On the Solvability of the Zaremba Problem in Infinite Sectors and the Invertibility of Associated Singular Integral Operators

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1 Introduction

Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and let $\Delta = \sum_{j=1}^{n} \partial_j^2$ stand for the Laplace operator in $\mathbb{R}^n$. The Zaremba problem, or the mixed problem for the Laplacian in Ω with L^p data, $1 < p < \infty$, has the form

$$(MBVP_p) \begin{cases} u \in \mathscr{C}^\infty(\Omega)\,, \\ \Delta u = 0 \quad \text{in} \quad \Omega\,, \\ u\Big|_D^{n.t.} = f_D \in \dot{L}_1^p(D) \quad \text{on } D\,, \\ \dfrac{\partial u}{\partial \nu}\Big|_N = f_N \in L^p(N) \quad \text{on } N\,, \\ \mathscr{N}(\nabla u) \in L^p(\partial\Omega)\,, \end{cases} \tag{1}$$

where D and N are disjoint open subsets of $\partial\Omega$ with the property that they share a common boundary $\partial D = \partial N$, and $\partial\Omega = \overline{D} \cup \overline{N}$. Above ν denotes the outward unit normal vector to Ω, which exists σ-a.e. on $\partial\Omega$, where $\sigma := \mathscr{H}^{n-1}\lfloor\partial\Omega$. Here $\mathscr{H}^{n-1}$ stands for the $n-1$-dimensional Hausdorff measure in $\mathbb{R}^n$. In (1), the non-tangential

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trace to $\partial\Omega$ denoted by $\cdot\Big|_{\partial\Omega}^{n.t.}$ is introduced in (42), $\mathcal{N}$ stands for the non-tangential maximal operator defined in (41) and $\dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$ stands for the normal derivative of u as in (43). In addition $L^p(\partial\Omega)$ and $L^p(N)$ are the Lebesgue spaces of p-th power integrable functions on $\partial\Omega$ and respectively N, with respect to σ, while $\dot{L}_1^p(D)$ is the homogeneous Sobolev-Lebesgue space of order 1 on D.

Boundary value problems with mixed Dirichlet and Neumann type conditions arise naturally in connection with physical phenomena such as conductivity, heat transfer, elastic deformations, electrostatics, etc., and there is a vast mathematical and engineering literature dealing with this topic, see e.g., [1], [12], [14], [19], [20], [21], [22], [23], [26], [32], [33], [34], [35], [36], [37], [39], [41]. In the setting of Lipschitz domains, in response to the question posed by C. Kenig on pp. 120 of [16] calling for characterizing the smoothness of the gradient of solutions of the Zaremba problem, R. Brown proved in [3] the well-posedness of the problem $(MBVP_2)$ and subsequently, in [40], R. Brown and J. Sykes establish

$$\begin{array}{l}\text{the well-posedness of } (MBVP_p) \text{ for } p\in(1,2] \\ \text{in the class of } \textit{creased} \text{ Lipschitz domains ,}\end{array} \tag{2}$$

where the creased condition roughly speaking indicates that D and N meet at an angle which is strictly less than π. Perturbation arguments then allow one to establish that

$$\begin{array}{c}\text{for each bounded creased Lipschitz domain the problem } (MBVP_p) \\ \text{is well-posed for } p\in(1,2+\varepsilon_\Omega)\text{ ,}\end{array} \tag{3}$$

where $\varepsilon_\Omega > 0$ depends on the Lipschitz character of the domain Ω, D, N and p. For additional work on the mixed problem in the Lipschitz or rougher settings we refer the interested reader to [4], [5], [6], [7], [8], [9], [24], [26], [28], [30], and the references therein.

One of the main goals of this paper is to establish sharp invertibility properties for a singular integral operator naturally associated with $(MBVP_p)$ when the domain Ω is an infinite sector in two dimensions and when a Dirichlet boundary condition is imposed on one ray of the sector, and a Neumann boundary condition is imposed on the other ray. Specifically, looking for a solution of $(MBVP_p)$, $p\in(1,\infty)$, expressed as a harmonic single layer potential operator with an L^p density leads to the issue of inverting the operator (see (59))

$$T := \begin{pmatrix} \partial_\tau S & \partial_\tau S \\ K^* & -\frac{1}{2}I + K^* \end{pmatrix} : L^p(D)\oplus L^p(N) \longrightarrow L^p(D)\oplus L^p(N)\ , \tag{4}$$

where S is the boundary-to-boundary harmonic single layer potential operator defined in (46), ∂_τ denotes differentiation in the tangential direction, K^* is the formal adjoint of the boundary-to-boundary harmonic double layer potential operator from (48), and I is the identity operator.

In the geometric context just described, we identify the set of critical integrability exponents $p \in (1,\infty)$ for which T fails to be invertible, and we establish an explicit characterization of its L^p spectrum for each $p \in (1,\infty)$. If the sector Ω has (full) aperture $\theta \in (0,2\pi)$, in Theorem 3 we establish that the singular integral operator T is invertible on the space $L^p(D) \oplus L^p(N)$ whenever $p \in (1,\infty)$ is such that

$$p \neq \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \text{if } \theta \in (0,\pi/2] \\ \frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi/2,\pi) \\ 2 & \text{if } \theta = \pi \\ \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi,3\pi/2] \\ \frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi} & \text{if } \theta \in (3\pi/2,2\pi)\,, \end{cases} \tag{5}$$

a result which has fundamental consequences for the solvability of $(MBVP_p)$ in the class of curvilinear polygons in $\mathbb{R}^2$.

The crux of the matter is that when Ω is the interior of an infinite angle in $\mathbb{R}^2$ the operator T is of Mellin convolution type. This enables the employment of the Mellin transform to identify the critical integrability exponents from the right-hand side of (5) as the zeroes of the determinant of the Mellin transform of the matrix kernel for the operator T. One remarkable byproduct of our main invertibility result is an example of a natural operator (i.e. T) that is linear and bounded on the entire Lebesgue scale L^p for all $p \in (1,\infty)$, which happens to be invertible for all but finitely many values of the parameter p without actually being invertible for every p. This phenomenon highlights the pathology that, as opposed to interpolation of boundedness, the property of being invertible does not interpolate. The failure of invertibility to interpolate is rooted in the fact that, as opposed to the operator T, its inverse T^{-1} does not necessarily act in a compatible fashion when considered on two different L^p spaces. This being said, it is known (from general principles of functional analytic nature) that T^{-1} exists and acts in a coherent fashion locally, i.e. on L^p spaces for all p's sufficiently close to some value p_o for which T is known to be invertible on the space L^{p_o}.

Building on the sharp invertibility results established for the operator T from Theorem 3 we are able to establish the well-posedness of $(MBVP_p)$ when Ω is the interior of an infinite angle in $\mathbb{R}^2$ of aperture $\theta \in (0,2\pi)$ whenever

$$p \neq \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \text{if } \theta \in (0,\pi/2] \\ \frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi/2,\pi) \\ 2 & \text{if } \theta = \pi \\ \frac{\theta}{\theta-\pi}, \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi,3\pi/2] \\ \frac{\theta}{\theta-\pi}, \frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi} & \text{if } \theta \in (3\pi/2,2\pi)\,, \end{cases} \tag{6}$$

see Theorem 5. While the existence of a solution follows immediately from Theorem 3, the uniqueness relies on a new integral representation formula for the gradient of harmonic functions u in a graph Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ satisfying $\mathcal{N}(\nabla u) \in L^p(\partial\Omega)$ for some $p \in (1,\infty)$, to the effect that

$$\partial_1 u = \partial_2 \mathscr{S}(\partial_\tau u) - \partial_1 \mathscr{S}\Big(\frac{\partial u}{\partial \nu}\Big) \ \text{ in } \ \Omega\,, \tag{7}$$

$$\partial_2 u = -\partial_1 \mathscr{S}(\partial_\tau u) - \partial_2 \mathscr{S}\Big(\frac{\partial u}{\partial \nu}\Big) \ \text{ in } \ \Omega\,. \tag{8}$$

Here $\mathscr{S}$ stands for the boundary-to-domain single layer potential operator introduced in (45). Formulas (7)-(8) are, in turn, proved using a sharp divergence theorem for functions with non-tangential traces established in [27].

The techniques employed in proving our main results allow, in particular, for a more nuanced analysis of the nature of ε_Ω in (3) in the case of creased curvilinear polygons in $\mathbb{R}^2$ (which is not achievable through the methods employed by Brown and Sykes). The work undertaken here clarifies how the geometry of the sector Ω affects the range of integrability exponents p for which $(MBVP_p)$ is well-posed. In order to emphasize some of the properties of the critical indices identified in the right-hand side of (5), for each $\theta \in (0,2\pi)$ introduce $p_{critic}(\theta)$ as

$$p_{critic}(\theta) := \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \text{if } \ \theta \in (0,\pi/2] \\ \min\big\{\frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi}\big\} & \text{if } \ \theta \in (\pi/2,\pi) \\ 2 & \text{if } \ \theta = \pi \\ \min\big\{\frac{2\theta}{2\theta-\pi}, \frac{\theta}{\theta-\pi}\big\} & \text{if } \ \theta \in (\pi,3\pi/2] \\ \min\{\frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi}, \frac{\theta}{\theta-\pi}\} & \text{if } \ \theta \in (3\pi/2,2\pi)\,. \end{cases} \tag{9}$$

Note that

$$\lim_{\theta\to 0^+} p_{critic}(\theta) = 2 \ \text{ and } \ \lim_{\theta\to\pi^-} p_{critic}(\theta) = 2\,, \tag{10}$$

and

$$p_{critic}(\theta) > 2 \ \text{ whenever } \ \theta \in (0,\pi)\,. \tag{11}$$

In the setting when the sector Ω is a creased Lipschitz domain (i.e. for $\theta \in (0,\pi)$) our results are, of course, in line with what the theory for generic creased Lipschitz domains predicts. In this scenario, the novelty and relevance of Theorem 3 stems from the explicit nature of the dependence of the critical exponent $p_{critic}(\theta)$ on Ω (via the aperture θ). For example, whenever $\theta \in (0,\pi)$ (which is precisely the range of θ's for which the Lipschitz domain $\Omega = \Omega_\theta$ is creased), we are able to identify the parameter ε_Ω appearing in (3) concretely as $\varepsilon_\Omega = p_{critic}(\theta) - 2 > 0$. In particular, in light of (10), this makes it clear that $\varepsilon_\Omega \to 0^+$ as $\theta \to 0^+$ or $\theta \to \pi^-$ (corresponding to the limiting cases when being Lipschitz or being creased is lost).

Compared to the earlier work of R. Brown and J. Sykes here we go beyond the class of creased Lipschitz domains by allowing sectors of aperture $\theta \in [\pi, 2\pi)$ in which case we continue to have solvability results for $(MBVP_p)$ of the sort described in Theorem 5. The new phenomenon that occurs in the latter case is a more restrictive range of p's than indicated in (2) since

$$\frac{4}{3} < p_{critic}(\theta) < 2 \text{ whenever } \theta \in (\pi, 2\pi) . \tag{12}$$

In fact, since also

$$\lim_{\theta \to \pi^+} p_{critic}(\theta) = 2 \text{ and } \lim_{\theta \to 2\pi^-} p_{critic}(\theta) = \frac{4}{3} , \tag{13}$$

our results help clarify the nature of the range of p's for which $(MBVP_p)$ is solvable in Lipschitz domains which are not necessarily creased. Indeed, this portion of our work should be compared with the results of R. Brown, L. Capogna, and L. Lanzani who have shown in [24] that in this scenario $(MBVP_p)$ is solvable for some $p > 1$.

