Jump Formulas for Tempered Distributions

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Preliminaries

The Schwartz class of Rapidly decreasing functions is

$$\mathcal{S}(\mathbb{R}^n) := \{\phi \in \mathcal{C}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty, \forall \alpha, \beta \in \mathbb{N}_0^n\}.$$

The class of Tempered distributions is the algebraic dual of $\mathcal{S}(\mathbb{R}^n)$

$$\mathcal{S}'(\mathbb{R}^n) := \{u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : \ u \text{ is linear and continuous } \},$$

where a linear functional u is continuous iff

$$u(\phi_j) \underset{j \to \infty}{\longrightarrow} 0 \quad \text{whenever} \quad \phi_j \underset{j \to \infty}{\longrightarrow} 0 \ \text{ in } \mathcal{S}(\mathbb{R}^n).$$

A function f defined on $\mathbb{R}^n$ is said to be Positive Homogeneous if

$$f(\alpha \cdot x) = \alpha^k f(x) \ \text{ for all } \ \alpha > 0 \ \text{ and for all } \ x \in \mathbb{R}^n.$$

Main Theorem

Theorem

For $\Phi \in \mathcal{C}^4(\mathbb{R}^n \backslash \{0\})$ *is odd and positive homogeneous of degree* $1-n$*, then*

$$\lim_{t \to 0^{\pm}} \Phi(x', t) = \pm \frac{i}{2} \widehat{\Phi}(0', 1)\delta(x') + P.V.\Phi(x', 0) \quad \text{in } \mathcal{S}'(\mathbb{R}^{n-1}).$$

Where $x' = (x_1, ..., x_{n-1})$, $\delta(\cdot)$ is the Dirac Delta distribution with weight at zero, $(P.V.\Phi) : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is given by

$$P.V.\Phi(\psi) := \lim_{\varepsilon \to 0^+} \int_{|x| > \varepsilon} \Phi(x)\psi(x)dx, \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^n),$$

and, $\widehat{\Phi}$ is the Fourier transform of Φ.

Example

We start by mentioning a simple example of this Phenomenon. For $\varepsilon \in (0, \infty)$, we define

$$f_{\varepsilon}^{\pm} := \frac{1}{x \pm i\epsilon} \quad \text{for} \quad x \in \mathbb{R}.$$

Then it is an easy exercise to see that we get the following

$$\boxed{\frac{1}{x \pm i\varepsilon} \longrightarrow \mp i\pi\delta + P.V.\frac{1}{x} \quad \text{as } \varepsilon \to 0^{+} \quad \text{in} \quad \mathcal{D}'(\mathbb{R})}.$$

Where $\mathcal{D}'(\mathbb{R})$ is the space of general distributions over $\mathbb{R}$.

This is the so-called Sokhotsky's formula.

Proof

Fix $\Phi \in C^4(\mathbb{R}^n \backslash \{0\})$ odd, positive homogeneous of degree $1-n$, and consider the function $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$. Let $\varepsilon > 0$, and write

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} \Phi(x',t)\psi(x')dx' = \underbrace{\lim_{t\to 0^+} \int_{|x'|>\varepsilon} \Phi(x',t)\psi(x')dx'}_{I_\varepsilon}$$

$$+ \underbrace{\lim_{t\to 0^+} \int_{|x'|<\varepsilon} \Phi(x',t)\big(\psi(x') - \psi(0')\big)dx'}_{II_\varepsilon}$$

$$+ \psi(0') \underbrace{\lim_{t\to 0^+} \int_{|x'|<\varepsilon} \Phi(x',t)dx'}_{III_\varepsilon}.$$

The Integral III_ε

Using the following change of variable

$$y' := \frac{x'}{t} \quad \text{which implies} \quad \begin{cases} dx' = t^{n-1}dy' \\ |x'| < \varepsilon \Leftrightarrow |y'| < \frac{\varepsilon}{t}, \end{cases}$$

and the homogeneity of Φ $\big($namely $\Phi(x', t) = t^{1-n}\Phi(y', 1)\big)$, we write

$$\psi(0') \lim_{t\to 0^+} \underbrace{\int_{|x'|<\varepsilon} \Phi(x', t)dx'}_{III_\varepsilon} = \psi(0') \lim_{t\to 0^+} \int_{|y'|<\frac{\varepsilon}{t}} \Phi(y', 1)dy'$$

$$= \boxed{\psi(0') \lim_{r\to\infty} \int_{|y'|<r} \Phi(y', 1)dy'}.$$

Where the limit in the box is independent of ε.

