Research Statement

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I am fifth year graduate student in the Department of Mathematics at Temple University, working at the interface between Harmonic Analysis, Partial Differential Equations, and Geometric Measure Theory, under the supervision of Professor Irina Mitrea. I have successfully completed my Qualifying and Oral Ph.D. Examinations and I am currently working on my Ph.D. thesis dealing with the study of *Singular Integral Operators associated with Elliptic Partial Differential Equations and Systems in Non-Smooth Domains.*

Many boundary value problems of mathematical physics are modelled by elliptic differential operators $\mathcal{L}$ in a given domain Ω. An effective method for treating such problems is the method of layer potentials, whose essence resides in reducing matters to solving a boundary integral equation. This, in turn, requires inverting a singular integral operator, naturally associated with $\mathcal{L}$ and Ω, on appropriate function spaces on $\partial\Omega$. When the operator $\mathcal{L}$ is of second order and the domain Ω is Lipschitz (i.e., Ω is locally the upper-graph of a Lipschitz function) the fundamental work of B. Dahlberg, C. Kenig, D. Jerison, E. Fabes, N. Rivière, G. Verchota, R. Brown, and many others, has opened the door for the development of a far-reaching theory in this setting, even though a number of very difficult questions still remain unanswered. In this regard the Spectral Radius Conjecture (that is, the question of whether the spectral radius of the boundary classical harmonic double layer potential operator is $< 1/2$ when acting on the space $L^2(\partial\Omega)$, of square integrable functions on $\partial\Omega$) has not been yet fully understood even for the case of the Laplacian in two dimensions.

My goal is to study spectral properties of singular integral operators associated with a variety of boundary value problems (with data of Dirichlet type, Neumann type, mixed type, Robin type, etc.) in uniformly rectifiable domains in the euclidean space. The motivation behind considering more complicated geometries than the Lipschitz ones resides in the fact that, even surfaces which appear smooth to the naked eye, exhibit in fact sophisticated (from the geometrical stand point) asperities and irregularities, thus the need of developing tools for a satisfactory mathematical theory in this setting. While boundedness properties on the scale of p-th power Lebesgue integrable functions of singular integral operators of convolution and non-convolution type in uniformly rectifiable domains has been developed in the work of G. David, S, Semmes, S. Hofmann, M. Mitrea and M. Taylor, spectral properties of these type of operators are still to be understood.

With an eye towards tackling more complicated geometric settings, I have recently studied spectral properties of singular integral operators associated with the mixed problems for the Laplacian and the Lamé system of elastostatics in polygonal domains in two dimensions. The mixed problems (which involve imposing Dirichlet and Neumann type boundary conditions on complementing pieces of $\partial\Omega$) have received considerable attention lately in the work of R. Brown, K. Ott, J. Taylor, I. Mitrea, M. Mitrea, M. Wright, G. Verchota, among others, and they arise naturally in modelling conductivity, heat transfer, wave phenomena, stamp problems in elasticity, etc. A concrete example is the case of a partially submerged iceberg Ω. In this case, the unknown function is u, where for each $x \in \Omega$, the value $u(x)$ denotes the temperature of the iceberg at x. The boundary of the iceberg submerged under the water, call it D, acts as a thermostat and on it one imposes a Dirichlet boundary condition. The part of $\partial\Omega$ above the water acts as a perfect insulator, hence one imposes a Neumann boundary condition on $N := \partial\Omega \setminus \overline{D}$. Thus, the mixed problem for the Laplacian in this setting takes the form:

$$\triangle u = 0 \ \text{ in } \ \Omega \qquad u\Big|_{D}^{n.t.} = f \in L^p_1(D) \ \text{ on } \ D \qquad \frac{\partial u}{\partial \nu}\Big|_{N}^{n.t.} = g \in L^p(N) \ \text{ on } \ N, \tag{1}$$

where $f \in L^p_1(D)$ and $g \in L^p(N)$ are given, ν stands for the outward unit normal vector to Ω, the notation $v\Big|_{E}^{n.t.}$ generically denotes the non-tangential restriction of the function v, originally defined in Ω, to $E \subseteq \partial\Omega$, and $p \in (1, \infty)$. A singular integral operator naturally associated with the boundary value problem (1) is $T : L^p(D) \oplus L^p(N) \longrightarrow L^p(D) \oplus L^p(N)$, with $T := \begin{pmatrix} \partial_\tau S & \partial_\tau S \\ K^* & -\frac{1}{2}I + K^* \end{pmatrix}$, where

S is the boundary harmonic single layer, K^* is the adjoint of the boundary harmonic double layer, and ∂_τ denotes differentiation in the tangential direction on $\partial\Omega$. Using Mellin transform techniques, I was able to identify the critical indexes p (which depend on the angles of the polygon Ω) for which the invertibility of T fails.

The situation for the mixed problem for the Lamé system is considerably more complicated, even in the case of polygonal domains in two dimensions. In this case, the critical indexes p also depend on the Lamé moduli and the nature of the conormal derivative appearing in K^*, and the nature of this dependence is quite intricate. However extensive numerical experimentations allowed me to formulate and partially prove a number of conjectures about this case.