On the Partial Fraction of the Cotangent Function, and its Application to Riemann's Zeta Function

Hussein Awala

Temple University, Philadelphia, US

Recall the celebrated Riemann-Zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s \in \mathbb{C},\ \operatorname{Re} s > 1.$$

- introduced and studied by L. Euler in the first half of the eighteenth century when $s \in (1, \infty)$;
- in 1859 B. Riemann extended the Euler definition to complex variables;
- values of the Riemann zeta function at even positive integers were computed by Euler;
- in 1979 Apéry proved the irrationality of $\zeta(3)$ and much more is known today;
- big conjecture (10^6 dollar problems): the non-trivial zeros of ζ all have real part $1/2$.

Our Goal: - go over Euler's argument for computing $\zeta(2s)$ for $s \in \mathbb{N}$.

Tools

The first step (non-trivial!) is to establish that for each $x \in \mathbb{R} \setminus \mathbb{Z}$:

$$\begin{aligned}\pi cot\left(\pi x\right) &= \frac{1}{x} + \sum_{n \in \mathbb{N}} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) \\ &= \lim_{N \to \infty} \left(\sum_{n=-N}^{N} \frac{1}{x+n} \right)\end{aligned}$$

Introducing the functions $f, g : \mathbb{R} \setminus \mathbb{Z} \longrightarrow \mathbb{R}$ given by

$$f\left(x\right) := \pi cot\left(\pi x\right) \quad \text{and} \quad g\left(x\right) := \lim_{N \to \infty} \left(\sum_{n=-N}^{N} \frac{1}{x+n} \right)$$

matters reduce to showing that $f = g$.

Domain of Definition

The functions f and g are both defined and continuous on $\mathbb{R} \setminus \mathbb{Z}$. Indeed:

1. Since $f(x) = \pi cot(\pi x) = \pi \frac{cos(\pi x)}{sin(\pi x)}$, one can easily see that it is defined and continuous on $\mathbb{R} \setminus \mathbb{Z}$.
2. Write
$g(x) = \lim_{N\to\infty} \left(\sum_{n=-N}^{N} \frac{1}{x+n} \right) = \frac{1}{x} + \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{2x}{x^2 - n^2} \right)$.This is because $\frac{1}{x+n} + \frac{1}{x-n} = \frac{2x}{x^2-n^2}$.

   It is enough to prove that the limit is uniformly convergent in a neighborhood of x, $\forall x \in \mathbb{R} \setminus \mathbb{Z}$ - both for the domain of definition and for continuity of g.
   We shall proceed point by point: let $x \in \mathbb{R} \setminus \mathbb{Z} \cup \{0\}$, $x \geq 0$ and let $\epsilon > 0$ be such that $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \setminus \mathbb{Z} \cup \{0\}$.

Domain of Definition - continued

The number of terms corresponding to $n \in \mathbb{N}$ such that $2n - 1 \leq (x+\epsilon)^2$ is finite – call this n_0. Consequently we can bound

$$\sum_{n=1}^{n_0} \frac{2y}{y^2 - n^2} \text{ uniformly for } y \in (x-\epsilon, x+\epsilon).$$

Turning to the terms corresponding to $n \in \mathbb{N}$ such that $2n - 1 > (x+\epsilon)^2$ notice that for each $y \in (x-\epsilon, x+\epsilon)$ there holds

$$y^2 < (x+\epsilon)^2 < 2n - 1,$$

This implies: $n^2 - y^2 > n^2 - 2n + 1 = (n-1)^2$ and so

$$\frac{1}{n^2 - y^2} < \frac{1}{(n-1)^2}$$

And so the following bound on $(x-\epsilon, x+\epsilon)$ holds :

$$\sum_{n=n_0+1}^{\infty} \frac{2y}{n^2 - y^2} < \sum_{n=n_0+1}^{\infty} \frac{2y}{(n-1)^2} < (x+\epsilon) \sum_{n=n_0+1}^{\infty} \frac{1}{(n-1)^2}$$

Domain of Definition - Continued

So back to our initial limit :

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{2y}{y^2-n^2}\right)=\sum_{n=0}^{n_0}\frac{2y}{n^2-y^2}-\sum_{n=n_0+1}^{\infty}\frac{2y}{n^2-y^2}$$

So

$$\sum_{n=1}^{N}\frac{2y}{y^2-n^2}$$

is uniformly convergent on $(x-\epsilon, x+\epsilon)$, making the limit well defined, and continuous on $(x-\epsilon, x+\epsilon)$.
A similar argument will work for $x<0$.