The Integral II_ε

Next, we turn our attention to $\underbrace{\int_{|x'|<\varepsilon} \Phi(x',t)\big(\psi(x') - \psi(0')\big)\,dx'}_{II_\varepsilon}$, the fact that $\Phi \in C^4(\mathbb{R}^n \setminus \{0\})$ gives for $x' \in \mathbb{R}^{n-1},\ t \in \mathbb{R}\setminus\{0\}$

$$|\Phi(x',t)| \leq \frac{\|\Phi\|_{L^\infty(S^{n-1})}}{|(x',t)|^{n-1}} \leq \frac{\|\Phi\|_{L^\infty(S^{n-1})}}{|x'|^{n-1}} \quad (*)$$

On the other hand, the function ψ is in $\mathcal{S}(\mathbb{R}^{n-1})$ allowing us to get

$$|\psi(x') - \psi(0')| \leq \|\nabla\psi\|_{L^\infty(\mathbb{R}^{n-1})}|x'| \quad (**)$$

Using $(*)$ and $(**)$ in II_ε we can write

$$II_\varepsilon \leq \int_{|x'|<\varepsilon} \|\nabla\psi\|_{L^\infty(\mathbb{R}^{n-1})} \frac{\|\Phi\|_{L^\infty(S^{n-1})}}{|x'|^{n-2}}\,dx'.$$

But this shows that

$$\boxed{\lim_{\varepsilon\to 0^+} II_\varepsilon = 0}.$$

The integral I_ε

Finally, a direct application of Lebesgue Dominating Convergence Theorem yields

$$\lim_{t\to 0^+} \underbrace{\int_{|x'|>\varepsilon} \Phi(x',t)\psi(x')dx'}_{I_\varepsilon} = \int_{|x'|>\varepsilon} \Phi(x',0)\psi(x')dx'$$

Hence, passing to the limit as $\varepsilon \to 0$, and using the properties we proved of I_ε, II_ε and III_ε we obtain

$$\lim_{t\to 0^+} \int_{\mathbb{R}^{n-1}} \Phi(x',t)\psi(x')dx' = \lim_{\varepsilon\to 0^+} \int_{|x'|>\varepsilon} \Phi(x',0)\psi(x')dx' + \psi(0') \lim_{r\to\infty} \int_{|x'|<r} \Phi(x',1)dx'. \quad (\star).$$

The Limit at 0^-

Applying the change of variable $x' \to -x'$, in the integrals in the equation $(\star)$ we get the identity

$$\lim_{t\to 0^-}\int_{\mathbb{R}^{n-1}}\Phi(x',t)\psi(x')dx' = \lim_{\varepsilon\to 0^+}\int_{|x'|>\varepsilon}\Phi(x',0)\psi(x')dx'$$
$$+\psi(0')\lim_{r\to\infty}\int_{|x'|<r}\Phi(x',-1)dx' \quad (\star\star),$$

Where we re-denote t by $-t$, and we use the fact that Φ is an odd function. Putting $(\star)$ and $(\star\star)$ together, we can write

$$\lim_{t\to 0^\pm}\int_{\mathbb{R}^{n-1}}\Phi(x',t)\psi(x')dx' = a_\pm\psi(0') + \lim_{\varepsilon\to 0^+}\int_{|x'|>\varepsilon}\Phi(x',0)\psi(x')dx'$$

where

$$a_\pm := \lim_{r\to\infty}\int_{|x'|<r}\Phi(x',\pm 1)dx'.$$

Remark on the value of $a_\pm$

Remark

Let Φ be defined on $\mathbb{R}^{n-1}$ be as in the theorem, then

$$\lim_{r\to\infty}\int_{|x'|<r}\Phi(x',\pm 1)dx' \text{ exists in } \mathbb{C}.$$

To prove this, notice first that for $t \neq 0$ fixed we can use the homogeneity of Φ to write

$$|\Phi(x',t)| \leq \frac{\|\Phi\|_{L^\infty(S^{n-1})}}{|(x',t)|^{n-1}} \leq C_t\|\Phi\|_{L^\infty(S^{n-1})}$$

Which shows that $\int_{|x'|<r}|\Phi(x',1)|dx' < \infty$ for r fixed. (1)

With this in hand, we write

$$\int_{|x'|<r} \Phi(x',1)dx' = \frac{1}{2}\int_{|x'|<r} \Phi(x',1)dx' + \frac{1}{2}\int_{|x'|<r} \Phi(x',1)dx'$$
$$= \int_{|x'|<r} \frac{1}{2}\big(\Phi(x',1) - \Phi(x',-1)\big)dx'.$$

Next, the fact that $\nabla\Phi$ is homogeneous of degree $-n$ combined with the mean value theorem yields

$$|\Phi(x',1) - \Phi(x',-1)| \leq \frac{2\|\nabla\Phi\|_{L^\infty(S^{n-1})}}{|x'|^n} \text{ for } x' \text{ large.} \quad (2)$$

Finally,(1), (2) and LDCT together finish the proof of the remark.

