On Pólya's Theorem for Polynomials with one Complex Variables

Hussein Awala

Temple University, Philadelphia, US

Definitions

Consider the following polynomial f defined on $\mathbb{C}$ as follows:

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$
$$z \longrightarrow z^n + b_{n-1}z^{n-1} + ... + b_0,$$

such that $n \geq 1$.

Together with the polynomial f we will associate the set $\mathcal{C}_f$ defined as follows:

$$\mathcal{C}_f := \{z \in \mathbb{C} \mid |f(z)| \leq 2\}.$$

Our goal today is to prove the following property that Pólya proved for the set $\mathcal{C}$:

Let f be a polynomial with degree $n \geq 1$, and a leading coefficient 1. Consider any line L in the complex plane. The length of the orthogonal projection of $\mathcal{C}_f$ into L is always less than 4.

L is the Real axis

Theorem (1)

Let f be a polynomial with degree $n \geq 1$, and a leading coefficient 1. Let R_f be the projection of $\mathcal{C}_f$ into the real axis. Then R_f can be covered by intervals, $I_1, ..., I_t$ such that:

$$\ell(I_1) + ... + \ell(I_t) \leq 4.$$

Where $\ell(I)$ is the length of the interval I.

Example

Before going further let us study the following Example. Define the following function :

$$f(z) = z^2 - 2$$

Notice that the projection for any complex number $z = x + iy$ to the x-axis is x. Hence the set R_f described above can be given by the following identity:

$$R_f = \{x \in \mathbb{R} \mid x + iy \in \mathcal{C}_f \text{ for some } y \in \mathbb{R}\}$$

Recall the definition of $\mathcal{C}_f$:

$$\mathcal{C}_f = \{x + iy \in \mathbb{C} \mid |f(x + iy)| \leq 2\}$$

Together with the definition of $f(z)$, we deduce the following:

$$(x^2 + y^2)^2 \leq 4(x^2 - y^2)$$

Example Continuation

From the last inequality that we got :

$$(x^2 + y^2)^2 \leq 4(x^2 - y^2)$$

We can furthermore deduce that :

$$x^4 \leq (x^2 + y^2)^2 \leq 4x^2 \text{ Which implies } x^2 \leq 4$$

All in all, we get:

$$R_f \subset \{x \mid |x| \leq 2\}$$

On the other hand, for any $z = x + iy$ with $|x| < 2$, then

$$|z^2 - 2| \leq 2 \text{ and hence } \{x \mid |x| \leq 2\} = R_f$$

Thus we got the result we were expecting,indeed $\ell(R_f) \leq 4$.

Proof of Theorem (1)

Let us start be recalling what we have:

1. f a polynomial with degree $n \geq 1$, and leading coefficient 1.
2. R_f is the projection of $\mathcal{C}_f$ into the real axis.

We want to prove: $\ell(R_f) \leq 4$. Start by writing:

$$f(z) = (z - c_1)...(z - c_n) \quad \text{where} \quad c_k = a_k + ib_k$$

Consider the following real valued polynomial P

$$P(x) = (x - a_1)...(x - a_n)$$

For any $z = x + iy \in \mathcal{C}_f$:

$$|x - a_k|^2 + |y - b_k|^2 = |z - c_k|^2 \quad \text{Pythagoras theorem}$$

And we deduce for $z \in \mathcal{C}_f$:

$$|P(x)| = |x - a_1|...|x - a_k| \leq |z - c_1|...|z - c_k| = |f(z)| \leq 2$$

Finally : $R_f \subset \{x \in \mathbb{R} \mid |P(x)| \leq 2\}$

Chebyshev's Theorem and Corollary

Theorem (chebyshev's)

Let $P(x)$ be a real polynomial of degree $n \geq 1$ with leading coefficient 1. Then:

$$\max_{-1 \leq x \leq 1} |P(x)| \geq \frac{1}{2^{n-1}}.$$

Corollary

Let P be as in Chebyshev's Theorem, and suppose that $P(x) \leq 2$ for all $x \in [a, b]$. Then

$$(b - a) \leq 4$$

Proof of Corollary

We start by mapping the $[a, b]$ onto $[-1, 1]$ using the substitution:

$$y = \frac{2}{b-a}(x-a) - 1$$

Next, introduce the polynomial Q defined on $\mathbb{R}$ given by:

$$Q(y) = P(\frac{b-a}{2}(y+1) + a)$$

Hence, Q have the following properties:

1. The leading coefficient of Q is $\left(\frac{b-a}{2}\right)^n$.
2. $\max\limits_{-1\leq x\leq 1} |Q(x)| = \max\limits_{-a\leq x\leq b} |P(x)|$.