While in the present work we exclusively focus on the basic case of an infinite sector, localization techniques (see for example [29, Lemma 1]) may in principle be employed to produce solvability results for mixed problems in the class of curvilinear polygons. Concretely, starting with (5), it is expected that if Ω is a curvilinear polygon in $\mathbb{R}^2$ with angles $\theta_1, \theta_2, \dots, \theta_{2N} \in (0, \pi)$, for some $N \in \mathbb{N}$, then the mixed boundary value problem $(MBVP_p)$ in which one imposes alternating Dirichlet and Neumann boundary conditions on the sides of Ω is well-posed for

$$p \in \left(1, \min_{i \in \{1, \dots, 2N\}} p_{critic}(\theta_i)\right) . \tag{14}$$

For a systematic treatment of mixed boundary value problems in polygonal domains in which the size and regularity of the solution is expressed in terms of membership to Sobolev spaces the reader is referred to [18, § 4].

The layout of the paper is as follows. Section 2 contains notation and known results that are useful for the present goals. In particular, here we record a key spectral result, see Theorem 1, for elements in the algebra generated by Hardy kernel operators on $L^p(\mathbb{R}_+)$ and the truncated Hilbert transform, for $1 < p < \infty$. In Section 3 we compute the Mellin symbol of a singular integral operator naturally associated with the mixed problem in an infinite sector in two dimensions, and establish invertibility properties for this operator on the scale of Lebesgue spaces $L^p(D) \oplus L^p(N)$, with $1 < p < \infty$, where D and N are the left and right rays of the sector and, respectively, the Dirichlet and the Neumann pieces of the boundary. Our main invertibility result, Theorem 3, identifies the critical indexes p for which the invertibility fails. Section 3 also contains further results on the spectra of the aforementioned singular integral operator, including an explicit characterization of the spectra as parametric curves in the plane, cf. Theorem 4. Finally, the main well-posedness result is stated and proved in Section 4.

2 Preliminaries

This section contains notation and definitions used throughout the paper along with some useful results from the literature that pertain to our discussion. To get started, recall that if X is a Banach space and $T : X \to X$ is a linear and continuous operator, the spectrum of T acting on X is given by

$$\sigma(T;X) := \{w \in \mathbb{C} : wI - T \text{ is not invertible on } X\}\,, \tag{15}$$

where, throughout the paper, I stands for the identity operator.

Next we introduce the Hardy kernels for $L^p(\mathbb{R}_+)$, where $\mathbb{R}_+$ stands for the set of non-negative real numbers and, for each $p \in (0,\infty)$, the space $L^p(\mathbb{R}_+)$ denotes the space of p-th power integrable functions on $\mathbb{R}_+$.

Definition 1. Let $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a Lebesgue measurable function. Then k is a Hardy kernel for $L^p(\mathbb{R}_+)$ for $p \in [1,\infty)$ provided

(1) k is a positive homogeneous function of degree -1, i.e., for any $\lambda > 0$ and any $x, y \in \mathbb{R}_+$, there holds $k(\lambda x, \lambda y) = \lambda^{-1} k(x,y)$;

(2) $\displaystyle\int_0^\infty |k(1,y)| y^{-1/p} dy < \infty$.

Furthermore, a matrix-valued function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{\ell \times m}$, $k = (k_{ij})_{1 \le i \le \ell, 1 \le j \le m}$ for $\ell, m \in \mathbb{N}$, is called a Hardy kernel for $L^p(\mathbb{R}_+)$ provided each entry k_{ij}, with $i \in \{1, \dots, \ell\}$ and $j \in \{1, \dots, m\}$, is a Hardy kernel for $L^p(\mathbb{R}_+)$.

The collection of Hardy kernels for $L^p(\mathbb{R}_+)$ shall be denoted in the sequel by HK_p.

It is worth pointing out that, if $k \in HK_p$ for $p \in (1,\infty)$, the homogeneity of k permits us to write

$$\int_0^\infty |k(1,y)| y^{-1/p} dy = \int_0^\infty |k(x,1)| x^{1/p-1} dx = \int_0^\infty |k^T(1,x)| x^{-1/p'} dx\,, \tag{16}$$

where $k^T : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is the transpose of k, defined as $k^T(x,y) := k(y,x)$ for each $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$, and $\frac{1}{p} + \frac{1}{p'} = 1$. Since clearly k^T is also positive and homogeneous of degree -1 and $(k^T)^T = k$, in concert with (16) this shows that

$$k \in HK_p \iff k^T \in HK_{p'}, \text{ where } p, p' \in (1,\infty) \text{ are such that } \tfrac{1}{p} + \tfrac{1}{p'} = 1. \tag{17}$$

Moving on, with each scalar-valued Hardy kernel $k \in HK_p$, $p \in (1,\infty)$, associate an integral operator $\mathscr{T}$ acting on functions $f \in L^p(\mathbb{R}_+)$ according to

$$\mathscr{T} f(x) := \int_0^\infty k(x,y) f(y) dy, \quad x \in \mathbb{R}_+\,. \tag{18}$$

The operator $\mathscr{T}$ is called a Hardy kernel operator with kernel k. The setup of the vector-valued case follows a similar blueprint. Concretely, fix two integers $\ell, m \in \mathbb{N}$

and let $k = (k_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}$ be such that $k \in HK_p$ for some $p \in (1,\infty)$. Associate with k the operator $\mathscr{T}$ acting on functions $f \in \big(L^p(\mathbb{R}_+)\big)^m$ according to

$$\mathscr{T} f(x) := \int_0^\infty k(x,y)\cdot f(y)\,dy, \quad x \in \mathbb{R}_+\,, \tag{19}$$

where $\cdot$ denotes matrix multiplication.

The Mellin transform of a measurable function f on $\mathbb{R}_+$ is defined as

$$\mathscr{M} f(z) := \int_0^\infty x^{z-1} f(x)\,dx\,, \tag{20}$$

for those $z \in \mathbb{C}$ for which the integral is absolutely convergent. If this is the case for all z's in some strip

$$\Gamma_{\alpha,\beta} := \{z \in \mathbb{C} : \alpha < \mathfrak{Re}\, z < \beta\}\,, \tag{21}$$

with $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha < \beta$, we shall refer to $\Gamma_{\alpha,\beta}$ as a strip of holomorphy for f. It is straightforward to see that if f is a measurable function on $\mathbb{R}_+$ then for each z in a strip of holomorphy for f we have

$$\mathscr{M} g(z-1) = \mathscr{M} f(z), \quad \text{where } g(t) := t f(t)\,. \tag{22}$$

Whenever $\ell, m \in \mathbb{N}$ and $k = (k_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}$ is an $\ell \times m$ matrix-valued measurable function on $\mathbb{R}_+$ and there exists some some common strip of holomorphy for all individual entries k_{ij}, we set

$$\mathscr{M} k(z) := (\mathscr{M} k_{ij}(z))_{1\leq i\leq \ell, 1\leq j\leq m} \text{ for each } z \in \Gamma_{\alpha,\beta}\,. \tag{23}$$

The following result found in [2], [13] and [25] allows one to explicitly determine the spectrum of an entire class of bounded linear operators acting from $(L^p(\mathbb{R}_+))^m$ into $(L^p(\mathbb{R}_+))^m$ as described below.

Theorem 1. *Let $h = (h_{ij})_{1\leq i,j\leq m}$ be a Hardy kernel for $L^p(\mathbb{R}_+)$, for some $p \in (1,\infty)$ and $m \in \mathbb{N}$, and let A, B be $m \times m$ matrices with real entries. Consider the operator $\mathscr{R}$ defined as*

$$\mathscr{R} f(s) := Af + \int_0^\infty \mathfrak{K}(s,t) f(t)\,dt, \quad s \in \mathbb{R}_+\,, \tag{24}$$

with

$$\mathfrak{K}(s,t) := h(s,t) + \frac{1}{s-t}\cdot B, \qquad \forall s,t \in \mathbb{R}_+ \text{ such that } s \neq t\,. \tag{25}$$

Then the operator

$$\mathscr{R} : (L^p(\mathbb{R}_+))^m \longrightarrow (L^p(\mathbb{R}_+))^m \text{ is well-defined, linear and bounded} \tag{26}$$

with spectrum

$$\sigma(\mathscr{R}; (L^p(\mathbb{R}_+))^m) = \overline{S}\,, \tag{27}$$

the closure of

$$S := \Big\{ w \in \mathbb{C} : \det \Big(wI - (A + \mathscr{M}\mathfrak{K}(\cdot,1))(\tfrac{1}{p} + i\xi) \Big) = 0, \ \textit{for some} \ \xi \in \mathbb{R} \Big\} . \quad (28)$$

A useful consequence of of Theorem 1 is singled out below.

Corollary 1. *Retain the setting of Theorem 1 and make the additional assumption that*

$$\det(A - \pi i \cdot B) \neq 0 . \quad (29)$$

Then the operator $\mathscr{R}$ *is invertible on* $(L^p(\mathbb{R}_+))^m$ *if and only if*

$$\det \Big(A + \mathscr{M}\mathfrak{K}(\cdot,1)(\tfrac{1}{p} + i\xi) \Big) \neq 0, \quad \forall \xi \in \mathbb{R} . \quad (30)$$

Proof. In one direction, assume that the operator $\mathscr{R}$ is invertible on $(L^p(\mathbb{R}_+))^m$. Thus $0 \notin \sigma(\mathscr{R}; (L^p(\mathbb{R}_+))^m)$. Using the characterization of the spectrum of $\mathscr{R}$ from (27)-(28) in Theorem 1 this is further equivalent with $0 \notin \overline{S}$. Consequently $0 \notin S$, and hence (30) holds.

In the opposite direction, assume that (30) is valid. Seeking a contradiction, suppose $\mathscr{R}$ is not invertible on $(L^p(\mathbb{R}_+))^m$, or equivalently, $0 \in \sigma(\mathscr{R}; (L^p(\mathbb{R}_+))^m)$. Appealing again to Theorem 1 this yields $0 \in \overline{S}$ and thus there exist $\{w_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$ and $\{\xi_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}$ such that

$$\lim_{j\to\infty} w_j = 0 , \quad (31)$$

and

$$\det \Big(w_j I - (A + \mathscr{M}\mathfrak{K}(\cdot,1))(\tfrac{1}{p} + i\xi_j) \Big) = 0 \ \text{ for each } \ j \in \mathbb{N} . \quad (32)$$

In the case when $\{\xi_j\}_{j\in\mathbb{N}}$ has a bounded subsequence, by the Bolzano-Weierstrass theorem, there is no loss of generality in assuming that there exists $\xi_* \in \mathbb{R}$ such that $\lim_{j\to\infty} \xi_j = \xi_*$. Note that

$$\mathbb{R} \ni \xi \mapsto (A + \mathscr{M}\mathfrak{K}(\cdot,1))(\tfrac{1}{p} + i\xi) \ \text{ is continuous} , \quad (33)$$

and that the determinant function is continuous. Thus, by passing to the limit in (32) and using (31), we obtain that in this scenario

$$\det \Big(A + \mathscr{M}\mathfrak{K}(\cdot,1)(\tfrac{1}{p} + i\xi_*) \Big) = 0 , \quad (34)$$

contradicting (30).