Continuation of Proof

We aim to prove

$$\lim_{t\to 0^\pm} \Phi(x', \varepsilon) = \pm\frac{i}{2}\widehat{\Phi}(0', 1)\delta(x') + P.V.\Phi(x', 0) \text{ in } \mathcal{S}'(\mathbb{R}^{n-1}).$$

So far we have

$$\lim_{t\to 0^\pm} \int_{\mathbb{R}^{n-1}} \Phi(x', t)\psi(x')dx' = a_\pm \psi(0') + \underbrace{\lim_{\varepsilon\to 0^+} \int_{|x'|>\varepsilon} \Phi(x', 0)\psi(x')dx'}_{P.V.\Phi(x',0)}$$

where

$$a_\pm := \lim_{r\to\infty} \int_{|x'|<r} \Phi(x', \pm 1)dx'.$$

To finish the proof, we need is to find the values of $a_\pm$. We are going to do so in a series of Claims.

Claim (1)

For $u_0 \in \mathcal{E}'(\mathbb{R}^n)$*, and* $u_1 \in L^1(\mathbb{R}^n)$ *we have*

$$u := u_0 + u_1 \quad \textit{has the property that} \quad \widehat{u} \in \mathcal{C}^0(\mathbb{R}^n)$$

Where $\mathcal{E}'(\mathbb{R}^n)$ is the space of distributions with compact support.

To prove this claim, first notice that the Fourier operator distributes over addition. Next

$$u_0 \quad \text{a distribution with a compact support} \quad \Longrightarrow \boxed{\widehat{u_0} \in \mathcal{C}^\infty(\mathbb{R}^n)}.$$

Finally, $u_1 \in L^1(\mathbb{R}^n)$ ensures that $\boxed{\widehat{u_1} \in \mathcal{C}^0(\mathbb{R}^n)}$ finishing the proof of claim 1.

Claim (2)

Let $\xi_n \neq 0$, $\xi' \in \mathbb{R}^{n-1}$ *such that there exists a constant* C' *with* $|\xi'| < C'$. *Consider,* $\Theta \in \mathcal{C}^1(\mathbb{R}^n \backslash \{0\})$ *satisfying:*

$$\Theta(\lambda x) = \lambda^{-1}\Theta(x) \;\; \forall\, x \in \mathbb{R}^n \backslash \{0\} \;\; \forall\, \lambda \in \mathbb{R} - \{0\} \;\; (*).$$

Then,

$$|\Theta(\xi', \xi_n) + \Theta(0', \xi_n)| \leq \frac{C}{|\xi_n|^2}.$$

Using smoothness of Θ and the condition $(*)$ write

$$|\Theta(\xi', \xi_n) + \Theta(0', \xi_n)| = \frac{1}{\xi_n} \cdot |\Theta(\xi'/\xi_n, 1) + \Theta(0', 1)|$$

$$\leq \frac{1}{\xi_n} \cdot \frac{\xi'}{\xi_n} \cdot \sup_{t \in [0,1]} |(\nabla\Theta)(-t\xi'/\xi_n, 1)| \leq \frac{C'}{\xi_n^2} \cdot \|\nabla\Theta\|_{L^\infty(S^{n-1})}.$$

Setting $\boxed{C := C' \cdot \|\nabla\Theta\|_{L^\infty(S^{n-1})}}$ we get our desired result.

Claim (3)

For $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$ *such that* $\widehat{\psi}$ *has a compact support, then* F *defined on* $\mathbb{R}$ *by*

$$F(\xi_n) := -(2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \left[\widehat{\Phi}(\xi', -\xi_n) + \widehat{\Phi}(0', \xi_n)\right] \widehat{\psi}(\xi') d\xi'$$

is well-defined and integrable on $\mathbb{R} \backslash [-1, 1]$*.*

A straight forward calculation shows that $\widehat{\Phi}$ is positive homogeneous of degree $n - (1 - n) = -1$. Hence as a function on $\mathbb{R}^n \backslash \{0\}$,

$$\widehat{\Phi}\big|_{\mathbb{R}^n \backslash \{0\}} \text{ is positive homogeneous of degree -1.}$$

Moreover, Φ being odd function implies that $\widehat{\Phi}$ is odd.

