On the Proof of the Friendship Theorem

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Preliminaries

A graph in the plane is an ordered pair $G = (V, E)$ where:

- $V = V(G)$ is called the vertex set, and is the set of vertices or nodes in the graph G.
- $E = E(G)$ is called the edge set, and is the set of edges or connections between the vertices.

For $u, v \in V(G)$.

- The undirected edges are represented by $\{u, v\}$.
- The directed edges are represented by (u, v).

Finally let $G = (V, E)$ be an undirected graph.

- Two nodes that are connected by an edge are adjacent.
- The neighbourhood of a node v, denoted $N(v)$, is the set of all the nodes that are adjacent to it, and $|N(v)|$ is it's cardinality.

Main Theorem

Theorem

Suppose that $G = (V, E)$ is a finite graph in which

$$\forall\ u, v \in V(G) \qquad |N(u) \cap N(v)| = 1.$$

Then there exists a vertex $z \in V(G)$ that is adjacent to all other vertices.

Informally, one can say

Theorem

Suppose a group of at least three people has the property that every pair of people has exactly one friend in common, then there must be one person who is a friend to all the others.

History and Application

- It is not yet known who formulated this problem, or who gave it the human touch that it has.
- In 1966, Paul Erdös, Alfred Renyi and Vera Sos Published the first proof of this theorem.
- This Theorem has a lot of applications, it is mainly used in the fields of
  - Block Designs
  - Coding Theory
  - Set Theory
- The Friendship Theorem is listed among Paul and Jack Abad's "100 Greatest Theorems".
- In 2001, Aigner and Ziegler mentioned the Friendship Theorem in there book titled "*Proofs from the Book* "as one of the greatest theorems of Erdös of all time.

More History

Several different approaches have been used to prove this theorem:

- In 1971, Herbert Wilf provided a geometric proof.
- In 1972, Judith Longyear and Torrence Parsons gave a proof by counting neighbors, walks and cycles in regular graphs.
- In 1983, Hammersley provided a proof using numerical techniques. He later extended this theorem to the so called "Love Problem".
- In 2001, Douglus West gave a proof similar to the one by Longyear and Parsons, counting common neighbors and cycles.
- Finally, in 2002 Craig Huneke gave two proofs, one being more combinatorial and one that combines combinatorics and linear algebra.

Proof

Theorem

Suppose that $G = (V, E)$ is a finite graph in which

$$\forall\ u, v \in V(G) \qquad |N(u) \cap N(v)| = 1. \ (\star)$$

Then there exists a vertex $z \in V(G)$ that is adjacent to all other vertices.

Proof :

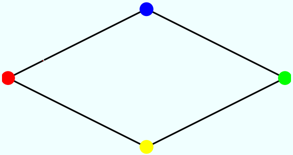
We will proceed by contradiction. Suppose that G is a finite graph satisfying $(\star)$. furthermore suppose that G has no vertex adjacent to all other vertices. Our proof today will be divided into two parts.

Proof-Part one

We claim that G is a regular graph. Here a graph $T = \big(V(T), E(T)\big)$ is called regular whenever

$$\forall\ u, v \in V(T) \qquad |N(u)| = |N(v)|.$$

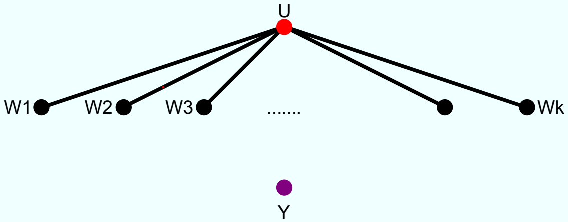
Turning our attention back to proving the Claim. Notice that the condition $(\star)$ ensures that G has no cycles of length 4, indeed consider the 4-cycle.



where the *RED*, and the *GREEN* vertices, have to vertices in common violating $(\star)$.

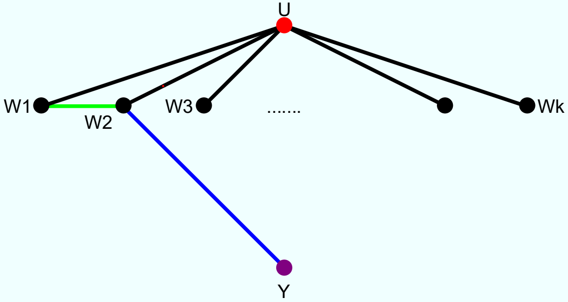
Continuation of Proof-Part one

Now let U and Y be two non-adjacent distinct vertices in G. Consider $|N(U)| = k$ for some $k \in \mathbb{N}$. Consider $N(U) = \{W1, W2, ..., Wk\}$



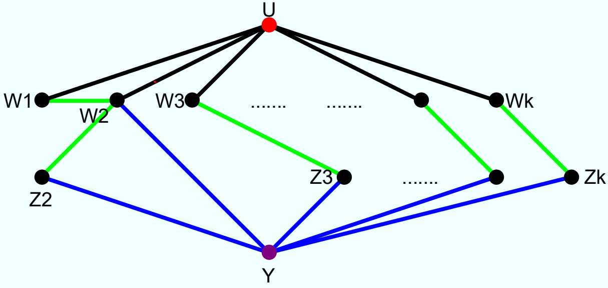
Continuation of Proof-Part one

U and Y must have exactly one vertex in common. Let us say that is $W2$. Similarly $W2$ and U must have exactly one vertex in common. Let us say that is $W1$.