The remaining case is the scenario when $\{\xi_j\}_{j\in\mathbb{N}}$ has a subsequence $\{\xi_{j_k}\}_{k\in\mathbb{N}}$ convergent to either $+\infty$ or $-\infty$ as $k \to \infty$. We start by introducing the space

$$L^1_*(\mathbb{R}_+) := \Big\{ f : \mathbb{R}_+ \longrightarrow \mathbb{C} : f \ \text{ is measurable and } \int_{\mathbb{R}_+} |f(x)| \frac{dx}{x} < \infty \Big\} . \quad (35)$$

Note that the property of h being a Hardy kernel ensures that the function h_p defined by $h_p(x) := x^{1/p} h(x)$ for each $x \in \mathbb{R}_+$ belongs to $L^1_*(\mathbb{R}_+)$. Since the Fourier transform on the Haar group is in fact the Mellin transform, the latter condition along with a version of the Riemann-Lebesgue lemma in this context guarantee that

$$\lim_{\xi\to\pm\infty} \mathscr{M}h(\cdot,1)(1/p+i\xi)=0\,, \tag{36}$$

where 0 here stands for the $m\times m$ zero matrix. In addition, an explicit elementary calculation based on Lemma 3 (proved independently later) shows that

$$\lim_{\xi\to\pm\infty} \mathscr{M}\left(\frac{1}{\cdot-1}\right)(\tfrac{1}{p}+i\xi)=-\pi i\,. \tag{37}$$

Consequently, using (36) and (37), we obtain

$$\lim_{\xi\to\pm\infty} \Big(A+\mathscr{M}\mathfrak{K}(\cdot,1)(\tfrac{1}{p}+i\xi)\Big)=A-\pi i\cdot B\,, \tag{38}$$

and thus, passing to the limit in (32) for the subsequence $\{j_k\}_{k\in\mathbb{N}}$ produces

$$\det(A-\pi i\cdot B)=0, \tag{39}$$

contradicting (29). This completes the proof of the corollary.

Throughout the paper, given a graph Lipschitz domain $\Omega\subseteq\mathbb{R}^2$, i.e., a domain Ω which is the upper-graph of a Lipschitz function $\phi:\mathbb{R}\longrightarrow\mathbb{R}$, we shall introduce the surface measure $\sigma:=\mathscr{H}^1\lfloor\partial\Omega$, where $\mathscr{H}^1$ stands for the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. Also $\nu=(\nu_1,\nu_2)$ will denote the outward unit normal vector to $\partial\Omega$ which, due to Radamacher's theorem, exists almost everywhere with respect to σ. For each $p>0$ we let $L^p(\partial\Omega)$ stand for the Lebesgue scale of p-th power integrable functions on $\partial\Omega$ with respect to σ. If $\mathrm{dist}(\cdot,\partial\Omega)$ is the distance function to $\partial\Omega$ and $a>1$ is fixed, for each $X\in\partial\Omega$ the non-tangential approach region with vertex at X is introduced as

$$\Gamma_a(X):=\Big\{Y\in\Omega:\,|X-Y|<a\cdot\mathrm{dist}(Y,\partial\Omega)\Big\}\,. \tag{40}$$

For a fixed $a>1$ and each function $w:\Omega\to\mathbb{R}$, the non-tangential maximal function of w, denoted by $\mathscr{N}w$, is set to be

$$\mathscr{N}w(X):=\sup_{Y\in\Gamma_a(X)}|w(Y)|,\qquad \text{for each }\ X\in\partial\Omega\,, \tag{41}$$

and the non-tangential limit of w at $X\in\partial\Omega$, denoted by $w\Big|_{\partial\Omega}^{n.t.}(X)$, is defined as

$$w\Big|_{\partial\Omega}^{n.t.}(X):=\lim_{\substack{Y\to X\\ Y\in\Gamma_a(X)}} w(Y)\,, \tag{42}$$

whenever the limit in the right-hand side of (42) exists. In addition, if w is differentiable in Ω and if $\langle\cdot,\cdot\rangle$ is the canonical inner product in $\mathbb{R}^2$, we set

$$\frac{\partial w}{\partial \nu} := \left\langle (\nabla w)\Big|_{\partial\Omega}^{n.t.}, \nu \right\rangle . \tag{43}$$

Moving on, recall the classical radial fundamental solution for the Laplacian E in $\mathbb{R}^2$ given by

$$E(X) = \tfrac{1}{2\pi} \cdot \ln|X|, \qquad \forall X \in \mathbb{R}^2 \setminus \{0\} . \tag{44}$$

In particular, $E \in \mathscr{C}^\infty(\mathbb{R}^2 \setminus \{0\})$ and $\Delta E = \delta_0$ in the sense of distributions, where δ_0 is the Dirac delta distribution with mass at the origin. Given a graph Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ we fix a point $X_o \in \mathbb{R}^2 \setminus \overline{\Omega}$ and introduce the boundary-to-domain single layer potential operator acting on a measurable function g defined on $\partial\Omega$ as

$$\mathscr{S} g(X) := \int_{\partial\Omega} [E(X-Q) - E(X_o - Q)]\, g(Q)\, d\sigma(Q), \qquad X \in \Omega . \tag{45}$$

In the same vein, the boundary-to-boundary single layer acting on a measurable function g defined on $\partial\Omega$ is set to be

$$Sg(X) := \int_{\partial\Omega} [E(X-Q) - E(X_o - Q)]\, g(Q)\, d\sigma(Q), \qquad X \in \partial\Omega . \tag{46}$$

A word of caution is in order. The singular integral operators in (45) and (46) are closely related versions of the standard harmonic single layers used in the literature, whose integral kernels are simply $E(X-Q)$. The reason we have altered the standard definitions is that while for a fixed $X \in \overline{\Omega}$ the function $E(X - \cdot)$ belongs to $L^q_{loc}(\partial\Omega)$ for each $q \in (1,\infty)$, this lacks decay at infinity. By way of contrast, the kernel $E(X-Q) - E(X_o - Q)$ has appropriate decay, namely $\mathscr{O}(|Q|^{-1})$ as $|Q| \to \infty$, as a simple application of the Mean Value Theorem shows.

Next we recall the principal value harmonic double layer potential operator acting on a measurable function g defined on $\partial\Omega$, given by

$$Kg(P) := p.v. \int_{\partial\Omega} \frac{\partial}{\partial \nu(Q)} [E(P-Q)] g(Q)\, d\sigma(Q), \quad \text{for } \sigma\text{-a.e. } P \in \partial\Omega . \tag{47}$$

The formal adjoint of the operator K from (47) can then be expressed as

$$K^* g(P) := p.v. \int_{\partial\Omega} \frac{\partial}{\partial \nu(P)} [E(P-Q)] g(Q)\, d\sigma(Q), \quad \text{for } \sigma\text{-a.e. } P \in \partial\Omega . \tag{48}$$

We wish to record the mapping properties of these singular integral operators on L^p spaces considered on the boundary of graph Lipschitz domains. A basic result that follows from [10] and standard techniques is

Theorem 2. *Let Ω be a graph Lipschitz domain in $\mathbb{R}^2$ and fix $p \in (1,\infty)$.*
(1) The following operators are well-defined, linear and bounded:

$$S : L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega)\,, \tag{49}$$

$$K : L^p(\partial\Omega) \to L^p(\partial\Omega)\,, \tag{50}$$

$$K^* : L^p(\partial\Omega) \to L^p(\partial\Omega)\,. \tag{51}$$

(2) For every $f \in L^p(\partial\Omega)$ one has $\mathscr{N}\big(\nabla\mathscr{S}f\big) \in L^p(\partial\Omega)$. Moreover there exists a finite constant $C > 0$ depending only on the Lipschitz character of Ω such that

$$\|\mathscr{N}\big(\nabla\mathscr{S}f\big)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}\,. \tag{52}$$

(3) For each $f \in L^p(\partial\Omega)$, the single layer potential operator satisfies

$$\mathscr{S}f\Big|^{n.t}_{\partial\Omega} = Sf\,. \tag{53}$$

(4) For every $f \in L^p(\partial\Omega)$ and $j \in \{1,2\}$ there holds

$$\begin{gathered}\partial_j\mathscr{S}f\Big|^{n.t.}_{\partial\Omega}(P) = -\tfrac{1}{2}\nu_j(P)f(P) + p.v.\int_{\partial\Omega}(\partial_jE)(P-Q)f(Q)\,d\sigma(Q)\\ \text{for}\;\; \sigma - a.e.\;\; P \in \partial\Omega\,,\end{gathered} \tag{54}$$

and thus

$$\frac{\partial\mathscr{S}f}{\partial\nu}(P) = \big(-\tfrac{1}{2}I + K^*\big)f(P), \quad \sigma - a.e.\;\; P \in \partial\Omega\,. \tag{55}$$

3 Mellin Analysis of Singular Integral Operators for the Mixed Problem

In this section we shall consider the case when Ω is the domain consisting of the interior of an infinite angle in $\mathbb{R}^2$ of aperture $\theta \in (0, 2\pi)$ with vertex at the origin (in particular Ω is a graph Lipschitz domain in $\mathbb{R}^2$). Hereafter we shall denote by $(\partial\Omega)_1$ and $(\partial\Omega)_2$ the left and the right side of the angle $\partial\Omega$, respectively. In this notation one can naturally identify the two pieces of the boundary $(\partial\Omega)_j$, $j = 1,2$, with $\mathbb{R}_+$ via the mapping $(\partial\Omega)_j \ni P \mapsto |P| \in \mathbb{R}_+$ and for each $p \in [1,\infty)$, identify $L^p(\partial\Omega)$ with the space $L^p(\mathbb{R}_+) \oplus L^p(\mathbb{R}_+)$.

In the sequel we shall assume that $D = (\partial\Omega)_1$ and $N = (\partial\Omega)_2$ which allows us to further identify the space $L^p(\partial\Omega)$ with $L^p(D) \oplus L^p(N)$. Seeking a solution of the mixed boundary value problem (1) of the form $u = \mathscr{S}h$ for some function $h \in L^p(\partial\Omega)$ and using Theorem 2 leads to the following system of boundary integral equations

$$\partial_\tau Sh\Big|_D = \partial_\tau f_D \;\text{ and }\; (-\tfrac{1}{2}I + K^*)h\Big|_N = f_N\,. \tag{56}$$

Here ∂_τ denotes differentiation in the tangential direction.