Finally, Claim (2) applied to $\widehat{\Phi}$ finishes the proof of claim (3).(The fact that $\Phi \in \mathcal{C}^4(\mathbb{R}^n \backslash \{0\})$ and is positive homogeneous of degree $1 - n$ ensure that $\widehat{\Phi} \in \mathcal{C}^1(\mathbb{R}^n \backslash \{0\})$ follows from)

Claim (4)

For $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$ *such that* $\widehat{\psi}$ *has a compact support, the function* $f = f_1 + f_2$ *defined on* $\mathbb{R}\backslash\{0\}$ *by*

$$f(t) := \underbrace{\int_{\mathbb{R}^{n-1}} \Phi(x', t)\psi(x')dx'}_{f_1} + \underbrace{\frac{1}{2i}(\mathrm{sgn}t)\widehat{\Phi}(0', 1)\psi(0')}_{f_2},$$

has a continuous extension to all of $\mathbb{R}$*.*

Starting with f_2, we will use the following 2 facts

$$\widehat{\mathrm{sgn}}(\xi_n) = -2i \cdot \mathrm{P.V.}(\frac{1}{\xi_n}), \quad \text{and} \quad \psi(0') = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \widehat{\psi}(\xi')d\xi',$$

to write

$$\widehat{f_2}(\xi_n) = -(2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \widehat{\Phi}(0', \xi_n)\widehat{\psi}(\xi')d\xi' \quad \text{for} \quad \xi_n \in \mathbb{R}\backslash\{0\}. \ (*)$$

Continuation of Proof for Claim 4

Turning our attention to computing the Fourier Transform of f_1 given by

$$f_1(t) = \int_{\mathbb{R}^{n-1}} \Phi(x', t)\psi(x')dx'$$

First, Notice that $f_1 \in \mathcal{S}'(\mathbb{R})$, then for $\eta \in \mathcal{S}(\mathbb{R})$ we write

$$\begin{aligned}\langle \widehat{f_1}, \eta\rangle &= \langle f_1, \widehat{\eta}\rangle = \Big\langle \langle \Phi, \psi\rangle, \widehat{\eta}\Big\rangle = \langle \Phi, \psi \otimes \widehat{\eta}\rangle \\ &= (2\pi)^{1-n}\langle \Phi, (\widehat{(\widehat{\psi})^{\mathrm{v}} \otimes \eta})\rangle = (2\pi)^{1-n}\langle \widehat{\Phi}, (\widehat{\psi})^{\mathrm{v}} \otimes \eta\rangle \\ &= (2\pi)^{1-n}\Big\langle \langle \widehat{\Phi}(\xi', \xi_n), \widehat{\psi}(-\xi')\rangle, \eta(\xi_n)\Big\rangle\end{aligned}$$

where $g^{\mathrm{v}}(t) = g(-t)$. All in all, we deduce

$$\widehat{f_1}(\xi_n) = (2\pi)^{1-n}\int_{\mathbb{R}^{n-1}} \widehat{\Phi}(\xi', \xi_n)\widehat{\psi}(-\xi')d\xi' \text{ for } \xi_n \in \mathbb{R}\backslash\{0\}. \quad (**)$$

Continuation of Proof for Claim 4

Putting $(*)$ and $(**)$ we get

$$\widehat{f}(\xi_n) = -(2\pi)^{1-n}\Big(\int_{\mathbb{R}^{n-1}} \widehat{\Phi}(\xi', -\xi_n)\widehat{\psi}(\xi')d\xi' - \int_{\mathbb{R}^{n-1}} \widehat{\Phi}(0', \xi_n)\widehat{\psi}(\xi')d\xi'\Big),$$

Which is the function F introduced in claim 3. Moving on, we introduce the function $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$ s.t $\theta \equiv 1$ on $[-1, 1]$, and we write

$$\widehat{f} = (1-\theta)\widehat{f} + \theta\widehat{f}$$

Using Claim 3 $(1-\theta)\widehat{f} = (1-\theta)\widehat{F} \in L^1(\mathbb{R})$. On the other hand, multiplying $\widehat{f}$ by θ ensures that $\theta\widehat{f} \in \mathcal{E}'(\mathbb{R})$. Next, use Claim 1 to deduce that $\widehat{f} \in \mathcal{C}^0(\mathbb{R})$, which implies that $f \in \mathcal{C}^0(\mathbb{R})$ finishing the proof of Claim 4.