Wish List for f and g

Assume for the moment that f and g satisfy the following properties :

- i f and g are periodic with period 1.
- ii f and g are odd functions.
- iii f and g satisfy the property $(\star)$ for each $x \in \mathbb{R} \setminus \mathbb{Z}$:

$$(\star) \qquad f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) = 2f(x) \ \text{ and } \ g\left(\frac{x}{2}\right) + g\left(\frac{x+1}{2}\right) = 2g(x).$$

Herglotz Trick

Define the function $h : \mathbb{R} \setminus \mathbb{Z} \longrightarrow \mathbb{R}$ given by $h(x) := f(x) - g(x)$ and recall that we want to show that $h \equiv 0$. Then:

- h is **continuous** on $\mathbb{R} \setminus \mathbb{Z}$;
- h is **periodic** of period 1 (by(i));
- h is **odd** (by(ii));
- h satisfies $h\left(\frac{x}{2}\right) + h\left(\frac{x+1}{2}\right) = 2h(x)$ for each $x \in \mathbb{R} \setminus \mathbb{Z}$ (by(iii)).

Moreover by L'Hospital's rule:

$$(\star\star)\ \lim_{x \to 0} \left(\pi \cot(\pi x) - \frac{1}{x}\right) = 0$$

So

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \left(\left(\pi \cot(\pi x) - \frac{1}{x}\right) - \left(\sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2} \right) \right) = 0$$

This is because for the first term we use $(\star\star)$, and the second term you change the limit order since the Sum is uniformly convergent around zero.

Herglotz Trick-continued

By Periodicity of h,

$$\lim_{x \to n} h(x) = 0$$

for all $n \in \mathbb{Z}$. In conclusion if we define h to be zero on $\mathbb{Z}$, h will be continuous on all of $\mathbb{R}$. Now since h is periodic, and continuous on $\mathbb{R}$, we can find x_0 such that $h(x_0) = M$ is the maximum of h on $\mathbb{R}$. Finally using the condition $\star$ we get

$$2h(x_0) = h\left(\frac{x_0}{2}\right) + h\left(\frac{x_0 + 1}{2}\right)$$

This implies that

$$M = h\left(\frac{x_0}{2}\right) = h\left(\frac{x_0 + 1}{2}\right)$$

by iteration, and using continuity of h, we deduce that $h(0) = M$, making $h \equiv 0$.

(i) f and g periodic of period 1

1. $$cos\left(\pi\left(x+1\right)\right) = -cos\left(\pi x\right) \ \& \ sin\left(\pi\left(x+1\right)\right) = -sin\left(\pi x\right)$$

   so

   $$cot\left(\pi\left(x+1\right)\right) = cot\left(\pi x\right)$$

   This implies that f is periodic of period 1.

2. $$g\left(x+1\right) = \sum_{n\in\mathbb{Z}} \frac{1}{x+1+n} = \lim_{n\to\infty}\left(\sum_{i=-n}^{i=n} \frac{1}{x+1+i}\right)$$

   $$= \lim_{n\to\infty}\left(\sum_{i=-n+1}^{i=n+1} \frac{1}{x+i}\right) = \lim_{n\to\infty}\left(\left(\sum_{i=-n}^{i=n} \frac{1}{x+i}\right) - \frac{1}{x-n} + \frac{1}{x+n+1}\right)$$

   $$= \lim_{n\to\infty}\left(\sum_{i=-n}^{i=n} \frac{1}{x+i}\right) = g\left(x\right)$$

   which implies that g is also periodic of period 1.

(ii) f and g Odd Functions

1. f is odd, since $sin(x)$ is odd and $cos(x)$ is even.
2. As for g we let

$$g_N(x) = \sum_{i=-N}^{i=N} \frac{1}{x+n}$$

$$g_N(-x) = \sum_{n=-N}^{n=N} \frac{1}{-x+n} = \sum_{n=-N}^{n=N} \frac{1}{-x-n}$$

$$= \sum_{n=-N}^{n=N} \frac{-1}{x+n} = -\sum_{n=-N}^{n=N} \frac{1}{x+n} = -g_N(x)$$

So g_N is odd, but

$$g = \lim_{N\to\infty} g_N$$

So g is odd as well.

f and g Satisfy the Condition $(\star)$

1. $$f\left(\frac{x}{2}\right)+f\left(\frac{x+1}{2}\right)=\pi\frac{\cos\left(\frac{\pi x}{2}\right)}{\sin\left(\frac{\pi x}{2}\right)}-\frac{\sin\left(\frac{\pi x}{2}\right)}{\cos\left(\frac{\pi x}{2}\right)}$$
$$=\pi\frac{\cos^2\left(\frac{\pi x}{2}\right)-\sin^2\left(\frac{\pi x}{2}\right)}{\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)}=2\pi\frac{\cos(\pi x)}{\sin(\pi x)}=2f(x)$$
So $(\star)$ holds for f