Identifying $L^p(\partial\Omega)$ with $L^p(D) \oplus L^p(N)$, and taking into consideration the system (56), we next introduce the operator

$$R : L^p(D) \oplus L^p(N) \longrightarrow L^p(D) \oplus L^p(N) \text{ given by}$$

$$R \begin{pmatrix} \phi_D \\ \phi_N \end{pmatrix} := \begin{pmatrix} \partial_\tau S & \partial_\tau S \\ K^* & K^* \end{pmatrix} \cdot \begin{pmatrix} \phi_D \\ \phi_N \end{pmatrix}. \tag{57}$$

In terms of this operator, the boundary integral system (56) becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} h|_D \\ h|_N \end{pmatrix} + R \begin{pmatrix} h|_D \\ h|_N \end{pmatrix} = \begin{pmatrix} \partial_\tau f_D \\ f_N \end{pmatrix}. \tag{58}$$

Introduce next the following operator

$$T : L^p(D) \oplus L^p(N) \longrightarrow L^p(D) \oplus L^p(N) \text{ given by } T = M + R,$$
$$\text{where } M \text{ is the operator of multiplication to the left by } \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{59}$$

As such, (58) (thus the system (56)) becomes

$$T \begin{pmatrix} h|_D \\ h|_N \end{pmatrix} = \begin{pmatrix} \partial_\tau f_D \\ f_N \end{pmatrix}, \tag{60}$$

reducing matters to the study of invertibility of the operator T.

A simple calculation based on (48) and (44) gives that the kernel of K^*, denoted by $k_{K^*}(\cdot,\cdot)$ satisfies

$$k_{K^*}(P,Q) = \frac{1}{2\pi} \cdot \frac{\langle P-Q, \nu(P)\rangle}{|P-Q|^2}, \qquad \forall P,Q \in \partial\Omega,\ P \neq Q. \tag{61}$$

Also, using (46) we may express the kernel of the operator $\partial_\tau S$ in the form

$$k_{\partial_\tau S}(P,Q) = \frac{1}{2\pi} \cdot \frac{\langle P-Q, \tau(P)\rangle}{|P-Q|^2}, \qquad \forall P,Q \in \partial\Omega,\ P \neq Q, \tag{62}$$

where $\tau(P) := (-\nu_2(P), \nu_1(P))$. Thus, collectively (61) and (62) imply that the kernel of the operator T may be written as

$$k(P,Q) = \begin{pmatrix} k_{\partial_\tau S}(P,Q) & k_{\partial_\tau S}(P,Q) \\ k_{K^*}(P,Q) & k_{K^*}(P,Q) \end{pmatrix}. \tag{63}$$

In turn, the kernel k from (63) can be regarded as a kernel on $\mathbb{R}_+ \times \mathbb{R}_+$. Specifically the function $k(\cdot,\cdot)$ on $\partial\Omega \times \partial\Omega$ shall be identified with the following 2×2 kernel matrix $\widetilde{k} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow M_{2\times 2}(\mathbb{R})$ given by

$$\widetilde{k}(s,t) := \begin{pmatrix} \widetilde{k}_{11}(s,t) & \widetilde{k}_{12}(s,t) \\ \widetilde{k}_{21}(s,t) & \widetilde{k}_{22}(s,t) \end{pmatrix}, \tag{64}$$

where $M_{2\times 2}(\mathbb{R})$ stands for the set of 2×2 matrices with real-valued entries, and

$$\widetilde{k}_{11}(s,t) := k_{\partial_\tau S}(P,Q), \ \text{ if } \ P,Q \in (\partial\Omega)_1 \ , \tag{65}$$

$$\widetilde{k}_{12}(s,t) := k_{\partial_\tau S}(P,Q), \ \text{ if } \ P \in (\partial\Omega)_1 \ \text{ and } \ Q \in (\partial\Omega)_2 \ , \tag{66}$$

$$\widetilde{k}_{21}(s,t) := k_{K^*}(P,Q), \ \text{ if } \ P \in (\partial\Omega)_2 \ \text{ and } \ Q \in (\partial\Omega)_1 \ , \tag{67}$$

$$\widetilde{k}_{22}(s,t) := k_{K^*}(P,Q), \ \text{ if } \ P,Q \in (\partial\Omega)_2 \ . \tag{68}$$

In what follows, we will assume that Ω is the region above the graph of the function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(x) := |x|\cot(\theta/2), \quad x \in \mathbb{R} \ . \tag{69}$$

Concretely, we can now write $(\partial\Omega)_1$ and $(\partial\Omega)_2$ as:

$$\begin{aligned} &(\partial\Omega)_1 := \left\{(-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}) : s \in \mathbb{R}_+\right\} \ \text{ and} \\ &(\partial\Omega)_2 := \left\{(s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}) : s \in \mathbb{R}_+\right\} \ . \end{aligned} \tag{70}$$

The goal of our next result is to provide explicit formulas for $\widetilde{k}_{ij}$, $i,j \in \{1,2\}$.

Lemma 1. *Let* $\widetilde{k} = \left(\widetilde{k}_{ij}\right)_{i,j\in\{1,2\}}$ *be as in* (64)-(68)*. Then for each* $s,t \in \mathbb{R}_+$ *such that* $s \neq t$ *there holds*

$$\widetilde{k}_{11}(s,t) = -\frac{1}{2\pi}\cdot\frac{1}{s-t} \ , \tag{71}$$

$$\widetilde{k}_{12}(s,t) = -\frac{1}{2\pi}\cdot\frac{s-t\cos\theta}{s^2+t^2-2st\cos\theta} \ , \tag{72}$$

$$\widetilde{k}_{21}(s,t) = \frac{1}{2\pi}\cdot\frac{t\sin\theta}{s^2+t^2-2st\cos\theta} \ , \tag{73}$$

$$\widetilde{k}_{22}(s,t) = 0 \ . \tag{74}$$

Proof. Let P and Q be such that $P,Q \in \partial\Omega$ and let $|P| = s \in \mathbb{R}_+$ and $|Q| = t \in \mathbb{R}_+$. We start our analysis by first assuming that $P \in (\partial\Omega)_2$ and $Q \in (\partial\Omega)_1$. Thus, in this case

$$P = \left(s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}\right) \ \text{ and } \ Q = \left(-t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2}\right) \ , \tag{75}$$

and simple geometric considerations show that

$$\nu(P) = \left(\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2}\right) \ \text{ and } \ \nu(Q) = \left(-\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2}\right) \ . \tag{76}$$

Consequently,

$$\left\langle P-Q, \nu(P)\right\rangle = t\sin\theta \ . \tag{77}$$

Thus, using (67) and (61) it follows that

$$\widetilde{k}_{21}(s,t) = k_{K^*}\big((s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}), (-t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2})\big) \qquad (78)$$
$$= \frac{1}{2\pi}\cdot\frac{t\sin\theta}{(s^2-2st\cos\theta+t^2)}\,,$$

which completes the justification of (73).

Second, consider the case when $P,Q \in (\partial\Omega)_2$. Then $\langle P-Q, \nu(P)\rangle = 0$, since in this case $P-Q$ is orthogonal to $\nu(P)$. Thus, based on (61), $\widetilde{k}_{22}(s,t) = 0$, proving (74).

Next, let $P,Q \in (\partial\Omega)_1$. Then,

$$P = \big(-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}\big) \text{ and } Q = \big(-t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2}\big)\,, \qquad (79)$$

and consequently we obtain

$$P-Q = (s-t)\big(-\sin\tfrac{\theta}{2}, \cos\tfrac{\theta}{2}\big)\,, \qquad (80)$$

$$|P-Q|^2 = (s-t)^2\sin^2\tfrac{\theta}{2} + (s-t)^2\cos^2\tfrac{\theta}{2} = (s-t)^2\,, \qquad (81)$$

$$\tau(P) = (\sin\tfrac{\theta}{2}, -\cos\tfrac{\theta}{2})\,, \qquad (82)$$

$$\langle P-Q, \tau(P)\rangle = -(s-t)\big(\sin^2\tfrac{\theta}{2} + \cos^2\tfrac{\theta}{2}\big) = -(s-t)\,. \qquad (83)$$

Thus, using (65), (62) and (80)-(83) it follows that

$$\widetilde{k}_{11}(s,t) = k_{\partial_\tau S}\big((-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}), (-t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2})\big) = -\frac{1}{2\pi}\cdot\frac{1}{s-t}\,, \qquad (84)$$

yielding the equality in (71).

Finally, consider $P \in (\partial\Omega)_1$, and $Q \in (\partial\Omega)_2$. Then,

$$P = \big(-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}\big) \text{ and } Q = \big(t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2}\big)\,, \qquad (85)$$

and consequently

$$P-Q = \big(-(s+t)\sin\tfrac{\theta}{2}, (s-t)\cos\tfrac{\theta}{2}\big)\,, \qquad (86)$$

$$|P-Q|^2 = (s+t)^2\sin^2\tfrac{\theta}{2} + (s-t)^2\cos^2\tfrac{\theta}{2} = s^2+t^2-2st\cos\theta\,, \qquad (87)$$

$$\tau(P) = (\sin\tfrac{\theta}{2}, -\cos\tfrac{\theta}{2})\,. \qquad (88)$$

Using next (66), (62) and (86)-(88) it follows that

$$
\begin{aligned}
\widetilde{k}_{12}(s,t) &= k_{\partial_\tau S}\big((-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}), (t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2})\big) \\
&= -\frac{1}{2\pi}\cdot\frac{(s+t)\sin^2\frac{\theta}{2}+(s-t)\cos^2\frac{\theta}{2}}{s^2+t^2-2st\cos(\theta)} \\
&= -\frac{1}{2\pi}\cdot\frac{s-t\cos(\theta)}{s^2+t^2-2st\cos(\theta)}\,.
\end{aligned}
\tag{89}
$$

This gives the identity in (72) and completes the proof of the lemma.

Clearly (71) gives that $\widetilde{k}_{11}(\cdot,\cdot)$ is the kernel of the Hilbert transform on $\mathbb{R}_+$. Our next result brings light on the nature of the other entries of the matrix-valued function $\widetilde{k}$.