Continuation of Proof-Part one

Turning our attention to Y , it must have exactly one vertex in common with each Wi. Call these common vertices $Zi's$ for $i \in \{2, ..., k\}$.



Continuation of Proof-Part one

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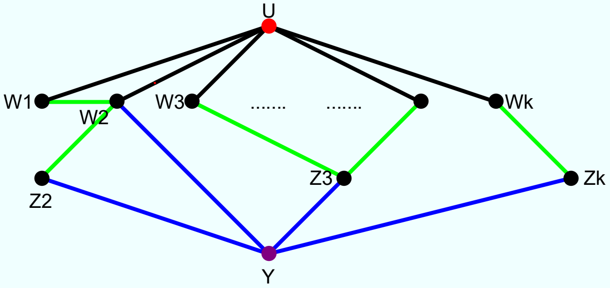
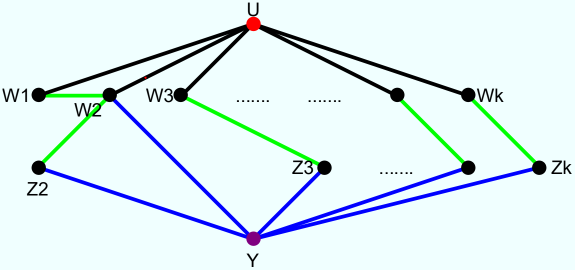


The Zi's that we just added have to be distinct other wise we get a 4-cycle , which we proved can't exist.

Continuation of Proof-Part one

Hence this is what our graph looks like now. Moving on, we count the edges of Y.



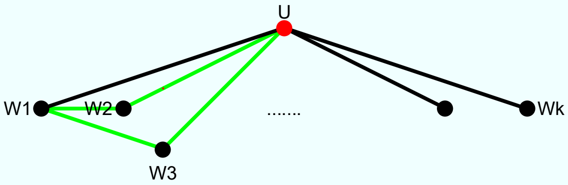
We get that $|N(Y)| \geq |N(U)|$. Finally, by symmetry we conclude that $|N(Y)| = |N(U)|$. To finish the proof of the claim, Notice that aside from $W2$ any vertex is either non-adjacent to Y or non-adjacent to U.

Continuation of Proof-Part one

To finish the first part of the proof we will count the number of vertices we have in total in G.

- Starting from U we have k adjacent vertices $Wi's$
- Each Wi also has k adjacent vertices, so we get k^2.
- But one of the vertices adjacent to each Wi is U. Hence the total is $k^2 - k + 1$. Since we counted U k-times.

Notice that each Wi is adjacent to exactly one other Wj other wise if adjacent to more we run again into a 4-cycle.



Proof-Part two

Moving on, we will present the second part of the proof which is a beautiful algebraic argument to reach contradiction. Start by introducing the adjacency matrix A, which for $G = \big(V(G), E(G)\big)$ is defined as follows:

$$a_{ij} := \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Here, $V(G) = \{v_1, ..., v_n\}$, where $n = k^2 - k + 1$. With this in hand, and using the condition $(\star)$ we have

$$A^2 = \begin{pmatrix} k & 1 & \cdots & 1 \\ 1 & k & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & 1 & k \end{pmatrix} = (k-1)I_n + J_n \quad (\star\star)$$

Here I_n is the identity Matrix, and J_n is a matrix of $1's$.

Continuation of Proof-Part two

Now Using $(\star\star)$, we deduce that the eigenvalues if A^2 are

$$\begin{cases} k-1+n=K^2 & \text{of multiplicity } 1 \\ K-1 & \text{of multiplicity } n-1 \end{cases}$$

So A has the following eigenvalues

$$\begin{cases} k & \text{of multiplicity } 1 \\ \sqrt{K-1} & \text{of multiplicity } r \\ -\sqrt{K-1} & \text{of multiplicity } s \end{cases}$$

where $r+s=n-1$.

Continuation of Proof-Part two

Finally, we use the fact that the trace of A (which is 0)is equal to the sum of its eigenvalues , we have

$$K + r\sqrt{k-1} - s\sqrt{k-1} = 0$$

This implies that $r \neq s$, and we can write

$$\sqrt{k-1} = \frac{k}{s-r}$$

Using the fact that, if the square root of a natural number is rational then it must be an integer, we deduce that $\exists h \in \mathbb{N}$ s.t.

$$h = \sqrt{k-1} \quad \text{which implies} \quad h(s-r) = k = h^2 + 1.$$

Then, h divides $h^2 + 1$, and h^2. This forces $h = 1$ and thus $k = 2 \implies n = 3$, which satisfies the theorem. Hence a contradiction.

Kotzig's Conjecture

Let us first rephrase our theorem, the condition $(\star)$ can be formulated as :

Suppose G is a graph with the property that between any two vertices there is exactly one path of length 2.

This lead to the consideration of the following problem:

Let $\ell \geq 2$. Is there a finite graphs with the property that between any two vertices there is precisely one path of length ℓ.

- Kotzig conjectured in 1974 that there exists no such graphs. He proved this conjecture for $3 \leq \ell \leq 8$.
- Kostochka proved in 1988 that the conjecture is true for $\ell \leq 20$.
- Xing and Hu proved the Kotzig's conjecture in 1994 for $\ell \geq 12$.

Thus, the Kotzig's conjecture is valid now as a theorem.

THANK YOU