Finally applying Chebyshev's Theorem to $Q/\left(\frac{b-a}{2}\right)^n$ we get:

$$2 \geq \max_{-a\leq x\leq b} |P(x)| \geq \left(\frac{b-a}{2}\right)^n \cdot \frac{1}{2^{n-1}} = 2\left(\frac{b-a}{4}\right)^n$$

Thus we obtain our result : $b - a \leq 4$.

Theorem 2

Theorem (2)

Let P be a real polynomial of degree $n \geq 1$, and a leading coefficient 1, such that all the roots of P are real. define the set A_p:

$$A_p := \{x \in \mathbb{R} \mid |P(x)| \leq 2\}$$

Then:

1. *A_p is covered by interval $I_1, ..., I_t$ for some t .*
2. $\ell(I_1) + ... + \ell(I_t) \leq 4$

Preliminaries

Lemma (1)

Let P be a non-constant polynomial with all real roots. If b is a multiple root of $P'(x)$, then b is also a root of $P(x)$.

let $b_1, ..., b_r$ be the roots of P, with multiplicities: $s_1, ..., s_r$. $(\sum_{j=1}^{j=r} s_j = n)$

We can write:

$$P(x) = (x - b_j)^{s_j} h_j(x)$$

Notice that b_j is a root of P', with multiplicity $s_j - 1$. Moreover, P' has a root between b_j and b_{j+1} for all $j \in 1, ...r - 1$. Hence we can count :

$$\left(\sum_{j=1}^{j=r} s_j - 1 \right) + (r-1) = (n-r) + (r-1) \text{ roots for } P' \text{ so far.}$$

But P' has only $n - 1$ roots. Thus all multiple roots of P' are roots for P.

Lemma (2)

Let P be a non-constant polynomial with all real roots. We have $P'(x)^2 \geq P(x) \cdot P''(x)$ for all $x \in \mathbb{R}$.

let $a_1, ..., a_n$ be the roots of P counted with multiplicity. Notice that if $x = a_j$ for $1 \leq j \leq n$, then we are done. Hence, assume x is not a root, and write:

$$P'(x) = \sum_{k=1}^{n} \frac{P(x)}{x - a_k}, \quad \text{which can be written as} \quad \frac{P'(x)}{P(x)} = \sum_{k=1}^{n} \frac{1}{x - a_k}$$

Differentiating both sides of the last equality we get:

$$\frac{P''(x) \cdot P(x) - P'(x)^2}{P(x)^2} = -\sum_{k=1}^{n} \frac{1}{(x - a_k)^2} < 0.$$

Thus: $P''(x) \cdot P(x) - P'(x)^2 < 0$, as desired.

Proof of Theorem (2)

Consider $p(x) = (x - a_1) \cdot ... \cdot (x - a_n)$, recall the set $A_p = \{x \in \mathbb{R} \mid |P(x)| \leq 2\}$. First, since p is a real polynomial, then A_p must be the union of intervals, call them $I_1, ..., I_t$. Rearrange the intervals such that I_1 is the leftmost and I_t is the rightmost interval.

Here are some properties of those intervals:

1. $P(x)$ assumes the values 2 or -2 on the endpoints of $I_J{}'s$.
2. By using Lemma $(1) - (2)$ one can prove that each interval I_j has at least one root.
3. All the roots are contained in the the intervals I_j for $j \in \{1, ..., t\}$.
4. $\ell(I_j) \leq 4$. (by the corollary to Chebyshev's Theorem)

The idea now find a polynomial T such that A_T is contained in one Interval and $A_P \subset A_T$

Proof of Theorem (2) continued

Consider that I_t contains the roots $\{b_1, ..., b_m\}$. If $m = n$, we are done.

Otherwise, if $m < n$, let the distance between I_t and I_{t-1} be d. and let $\{c_1, ..., c_{n-m}\}$, be all the other roots of P. Define the following polynomials:

$$Q(x) = (x - b_1) \cdot ... \cdot (x - b_m) \quad \text{and} \quad R(x) = (x - c_1) \cdot ... \cdot (x - c_{n-m})$$

And consider the polynomial:

$$P_1(x) = Q(x + d) \cdot R(x)$$

Then:

1. P_1 is a polynomial of degree n.
2. $|P_1(x)| \leq |P(x)| \leq 2$ for all $x \in I_1 \cup ... \cup I_{t-1}$.
3. $|P_1(x - d)| \leq |P(x)| \leq 2$ for all $x \in I_t$.

Proof of Theorem (2) continued

Property (2) above show that:

$$I_t \subset A_{P_1} \text{ for } j \in \{1, ..., t-1\}.$$

Moreover, property (3) shows:

$$I_t - d \subset A_{P_1}.$$

All in all,

$$\Big(I_1 \cup ... \cup I_{t-1} \cup \big(I_t - d\big)\Big) \subset A_{P_1}.$$

And hence we were able to merge the last two interval I_t and I_{t-1}

Repeating this process $t-1$ times should yield the polynomial T we desire, thus proving the theorem.

THANK YOU