Lemma 2. *Fix $p\in(1,\infty)$. Then the kernels $\widetilde{k}_{12}$, $\widetilde{k}_{21}$ and $\widetilde{k}_{22}$ from* (72)-(74) *are Hardy kernels for $L^p(\mathbb{R}_+)$.*

Proof. Since $\widetilde{k}_{22}\equiv 0$ this is clearly a Hardy kernel for $L^p(\mathbb{R}_+)$. Observe next that

$$
\widetilde{k}_{12} = -\frac{1}{2\pi}(\kappa_1-(\cos\theta)\cdot\kappa_2) \text{ and } \widetilde{k}_{21} = \frac{1}{2\pi}(\sin\theta)\cdot\kappa_2\,,
\tag{90}
$$

are measurable functions on $\mathbb{R}_+\times\mathbb{R}_+$, where

$$
\begin{aligned}
\kappa_1(s,t) &:= \frac{s}{s^2-2st\cos\theta+t^2}\,, \quad \forall (s,t)\in\mathbb{R}_+\times\mathbb{R}_+, s\neq t\,, \\
\kappa_2(s,t) &:= \frac{t}{s^2-2st\cos\theta+t^2}\,, \quad \forall (s,t)\in\mathbb{R}_+\times\mathbb{R}_+, s\neq t\,.
\end{aligned}
\tag{91}
$$

Thus, the statements in the lemma about $\widetilde{k}_{12}$ and $\widetilde{k}_{21}$ follow immediately if we show that κ_1 and κ_2 are Hardy kernels for $L^p(\mathbb{R}_+)$. To this end we start with the immediate observation that, based on (91), the functions κ_i, $i=1,2$, are homogeneous of degree -1. Going further note that

$$
t^2-2t\cos\theta+1 = (t-\cos\theta)^2+\sin^2\theta > 0 \text{ for } \theta\in(0,2\pi)\,.
\tag{92}
$$

In addition, a quick inspection of (91) reveals that

$$
|\kappa_1(1,t)|t^{-1/p} \approx t^{-1/p} \text{ and } |\kappa_2(1,t)|t^{-1/p} \approx t^{1-1/p} \text{ on } (0,1)\,,
\tag{93}
$$

and

$$
|\kappa_1(1,t)|t^{-1/p} \approx t^{-(2+1/p)} \text{ and } |\kappa_2(1,t)|t^{-1/p} \approx t^{-(1+1/p)} \text{ on } (M,\infty)\,,
\tag{94}
$$

if $M>0$ is large enough. Combining (92)-(94) ultimately yields

$$
\int_0^\infty |\kappa_1(1,t)|t^{-1/p}dt < +\infty \text{ and } \int_0^\infty |\kappa_2(1,t)|t^{-1/p}dt < +\infty\,.
\tag{95}
$$

Above we have used the notation $f(t)\approx g(t)$ on an arbitrary set E provided there exist positive finite constants c_1,c_2 such that $c_1g(t)\le f(t)\le c_2g(t)$ for each $t\in E$.

In turn, (95) completes the proof of the fact that κ_1 and κ_2 are Hardy kernels on $L^p(\mathbb{R}_+)$, finishing the proof of the lemma.

Consider next the operator

$$\mathfrak{T} : \big(L^p(\mathbb{R}_+)\big)^2 \longrightarrow \big(L^p(\mathbb{R}_+)\big)^2 \tag{96}$$

given by

$$\mathfrak{T}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \int_{\mathbb{R}_+} \begin{pmatrix} \widetilde{k}_{11}(s,t) & \widetilde{k}_{12}(s,t) \\ \widetilde{k}_{21}(s,t) & \widetilde{k}_{22}(s,t) \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} dt\,, \tag{97}$$

where $p \in (1,\infty)$. By Lemma 1 and Lemma 2, the operator $\mathfrak{T}$ is of the form discussed in Theorem 1 with

$$h := \begin{pmatrix} 0 & \widetilde{k}_{12} \\ \widetilde{k}_{21} & 0 \end{pmatrix},\ B := \begin{pmatrix} \frac{1}{2\pi} & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } A := \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{98}$$

In particular, $A - \pi i \cdot B = \begin{pmatrix} -\frac{i}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and thus its determinant is $\neq 0$. As such Corollary 1 applies and yields

$$\begin{aligned} &\mathfrak{T} \text{ is invertible on } (L^p(\mathbb{R}_+))^2,\ 1 < p < \infty, \text{ if and only if} \\ &\det\left(\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + \mathscr{M}\widetilde{k}(\cdot,1)(1/p + i\xi)\right) \neq 0 \quad \forall \xi \in \mathbb{R}\,. \end{aligned} \tag{99}$$

Our next goal is to compute $\mathscr{M}\widetilde{k}$, the Mellin symbol of $\widetilde{k}$. In this regard we have the following.

Lemma 3. *Let $\Omega \subseteq \mathbb{R}^2$ be the domain consisting of the interior of an infinite angle of aperture $\theta \in (0,2\pi)$ with vertex at the origin and suppose $\widetilde{k}$ is as in* (64) *and its entries are as in* (71)-(74)*. Then, for each $z \in \mathbb{C}$ with $\mathfrak{Re}\, z \in (0,1)$, there holds*

$$\mathscr{M}\widetilde{k}(\cdot,1)(z) = \frac{1}{2\sin(\pi z)} \begin{pmatrix} \cos(\pi z) & \cos(\theta + z(\pi - \theta)) \\ \sin(\theta + z(\pi - \theta)) & 0 \end{pmatrix}. \tag{100}$$

Proof. With an eye toward computing the Mellin transforms of $\widetilde{k}_{ij}(\cdot,1)$ for indices $i,j \in \{1,2\}$, fix $\theta \in (0,2\pi)$ and introduce the following measurable functions on $\mathbb{R}_+$ denoted by f, g, and h, and given by

$$f(\lambda) := \frac{1}{\lambda - 1}, \qquad \forall \lambda \in \mathbb{R}_+ \text{ such that } \lambda \neq 1\,, \tag{101}$$

and

$$g(\lambda) := \frac{1}{\lambda^2 + 2\lambda\cos(\pi-\theta)+1}, \text{ and } h(\lambda) := \lambda g(\lambda), \qquad \forall \lambda \in \mathbb{R}_+ . \tag{102}$$

Using formula 2.12 from page 14 in [31] we have

$$\mathscr{M} f(z) = -\pi\cot(\pi z) \text{ whenever } \mathfrak{Re}\, z \in (0,1)\,, \tag{103}$$

and as such, since $\frac{1}{2\pi} f(s) = -\widetilde{k}_{11}(s,1)$ for $s \in \mathbb{R}_+ \setminus \{1\}$,

$$\mathscr{M}\widetilde{k}_{11}(\cdot,1)(z) = \tfrac{1}{2}\cot(\pi z) \text{ whenever } \mathfrak{Re}\, z \in (0,1)\,. \tag{104}$$

Going further, note that

$$\widetilde{k}_{12}(s,1) = -\tfrac{1}{2\pi}\big[h(s) - (\cos\theta)\cdot g(s)\big] \text{ for } s \in \mathbb{R}_+\,, \tag{105}$$

$$\widetilde{k}_{21}(s,1) = \tfrac{1}{2\pi}\cdot(\sin\theta)\cdot g(s) \text{ for } s \in \mathbb{R}_+\,, \tag{106}$$

and as such, for each $z \in \mathbb{C}$ belonging to the intersection of the two strips of convergence for the Mellin transforms of the functions g and h, there holds

$$\mathscr{M}\widetilde{k}_{12}(\cdot,1)(z) = -\tfrac{1}{2\pi}\big[\mathscr{M}h(z) - (\cos\theta)\cdot\mathscr{M}g(z)\big]\,, \tag{107}$$

$$\mathscr{M}\widetilde{k}_{21}(\cdot,1)(z) = \tfrac{1}{2\pi}\cdot(\sin\theta)\cdot\mathscr{M}g(z)\,. \tag{108}$$

Appealing again to [31], this time to formula 2.54 on page 23 and formula 2.5 on page 13, the strip of convergence for g is $\varGamma_{0,2}$ and that of h is $\varGamma_{-1,1}$. In addition the following hold

$$\text{for each } z \in \mathbb{C} \text{ such that } \mathfrak{Re}\, z \in (0,2)\,,$$
$$\mathscr{M}g(z) = \begin{cases} \pi\dfrac{\csc\theta}{\sin(\pi z)}\sin(\theta + z(\pi-\theta)), \text{ for } \theta \in (0,2\pi)\setminus\{\pi\}\,, \\ -\dfrac{\pi(z-1)}{\sin(\pi z)}, \text{ for } \theta = \pi\,, \end{cases} \tag{109}$$

and $\mathscr{M}h(z) = \mathscr{M}g(z+1)$ for $z \in \varGamma_{-1,1}$. As such, using (109),

$$\text{for each } z \in \mathbb{C} \text{ such that } \mathfrak{Re}\, z \in (-1,1)\,,$$
$$\mathscr{M}h(z) = \begin{cases} \pi\dfrac{\csc\theta}{\sin(\pi z)}\sin(z(\pi-\theta)), \text{ for } \theta \in (0,2\pi)\setminus\{\pi\}\,, \\ \dfrac{\pi z}{\sin(\pi z)}, \text{ for } \theta = \pi\,, \end{cases} \tag{110}$$

where we have used that $\sin(\pi(z+1)) = -\sin(\pi z)$.

Consequently, based on (107), (109), (110), and elementary trigonometric formulas, we obtain that

$$\mathscr{M}\widetilde{k}_{12}(\cdot,1)(z)=\frac{\cos(\theta+z(\pi-\theta))}{2\sin(\pi z)},\quad \text{for }\ \mathfrak{Re}\,z\in(0,1)\ \text{ and }\ \theta\in(0,2\pi)\,. \tag{111}$$

Next, appealing again to (107) and (109), we immediately arrive at

$$\mathscr{M}\widetilde{k}_{21}(\cdot,1)(z)=\frac{\sin(\theta+z(\pi-\theta))}{2\sin(\pi z)},\quad \text{for }\ \mathfrak{Re}\,z\in(0,1)\ \text{ and }\ \theta\in(0,2\pi)\,. \tag{112}$$

Finally, since $\widetilde{k}_{22}\equiv 0$ on $\mathbb{R}_+\times\mathbb{R}_+$ (cf. (74)), there holds

$$\mathscr{M}\widetilde{k}_{22}(\cdot,1)(z)=0 \qquad \forall z\in\mathbb{C}\,, \tag{113}$$

and combining this with (104), (111) and (112) we obtain that (100) holds for each $z\in\mathbb{C}$ such that $\mathfrak{Re}\,z\in(0,1)$ and each $\theta\in(0,2\pi)$, completing the proof of the lemma.

At this point we are ready to present the main invertibility result of the paper.

Theorem 3. *Let $\Omega\subseteq\mathbb{R}^2$ be the interior of an infinite angle of aperture $\theta\in(0,2\pi)$. Denote by $D:=(\partial\Omega)_1$ and $N:=(\partial\Omega)_2$ the left ray, and respectively the right ray of Ω. Recall the operator T from* (59) *acting in a linear and bounded fashion from $L^p(D)\oplus L^p(N)$ into itself, for $1<p<\infty$. Then T is an isomorphism whenever $p\in(1,\infty)$ is such that*

$$p\neq\begin{cases}\frac{2\pi-\theta}{\pi-\theta} & \textit{if }\ \theta\in(0,\pi/2]\\ \frac{2\pi-\theta}{\pi-\theta},\,\frac{2\theta}{2\theta-\pi} & \textit{if }\ \theta\in(\pi/2,\pi)\\ 2 & \textit{if }\ \theta=\pi\\ \frac{2\theta}{2\theta-\pi} & \textit{if }\ \theta\in(\pi,3\pi/2]\\ \frac{2\theta}{2\theta-\pi},\,\frac{2\theta}{2\theta-3\pi} & \textit{if }\ \theta\in(3\pi/2,2\pi)\,.\end{cases} \tag{114}$$

Proof. We begin the proof by observing that, given the structure of the kernel of the operator T from (61)-(63), the operator T is invariant under translations and rotations in $\mathbb{R}^2$. Indeed, if $\mathfrak{R}$ is an arbitrary rotation in $\mathbb{R}^2$ then for each $P,Q\in\mathbb{R}^2$ there holds

$$\langle\mathfrak{R}(P),\mathfrak{R}(Q)\rangle=\langle P,Q\rangle\,. \tag{115}$$

Then, based on this and the equations (61) and (62), it follows that the matrix in (63) is invariant under the rotation $\mathfrak{R}$ and any translation. Thus we may assume without loss of generality that $\Omega\subseteq\mathbb{R}^2$ is the upper graph of the function ϕ from (69) with $\theta\in(0,2\pi)$. In this scenario, denote again by $D=(\partial\Omega)_1$ and $N=(\partial\Omega)_2$ the left, and respectively the right ray of Ω as in (70).

