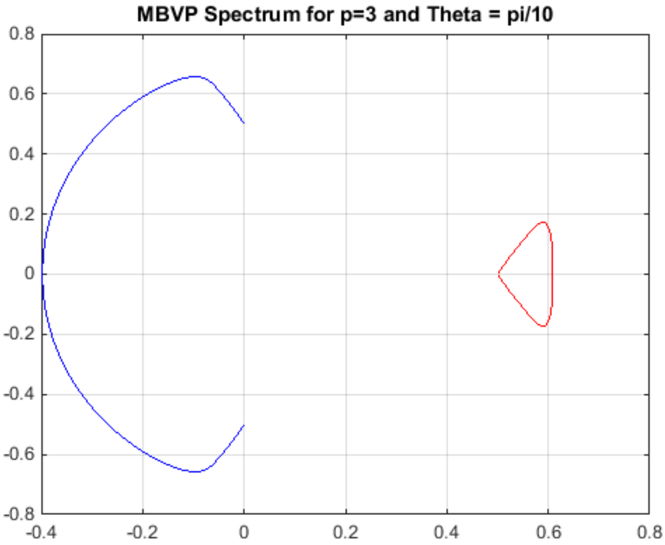
Let Ω an sector in $\mathbb{R}^2$ with apretue theta $\in (0, 2\pi)$, then $(MBVP_P)$ has a solution if:

$$p \neq \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \textit{if} \quad \theta \in (0, \pi/2) \\ \frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi} & \textit{if} \quad \theta \in (\pi/2, \pi) \\ \frac{2\theta}{2\theta-\pi} & \textit{if} \quad \theta \in (\pi, 3\pi/2) \\ \frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi} & \textit{if} \quad \theta \in (3\pi/2, 2\pi) \\ 3 & \textit{if} \quad \theta = \pi/2 \\ 3/2 & \textit{if} \quad \theta = 3\pi/2 \\ 2 & \textit{if} \quad \theta = \pi. \end{cases}$$

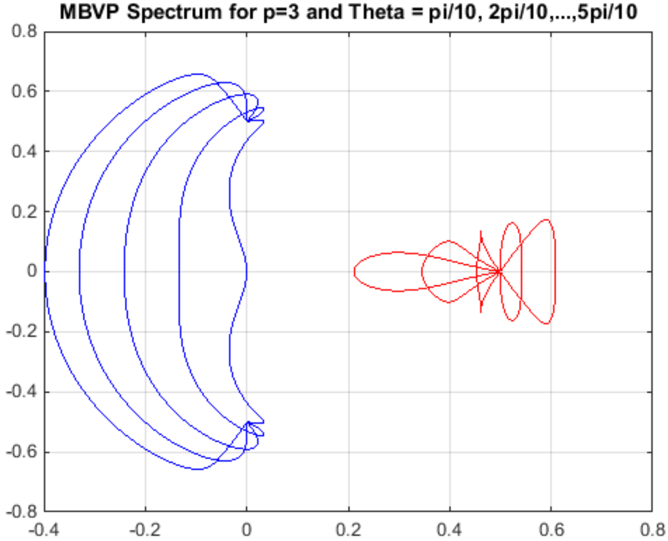
Graph of critical indexes in Main Theorem for *MBVP*



Spectrum for $MBVP_p$



Spectrum for $MBVP_p$



THANK YOU