Claim (5)

Recall $a_{\pm} := \lim_{r\to\infty} \int_{|x'|<r} \Phi(x', \pm 1)dx'$ *then*

$$a_{\pm} = \pm \frac{i}{2}\widehat{\Phi}(0', 1).$$

we start by recalling the equation

$$\lim_{t\to 0^{\pm}} \int_{\mathbb{R}^{n-1}} \Phi(x', t)\psi(x')dx' = a_{\pm}\psi(0') + \lim_{\varepsilon\to 0^{+}} \int_{|x'|>\varepsilon} \Phi(x', 0)\psi(x')dx'$$

we deduce that

$$(a_{+} - a_{-})\psi(0') = \lim_{t\to 0^{+}} \int_{\mathbb{R}^{n-1}} \Phi(x', t)\psi(x')dx' - \lim_{t\to 0^{-}} \int_{\mathbb{R}^{n-1}} \Phi(x', t)\psi(x')dx'.$$

But by Claim 4 we have

$$f(t) = \int_{\mathbb{R}^{n-1}} \Phi(x', t)\psi(x')dx' + \frac{1}{2i}(\operatorname{sgn} t)\widehat{\Phi}(0', 1)\psi(0') \quad (*)$$

can be continuously extended to all $\mathbb{R}$

Continuation of Proof for Claim 5

Using $(*)$ we get that

$$(a_+ - a_-)\psi(0') = i\widehat{\Phi}(0', 1)\psi(0').$$

Where this is true for all $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$. Hence, choosing the right ψ (namely s.t. $\psi(0') \neq 0$) yields

$$(a_+ - a_-) = i\widehat{\Phi}(0', 1).$$

Finally, to finish the proof, the fact that Φ is an odd function and the definition of $a_\pm$ given by

$$a_\pm := \lim_{r\to\infty} \int_{|x'|<r} \Phi(x', \pm 1)dx',$$

ensure that, $\boxed{a_+ = -a_-}$, thus finishing the proof of the Main theorem.

Corollary

For $\Phi \in \mathcal{C}^4(\mathbb{R}^n \backslash \{0\})$ *is odd and positive homogeneous of degree* $1-n$*, let* $\psi \in \mathcal{S}(\mathbb{R}^{n-1})$*. Then for all* $x' \in \mathbb{R}^{n-1}$ *one has*

$$\lim_{t \to 0^{\pm}} \int_{\mathbb{R}^{n-1}} \Phi(x'-y',t)\psi(y')dy' = \pm \frac{i}{2}\widehat{\Phi}(0',1)\psi(x') + \lim_{\varepsilon \to 0^+} \int_{\substack{y' \in \mathbb{R}^{n-1} \\ |x'-y'|>\varepsilon}} \Phi(x'-y',0)\psi(y')dy'.$$

To prove this use the change of variable $z' = x' - y'$ to write

$$\lim_{t \to 0^{\pm}} \int_{\mathbb{R}^{n-1}} \Phi(x'-y',t)\psi(y')dy' = \lim_{t \to 0^{\pm}} \int_{\mathbb{R}^{n-1}} \Phi(z',t)\psi(x'-z')dy'$$

$$\lim_{t \to 0^{\pm}} \langle \Phi(\cdot,t), \psi(x'-\cdot) \rangle = \pm \frac{i}{2}\widehat{\Phi}(0',1)\psi(x') + \langle P.V.\Phi(\cdot,0), \psi(x'-\cdot) \rangle.$$

Where the last equality is die to above theorem.

An easy application for the above Corollary can be observed through the Cauchy operator, where for $\psi \in \mathcal{S}(\mathbb{R})$ it is given by

$$(\mathscr{C}\psi)(z) := \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\psi}{x-z}dx, \quad \text{for all} \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

Then, the following Plemelj jump-formula hold at every $x \in \mathbb{R}$:

$$\lim_{y\to 0^{\pm}} (\mathscr{C}\psi)(x+iy) = \pm\frac{1}{2}\psi(x) + \lim_{\varepsilon\to 0^{+}} \frac{1}{2\pi i}\int_{\substack{y\in\mathbb{R} \\ |x-y|>\varepsilon}} \frac{\psi(y)}{y-x}dy.$$

To prove this formula, one applies the above corollary to the function

$$\Phi(x,y) := \frac{-1}{2\pi i(x+iy)}.$$

Next, use the fact that

$$\widehat{\Phi}(\xi) = \frac{1}{\xi}, \quad \text{and thus } \widehat{\Phi}(0,1) = -i.$$

THANK YOU