Next, let us make the simple observation that, in the light of the identifications $L^p(D)\equiv L^p(\mathbb{R}_+)$ and $L^p(N)\equiv L^p(\mathbb{R}_+)$ for each $p\in(1,\infty)$ the following holds

$$T \text{ is invertible on } L^p(D)\oplus L^p(N) \iff \mathfrak{T} \text{ is invertible on } \big(L^p(\mathbb{R}_+)\big)^2\,. \tag{116}$$

However, using this, Lemma 3 and (99) we may further conclude that

$$\begin{gathered} T \text{ is invertible on } L^p(D)\oplus L^p(N),\ 1<p<\infty, \text{ if and only if} \\ \forall z \in \tfrac{1}{p}+i\mathbb{R} \\ \det\left[\frac{1}{2\sin(\pi z)}\begin{pmatrix} \cos(\pi z) & \cos(z(\pi-\theta)+\theta) \\ \sin(\theta+z(\pi-\theta)) & -\sin(\pi z)\end{pmatrix}\right] \neq 0\,. \end{gathered} \tag{117}$$

For each $\theta \in (0,2\pi)$ and each $z \in \frac{1}{p}+i\mathbb{R}$ introduce

$$A(\theta,z) := \frac{1}{2\sin(\pi z)}\begin{pmatrix} \cos(\pi z) & \cos(z(\pi-\theta)+\theta) \\ \sin(\theta+z(\pi-\theta)) & -\sin(\pi z)\end{pmatrix}, \tag{118}$$

and note that (using elementary trigonometric identities such as the double angle formula, etc.) for each pair θ, z we may write

$$\begin{aligned} \det A(\theta,z) &= -\frac{1}{4\sin^2(\pi z)}\cdot\Big[\cos(\pi z)\cdot\sin(\pi z)+\sin(\theta+z(\pi-\theta))\cdot\cos(\theta+z(\pi-\theta))\Big] \\ &= -\frac{1}{8\sin^2(\pi z)}\Big[\sin(2\pi z)+\sin(2\theta+2z(\pi-\theta))\Big] \\ &= -\frac{1}{4\sin^2(\pi z)}\cdot\sin(\theta-\theta z+2\pi z)\cdot\cos(-\theta+\theta z)\,. \end{aligned} \tag{119}$$

Consequently, using (117)-(119),

$$\begin{gathered} T \text{ is invertible on } L^p(D)\oplus L^p(N),\ 1<p<\infty, \text{ if and only if} \\ \sin(\theta-\theta z+2\pi z)\neq 0 \ \text{ and } \ \cos(-\theta+\theta z)\neq 0 \ \text{ for } \ z\in\tfrac{1}{p}+i\mathbb{R}\,. \end{gathered} \tag{120}$$

We shall first determine all those values $z \in \Gamma_{0,1}$, for which $\sin(\theta-\theta z+2\pi z)=0$, where as before,

$$\Gamma_{0,1} := \{z\in\mathbb{C} : 0<\mathfrak{Re}\, z<1\}\,. \tag{121}$$

Since

$$\sin(\theta-\theta z+2\pi z)=0 \iff \theta-\theta z+2\pi z = k\pi \ \text{ for some } k\in\mathbb{Z}\,, \tag{122}$$

if $\sin(\theta-\theta z+2\pi z)=0$ then z is necessarily of the form $z=\dfrac{k\pi-\theta}{2\pi-\theta}\in\mathbb{R}$ for some $k\in\mathbb{Z}$. Combining this with the fact that $z\in\Gamma_{0,1}$ this forces k to be 1 and θ to belong to $(0,\pi)$, and ultimately $z=\dfrac{\pi-\theta}{2\pi-\theta}$. Letting

$$\mathscr{Z}_1(\theta) := \{z\in\Gamma_{0,1} : \sin(\theta-\theta z+2\pi z)=0\}, \quad \theta\in(0,2\pi)\,, \tag{123}$$

we thus obtain

$$\mathscr{Z}_1(\theta) = \begin{cases} \Big\{\dfrac{\pi-\theta}{2\pi-\theta}\Big\}, \text{ when } \theta \in \Big(0,\pi\Big)\,; \\ \emptyset, \text{ when } \theta \in \Big[\pi,2\pi\Big)\,. \end{cases} \tag{124}$$

Next, we turn to the task of identifying all those values $z \in \Gamma_{0,1}$, for which the equation $\cos(\theta z - \theta) = 0$. Since

$$\cos(\theta z - \theta) = 0 \iff \theta z - \theta = \frac{\pi}{2} + k\pi \text{ for some } k \in \mathbb{Z}\,, \tag{125}$$

this implies that if $\cos(\theta z - \theta) = 0$ then necessarily $z = 1 + \dfrac{\pi}{2\theta} + \dfrac{k\pi}{\theta} \in \mathbb{R}$. Using that $z \in \Gamma_{0,1}$ this amounts to having

$$2\theta + (2k+1)\pi \in (0, 2\theta)\,. \tag{126}$$

Notice that $2\theta + (2k+1)\pi < 2\theta$ combined with the fact that $k \in \mathbb{Z}$ immediately gives

$$k \leq -1\,. \tag{127}$$

Going further, we shall divide our analysis in several cases. First consider the scenario in which the aperture $\theta \in (0, \frac{\pi}{2}]$. Then, combing this with (126) implies $-(2k+1)\pi < 2\theta \leq \pi$ and ultimately $k > -1$, contradicting (127). This shows that, when $\theta \in (0, \frac{\pi}{2}]$ the equation $\cos(\theta z - \theta) = 0$ has no roots $z \in \mathbb{C}$ with $\mathfrak{Re}\, z \in (0,1)$. Next, consider the scenario in which $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}]$. In concert with (126) this implies $-(2k+1)\pi < 2\theta \leq 3\pi$ and ultimately $k > -2$. Keeping in mind (127), this forces $k = -1$ and $z = \frac{2\theta-\pi}{2\theta} \in (0,1)$. Finally we shall analyze the case when $\theta \in (\frac{3\pi}{2}, 2\pi)$. As before, combing this with (126) implies $-(2k+1)\pi < 2\theta \leq 4\pi$ and thus $k > -2.5$. Since $k \in \mathbb{Z}$ and (127) holds this further gives $k \in \{-1,-2\}$ and hence $z \in \big\{\frac{2\theta-\pi}{2\theta}, \frac{2\theta-3\pi}{2\theta}\big\} \subset (0,1)$. Thus, setting

$$\mathscr{Z}_2(\theta) := \{z \in \Gamma_{0,1} : \cos(\theta z - \theta) = 0\}, \quad \theta \in (0, 2\pi)\,, \tag{128}$$

we may conclude that

$$\mathscr{Z}_2(\theta) = \begin{cases} \emptyset, \text{ when } \theta \in \Big(0, \dfrac{\pi}{2}\Big]\,; \\ \Big\{\dfrac{2\theta-\pi}{2\theta}\Big\}, \text{ when } \theta \in \Big(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\Big]\,; \\ \Big\{\dfrac{2\theta-\pi}{2\theta}, \dfrac{2\theta-3\pi}{2\theta}\Big\}, \text{ when } \theta \in \Big(\dfrac{3\pi}{2}, 2\pi\Big)\,. \end{cases} \tag{129}$$

Finally, the conclusion of the theorem follows combining (120) with (123)-(124) and (128)-(129).

Moving on, we shall discuss a number of results concerning the spectrum of the operator $\mathfrak{T}$ from (96), for $1 < p < \infty$. We start by recalling the complex square root defined using the principal branch of the logarithm, i.e.,

$$\sqrt{z} := e^{\frac{1}{2}\log(z)}, \quad \forall z \in \mathbb{C}, \tag{130}$$

where log stands for the principal branch of the logarithm in $\mathbb{C}$.

Then our result reads as follows.

Theorem 4. *Let $\Omega \subseteq \mathbb{R}^2$ be the interior of an infinite angle of aperture $\theta \in (0, 2\pi)$. Denote by $D := (\partial\Omega)_1$ and $N := (\partial\Omega)_2$ the left ray, and respectively the right ray of Ω. Recall the operator T from* (59) *acting in a linear and bounded fashion from $L^p(D) \oplus L^p(N)$ into itself, for $1 < p < \infty$. Then, for each $p \in (1, \infty)$ there holds*

$$\sigma(T; L^p(D) \oplus L^p(N)) = \mathscr{A} \cup \left\{ -\frac{1}{2}, \frac{i}{2}, -\frac{i}{2} \right\}, \tag{131}$$

where

$$\mathscr{A} := \left\{ \frac{\cos(\pi z) - \sin(\pi z) \pm \sqrt{1 + \sin(2\pi z) + 2\sin(2z(\pi - \theta) + 2\theta)}}{4\sin(\pi z)} : z \in \tfrac{1}{p} + i\mathbb{R} \right\}. \tag{132}$$

Here $\sqrt{\cdot}$ stands for the complex square root defined in (130)*.*

Moreover, the set $\sigma(T; L^p(D) \oplus L^p(N)) \subseteq \mathbb{C} \equiv \mathbb{R}^2$ is symmetric with respect to the x-axis, i.e., the following implication holds

$$w \in \sigma(T; L^p(D) \oplus L^p(N)) \iff \overline{w} \in \sigma(T; L^p(D) \oplus L^p(N)), \tag{133}$$

where bar denotes complex conjugation.