2. $$g_N\left(\frac{x}{2}\right)+g_N\left(\frac{x+1}{2}\right)=\sum_{n=-N}^{n=N}\frac{1}{\frac{x}{2}+n}+\sum_{n=-N}^{n=N}\frac{1}{\frac{x+1}{2}+n}$$
$$=\sum_{n=-N}^{n=N}\frac{2}{x+2n}+\sum_{n=-N}^{n=N}\frac{2}{x+2n+1}=2g_{2N}(x)+\frac{2}{x+2N+1}$$
taking the limit of N at infinity we conclude that $(\star)$ holds for g.

Application on the Riemann's Zeta Function

After proving the partial fraction expansion of the cotangent, Euler few years later used it to find the values of the Riemann's Zeta function at even integers.
so first let us review what we have proved so far :

$$\pi cot\left(\pi y\right) = \frac{1}{y} + \sum_{n=1}^{\infty} \frac{2y}{y^2 - n^2}$$

So for $|y| < \pi$

$$ycot\left(y\right) = 1 - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{2y\frac{y}{\pi}}{n^2 - \left(\frac{y}{\pi}\right)^2} = 1 - 2\sum_{n=1}^{\infty} \frac{y^2}{\pi^2 n^2 - y^2}$$

$$= 1 - 2\sum_{n=1}^{\infty} \frac{\frac{y^2}{\pi^2 n^2}}{1 - \frac{y^2}{\pi^2 n^2}} = 1 - 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\left(\frac{y}{\pi n}\right)^{2k}\right)$$

$$= 1 - 2\sum_{k=1}^{\infty} \frac{1}{\pi^{2k}}\left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}}\right) y^{2k}$$

Power Series Expansion for $ycot\left(y\right)$

So we proved so far that in the power series expansion of $ycot\left(y\right)$ the coefficient of y^{2k} is equal to

$$-2\frac{1}{\pi^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=-2\frac{1}{\pi^{2k}}\zeta\left(2k\right)$$

So all is left is to find the power series expansion of the cotangent and compare.

$$cos\left(y\right)=\frac{e^{iy}+e^{-iy}}{2}, sin\left(y\right)=\frac{e^{iy}-e^{-iy}}{2i}$$

So

$$ycot\left(y\right)=iy\frac{e^{iy}+e^{-iy}}{e^{iy}-e^{-}iy}=iy\frac{e^{2iy}+1}{e^{2iy}-1}$$

Let $x=2iy$ i.e. $y=\frac{x}{2i}$ then

$$ycot\left(y\right)=\frac{x}{2i}cot\left(\frac{x}{2i}\right)=\frac{x}{2}\frac{e^{x}+1}{e^{x}-1}=\frac{x}{2}+\frac{x}{e^{x}-1}$$

Bernoulli Numbers

Define:

$$\frac{x}{2i} cot\left(\frac{x}{2i}\right) - \frac{x}{2} = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

The numbers B_n are called the Bernoulli numbers. Since $ycot\,(y)$ is an even function then : $B_n = 0$ for all $n > 1$ odd, and $B_1 = \frac{-1}{2}$ Notice that

$$(e^x - 1)\left(\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\right) = \left(\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\right)\left(\sum_{n=1}^{\infty} \frac{x^n}{n!}\right) = x$$

And so for $n \neq 1$

$$\sum_{\ell=0}^{\ell=n-1} \frac{B_\ell}{\ell!\,(n-\ell)!} = 0 \qquad \text{and} \qquad B_0 = 1$$

And so we can calculate the Bernoulli numbers recursively.

Values of Riemann's Zeta Function at Even Integers

Recall that for $|y| < \pi$ we proved:

$$y\cot(y) = 1 - 2\sum_{k=1}^{\infty} \frac{1}{\pi^{2k}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \right) y^{2k}$$

and for $x = 2iy$

$$y\cot(y) = \frac{x}{2} + \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = iy + \sum_{k=0}^{\infty} B_k \frac{(2iy)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} y^{2k}$$

Hence:

$$1 - 2\sum_{k=1}^{\infty} \frac{1}{\pi^{2k}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \right) y^{2k} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} y^{2n}$$

And so we have the identity:

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} 2^{2k-1} B_{2k}}{(2k)!} (\pi)^{2k}$$

THANK YOU