Proof. Much as observed in the beginning of the proof of Theorem 3, without loss of generality matters can be reduced to considering Ω to be the upper graph of the function ϕ from (69) with $\theta \in (0, 2\pi)$ (with $D = (\partial\Omega)_1$ and $N = (\partial\Omega)_2$ as in (70)). Also, in light of the identifications $L^p(D) \equiv L^p(\mathbb{R}_+)$ and $L^p(N) \equiv L^p(\mathbb{R}_+)$ for each $p \in (1, \infty)$ the following holds

$$\sigma(T; L^p(D) \oplus L^p(N)) = \sigma\big(\mathfrak{T}; (L^p(\mathbb{R}_+))^2\big), \tag{134}$$

where $\mathfrak{T}$ is as in (96)-(97). Next, appealing to (97), Lemma 1 and Lemma 2, we may conclude that the operator $\mathfrak{T}$ is of the form discussed in Theorem 1 with h, B, A as in (98). Consequently, conclusion (27) in Theorem 1 ensures that $\mathfrak{T}$ is a linear bounded operator on $(L^p(\mathbb{R}_+))^2$ and

$$\sigma\big(\mathfrak{T}; (L^p(\mathbb{R}_+))^2\big) = \overline{\Lambda}, \tag{135}$$

where

$$\Lambda := \big\{ w \in \mathbb{C} : \det\Big(wI - \big(A + \mathscr{M}\widetilde{k}(\cdot, 1)\big)(z) \Big) = 0, \text{ for some } z \in \tfrac{1}{p} + i\mathbb{R} \big\}. \tag{136}$$

Note that (131)-(132) immediately follow from (134) and (135)-(136) as soon as we establish that

$$\mathscr{A} = \Lambda, \tag{137}$$

and

$$\overline{\mathscr{A}} = \mathscr{A} \cup \left\{ -\tfrac{1}{2}, \tfrac{i}{2}, -\tfrac{i}{2} \right\}. \tag{138}$$

We shall focus first on establishing (137). With this goal in mind, Lemma 3 gives that for each $z \in \frac{1}{p} + i\mathbb{R}$ there holds

$$\begin{aligned} &\big(A + \mathscr{M}\widetilde{k}(\cdot,1)\big)(z) \\ &\quad = \frac{1}{2\sin(\pi z)} \begin{pmatrix} \cos(\pi z) & \cos(z(\pi-\theta)+\theta) \\ \sin(\theta + z(\pi-\theta)) & -\sin(\pi z) \end{pmatrix}. \end{aligned} \tag{139}$$

Thus, for each $w \in \mathbb{C}$ we have

$$\begin{aligned} &wI - \big(A + \mathscr{M}\widetilde{k}(\cdot,1)\big)(z) = \\ &\quad \frac{1}{2\sin(\pi z)} \begin{pmatrix} 2w\sin(\pi z) - \cos(\pi z) & -\cos(z(\pi-\theta)+\theta) \\ -\sin(\theta + z(\pi-\theta)) & 2w\sin(\pi z) + \sin(\pi z) \end{pmatrix}. \end{aligned} \tag{140}$$

Consequently

$$\det\Big(wI - \big(A + \mathscr{M}\widetilde{k}(\cdot,1)\big)(z)\Big) = \frac{1}{4\sin^2(\pi z)} \cdot \mathscr{F}_\theta(w,z), \tag{141}$$

where

$$\begin{aligned} \mathscr{F}_\theta(w,z) :=& 4w^2\sin^2(\pi z) + 2w\big(-\sin(\pi z)\cos(\pi z) + \sin^2(\pi z)\big) \\ &+ \big(-\cos(\pi z)\sin(\pi z) - \cos(z(\pi-\theta)+\theta)\sin(z(\pi-\theta)+\theta)\big). \end{aligned} \tag{142}$$

Thus

$$\begin{aligned} &\Big\{w \in \mathbb{C} : \det\Big(wI - \big(A + \mathscr{M}\widetilde{k}(\cdot,1)\big)(z)\Big) = 0, \text{ for some } z \in \tfrac{1}{p} + i\mathbb{R}\Big\} \\ &\qquad = \Big\{w \in \mathbb{C} : \mathscr{F}_\theta(w,z) = 0, \text{ for some } z \in \tfrac{1}{p} + i\mathbb{R}\Big\}. \end{aligned} \tag{143}$$

Now, since for each fixed $z \in \frac{1}{p} + i\mathbb{R}$, the equation $\mathscr{F}_\theta(w,z) = 0$ is quadratic in w, we obtain that

$$\begin{aligned} &\mathscr{F}_\theta(w,z) = 0 \quad \text{if and only if} \\ &w \in \left\{ \frac{\cos(\pi z) - \sin(\pi z) \pm \sqrt{1 + \sin(2\pi z) + 2\sin(2z(\pi-\theta)+2\theta)}}{4\sin(\pi z)} \right\}, \end{aligned} \tag{144}$$

which combined with (143) readily yields (137). Next, the equality in (138) follows from the definition of the set A, and Lemma 4 which we prove next, independent of the current considerations.

Turning attention to the implication (133), and using (131)-(132), matters reduce to showing that the set $\mathscr{A} \cup \{-\frac{1}{2}, \frac{i}{2}, -\frac{i}{2}\}$ is invariant under complex conjugation. Since clearly this is the case for $\{-\frac{1}{2}, \frac{i}{2}, -\frac{i}{2}\}$ we are left with showing that $w \in \mathscr{A}$ if and only if $\overline{w} \in \mathscr{A}$, where bar denotes complex conjugation. However, this is a consequence of the fact that

$$\overline{w_+(z)} = w_+(\overline{z}) \quad \text{and} \quad \overline{w_-(z)} = w_-(\overline{z}) \qquad \forall z \in \mathbb{C}\,, \tag{145}$$

where, for each $z \in \mathbb{C}$,

$$w_\pm(z) := \frac{\cos(\pi z) - \sin(\pi z) \pm \sqrt{1 + \sin(2\pi z) + 2\sin(2z(\pi - \theta) + 2\theta)}}{4\sin(\pi z)}\,. \tag{146}$$

Indeed, (145) immediately follows using elementary properties of complex conjugation, keeping in mind that for each $z \in \mathbb{C}$ there holds

$$\overline{\sin(z)} = \sin(\overline{z}) \;\text{ and }\; \overline{\cos(z)} = \cos(\overline{z})\,, \tag{147}$$

and

$$\sqrt{\overline{z}} = \overline{\sqrt{z}}\,. \tag{148}$$

Here $\sqrt{\cdot}$ is as in (130) and to see (148), we write $\sqrt{z} = e^{\frac{1}{2}\log(z)} = e^{\frac{1}{2}\ln|z|}(\cos(\arg(z)) + i\sin(\arg(z)))$, where log is the principal branch of the logarithm. Then (148) follows upon noticing that $\arg(\overline{z}) = -\arg(z)$. This finishes the proof of (133) and completes the proof of Theorem 4.

Next, we state and prove the lemma invoked in the proof of Theorem 4.

Lemma 4. *Let $\theta \in (0, 2\pi)$, $x \in (0,1)$ and $z = x + iy$ for $y \in \mathbb{R}$. Then the following hold*

$$\lim_{y\to\infty} \frac{\cos(\pi z)}{\sin(\pi z)} = -i \;\textit{ and }\; \lim_{y\to-\infty} \frac{\cos(\pi z)}{\sin(\pi z)} = i\,, \tag{149}$$

$$\lim_{y\to\infty} \sqrt{\frac{1 + \sin(2\pi z) + 2\sin(2z(\pi - \theta) + 2\theta)}{(\sin(\pi z))^2}} = 1 - i\,, \tag{150}$$

and

$$\lim_{y\to-\infty} \sqrt{\frac{1 + \sin(2\pi z) + 2\sin(2z(\pi - \theta) + 2\theta)}{(\sin(\pi z))^2}} = 1 + i\,. \tag{151}$$

Proof. This proof is elementary and is included here just for the sake of completeness. We start with the observation that whenever $x \in (0,1)$, $y \in \mathbb{R}$ and $z = x + iy$, then $\sin(\pi z) \neq 0$. Next, whenever $z = x + iy$ with $x, y \in \mathbb{R}$, we shall make use of the elementary identities

$$2\sin(\pi z) = \sin(\pi x)(e^{\pi y} + e^{-\pi y}) + i\cos(\pi x)(e^{\pi y} - e^{-\pi y})\,, \tag{152}$$

$$2\cos(\pi z) = \cos(\pi x)(e^{\pi y} + e^{-\pi y}) - i\sin(\pi x)(e^{\pi y} - e^{-\pi y})\,, \tag{153}$$

along with elementary algebra to write

$$\frac{\cos(\pi z)}{\sin(\pi z)} = \frac{1}{i} \cdot \frac{e^{\pi y}(\cos(\pi x) - i\sin(\pi x)) + e^{-\pi y}(\cos(\pi x) + i\sin(\pi x))}{e^{\pi y}(\cos(\pi x) - i\sin(\pi x)) + e^{-\pi y}(-\cos(\pi x) - i\sin(\pi x))}\,, \tag{154}$$

from which the equalities in (149) easily follow. Here we made use of the fact that $\sin(\pi z) \neq 0$ if $y \neq 0$.

Turning our attention to the equalities in (150) and in (151), we start with the observation that, based on (152), there holds

$$\lim_{y\to\infty} \frac{1}{(\sin(\pi z))^2} = \lim_{y\to-\infty} \frac{1}{(\sin(\pi z))^2} = 0\,. \tag{155}$$

Next, let us observe that since $\dfrac{\sin(2\pi z)}{(\sin(\pi z))^2} = 2 \cdot \dfrac{\cos(\pi z)}{\sin(\pi z)}$, based on (149), there holds

$$\lim_{y\to\infty} \frac{\sin(2\pi z)}{(\sin(\pi z))^2} = -2i, \ \text{ and } \lim_{y\to-\infty} \frac{\sin(2\pi z)}{(\sin(\pi z))^2} = 2i\,. \tag{156}$$

Our next goal is to find the limit of $\dfrac{\sin(2z(\pi-\theta)+2\theta)}{(\sin(\pi z))^2}$ as y approaches $\pm\infty$. To this end, first write

$$\sin(2z(\pi-\theta)+2\theta) = \sin(2(x+iy)(\pi-\theta)+2\theta) = \sin(a+iby)\,, \tag{157}$$

where

$$a := 2x(\pi-\theta)+2\theta \ \text{ and } \ b := 2(\pi-\theta)\,. \tag{158}$$

Consequently, squaring (152) and using (157), we obtain

$$\frac{\sin(2z(\pi-\theta)+2\theta)}{(\sin(\pi z))^2} = \tag{159}$$

$$2 \cdot \frac{e^{by}(\sin a + i\cos a) + e^{-by}(\sin a - i\cos a)}{e^{2y\pi}(\sin(2x\pi) + i\cos(2x\pi)) + e^{-2y\pi}(\sin(2x\pi) - i\cos(2x\pi)) + 2}\,.$$

Combining (159) with (158), and the fact that $\theta \in (0, 2\pi)$, allows us to conclude that

$$\lim_{y\to\infty} \frac{\sin(2z(\pi-\theta)+2\theta)}{(\sin(\pi z))^2} = \lim_{y\to-\infty} \frac{\sin(2z(\pi-\theta)+2\theta)}{(\sin(\pi z))^2} = 0\,. \tag{160}$$

Together (155), (156), and (160) yield

$$\lim_{y\to\infty} \frac{1+\sin(2\pi z)+2\sin(2z(\pi-\theta)+2\theta)}{(\sin(\pi z))^2} = -2i\,, \tag{161}$$

$$\lim_{y\to-\infty} \frac{1+\sin(2\pi z)+2\sin(2z(\pi-\theta)+2\theta)}{(\sin(\pi z))^2} = 2i\,. \tag{162}$$

Finally, (150) and (151) follow from (161) and (162), keeping in mind that $\sqrt{\cdot}$ stands for the complex square root defined in (130).

We conclude this section by presenting several spectral examples.

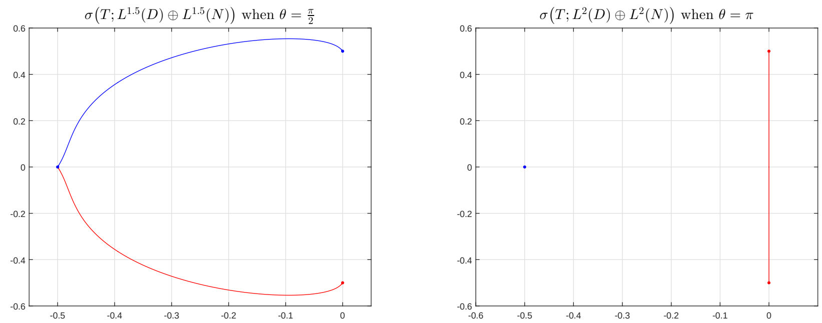

Fig. 1 The $L^p(D) \oplus L^p(N)$ spectrum of the operator T. The case $p = 1.5$ and $\theta = \frac{\pi}{2}$ is presented in the left figure. The case $p = 2$ and $\theta = \pi$ is presented in the right figure.

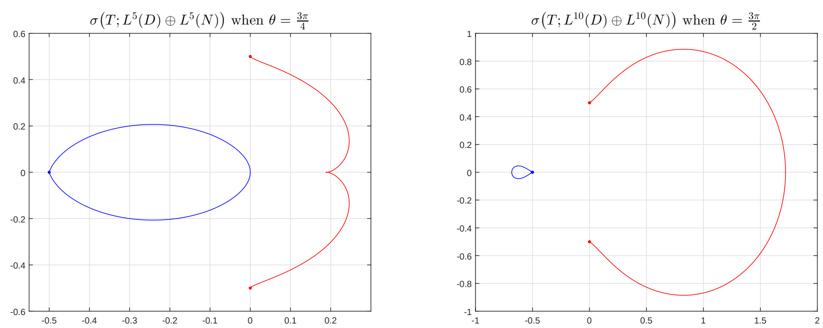

Fig. 2 The $L^p(D) \oplus L^p(N)$ spectrum of the operator T. The case $p = 5$ and $\theta = \frac{3\pi}{4}$ is presented in the left figure. The case $p = 10$ and $\theta = \frac{3\pi}{2}$ is presented in the right figure.

4 The well-posedness of $(MBVP_p)$ in a sector

In this section we study the well-posedness of the mixed boundary value problem (1) in an infinite sector in $\mathbb{R}^2$. Our main result in this direction is as follows.

Theorem 5. *Let $\Omega \subseteq \mathbb{R}^2$ be the interior of an infinite angle of aperture $\theta \in (0, 2\pi)$. Denote by $D := (\partial\Omega)_1$ and $N := (\partial\Omega)_2$ the left ray, and respectively the right ray* *of* Ω*. Then the mixed boundary value problem* $(MBVP_p)$ *from* (1) *is well-posed whenever*

$$p \neq \begin{cases} \frac{2\pi-\theta}{\pi-\theta} & \text{if } \theta \in (0,\pi/2] \\ \frac{2\pi-\theta}{\pi-\theta}, \frac{2\theta}{2\theta-\pi} & \text{if } \theta \in (\pi/2,\pi) \\ 2 & \text{if } \theta = \pi \\ \frac{2\theta}{2\theta-\pi}, \frac{\theta}{\theta-\pi} & \text{if } \theta \in (\pi,3\pi/2] \\ \frac{2\theta}{2\theta-\pi}, \frac{2\theta}{2\theta-3\pi}, \frac{\theta}{\theta-\pi} & \text{if } \theta \in (3\pi/2,2\pi) \,. \end{cases} \tag{163}$$

A key step in the proof of Theorem 5 is the following result. To state it, recall the operators $\mathscr{S}$ from (45) and K^* from (48).

Proposition 1. *Let* Ω *be a graph Lipschitz domain in* $\mathbb{R}^2$ *and* $p \in (1,\infty)$ *be such that the operators* $\frac{1}{2}I + K^*$ *and* $-\frac{1}{2}I + K^*$ *are invertible on* $L^p(\partial\Omega)$*. Then the following holds*

$$\begin{gathered} \Delta u = 0 \ \text{ in } \ \Omega \ \text{ and } \ \mathscr{N}(\nabla u) \in L^p(\partial\Omega) \Longleftrightarrow u = \mathscr{S} f + c \\ \text{for some function } \ f \in L^p(\partial\Omega) \ \text{ and some constant } \ c \,. \end{gathered} \tag{164}$$

Proof. The right-to-left implication in (164) is immediate using the fact that for each $f \in L^p(\partial\Omega)$ the function $\mathscr{S} f$ is harmonic in Ω and Theorem 2.

We turn now our attention to the left-to-right implication in (164) and fix for the moment a function

$$w \in \mathscr{C}^\infty(\Omega) \ \text{ such that } \ \Delta w = 0 \ \text{ in } \ \Omega \ \text{ and } \ \mathscr{N}(\nabla w) \in L^p(\partial\Omega) \,. \tag{165}$$

Using a Fatou type result for harmonic functions with pointwise finite non-tangential maximal operator in Lipschitz domains (obtained by combining [15, Theorem 6.4, pp. 112] and [11, Theorem 1]) it follows that

$$\text{there exists } (\nabla w)\Big|_{\partial\Omega}^{n.t} \ \sigma\text{-a.e. on } \partial\Omega \text{ and } (\nabla w)\Big|_{\partial\Omega}^{n.t} \in (L^p(\partial\Omega))^2 \,. \tag{166}$$

Consequently,

$$\text{there exist } \partial_\tau w = \nu_1(\partial_2 w)\Big|_{\partial\Omega}^{n.t} - \nu_2(\partial_1 w)\Big|_{\partial\Omega}^{n.t} \ \text{ and } \ \frac{\partial w}{\partial \nu} \ \sigma\text{-a.e. on } \partial\Omega \,, \tag{167}$$

and

$$\partial_\tau w \in L^p(\partial\Omega) \ \text{ and } \ \frac{\partial w}{\partial \nu} \in L^p(\partial\Omega) \,. \tag{168}$$

For each fixed $j \in \{1,2\}$ and $X \in \Omega$ consider next the vector field $\mathbf{F} = (F_1, F_2)$ in Ω given by

$$F_1 := -(\partial_1 E)(X-\cdot)(\partial_j w) + \delta_{1j}(\partial_k E)(X-\cdot)(\partial_k w) - (\partial_j E)(X-\cdot)(\partial_1 w) \,, \tag{169}$$

$$F_2 := -(\partial_2 E)(X-\cdot)(\partial_j w) + \delta_{2j}(\partial_k E)(X-\cdot)(\partial_k w) - (\partial_j E)(X-\cdot)(\partial_2 w) \,. \tag{170}$$

Above E is the fundamental solution for the Laplacian as given in (44). A straightforward calculation based on (169)-(170) gives that $\operatorname{div}\mathbf{F} = \delta_X \partial_j w$ in the sense of distributions in Ω, where δ_X denotes the Dirac delta distribution with mass at X. In addition, (167)-(168) combined with the fact whenever $X \in \Omega$ is fixed

$$\mathscr{N}\left(\frac{1}{|X-\cdot|}\right) \in L^q(\Omega) \text{ for each } q \in (1,\infty)\,, \tag{171}$$

guarantee that the remaining conditions of the Divergence Theorem from [27] are satisfied. In turn, this yields in the case $j = 1$

$$\partial_1 w = \partial_2 \mathscr{S}(\partial_\tau w) - \partial_1 \mathscr{S}\left(\frac{\partial w}{\partial \nu}\right) \text{ in } \Omega\,, \tag{172}$$

and in the case $j = 2$

$$\partial_2 w = -\partial_1 \mathscr{S}(\partial_\tau w) - \partial_2 \mathscr{S}\left(\frac{\partial w}{\partial \nu}\right) \text{ in } \Omega\,. \tag{173}$$

Assuming that in addition to (165) the function w also satisfies

$$\frac{\partial w}{\partial \nu} = 0 \;\; \sigma\text{-a.e. on } \partial\Omega\,, \tag{174}$$

taking non-tangential traces in (172) and (173), and using the jump relations for the gradient of $\mathscr{S}$ from (54), we arrive at

$$\partial_1 w\Big|_{\partial\Omega}^{n.t.} = -\tfrac{1}{2}\nu_2 \partial_\tau w + p.v.\int_{\partial\Omega} (\partial_2 E)(\cdot - Q)(\partial_\tau w)(Q)\, d\sigma(Q)\,, \tag{175}$$

$$\partial_2 w\Big|_{\partial\Omega}^{n.t.} = \tfrac{1}{2}\nu_1 \partial_\tau w - p.v.\int_{\partial\Omega} (\partial_1 E)(\cdot - Q)(\partial_\tau w)(Q)\, d\sigma(Q)\,. \tag{176}$$

Combining (175) and (176) together with the fact that, as seen from (48), the integral kernel of the operator K^* is $\nu_1(P)(\partial_1 E)(P-Q) + \nu_2(P)(\partial_2 E)(P-Q)$, we obtain

$$\partial_\tau w = \nu_1 \partial_2 w\Big|_{\partial\Omega}^{n.t.} - \nu_2 \partial_1 w\Big|_{\partial\Omega}^{n.t.} = \tfrac{1}{2}\partial_\tau w - K^*(\partial_\tau w) \text{ on } \partial\Omega\,. \tag{177}$$

Hence $(\frac{1}{2}I + K^*)(\partial_\tau w) = 0$ and using the membership of $\partial_\tau w$ in $L^p(\partial\Omega)$ from (168) along with the invertibility of the operator $\frac{1}{2}I + K^*$ on $L^p(\partial\Omega)$ this further yields $\partial_\tau w = 0$. Using this information together with the assumption (174) and identities (172)-(173) allows us to conclude that $\nabla w \equiv 0$ in Ω and thus w is constant in Ω. To summarize we have shown so far that

$$\text{if } w \text{ satisfies (165) and (174) then } w \text{ is constant in } \Omega\,. \tag{178}$$

Next, let u satisfy the conditions (165), i.e., be harmonic in Ω and such that $\mathscr{N}(\nabla u) \in L^p(\partial\Omega)$. On grounds of (168), we obtain that $\frac{\partial u}{\partial \nu} \in L^p(\partial\Omega)$. Using this and the $L^p(\partial\Omega)$ invertibility of the operator $-\frac{1}{2}I+K^*$ it is therefore meaningful to define

$$w := u - \mathscr{S}\left[(-\tfrac{1}{2}I+K^*)^{-1}\left(\frac{\partial u}{\partial\nu}\right)\right] \text{ in } \Omega\,. \tag{179}$$

Notice that clearly w is harmonic in Ω. Using the properties of u along with item *(2)* from Theorem 2, it immediately follows that $\mathscr{N}(\nabla w)\in L^p(\partial\Omega)$. In addition, using item *(4)* from Theorem 2 we may write

$$\frac{\partial w}{\partial\nu} = \frac{\partial u}{\partial\nu} - (-\tfrac{1}{2}+K^*)(-\tfrac{1}{2}I+K^*)^{-1}\left(\frac{\partial u}{\partial\nu}\right) = 0, \tag{180}$$

i.e. w also satisfies (174). Consequently, using (178), we further obtain that w is a constant c in Ω. This and (179) immediately yield that $u=\mathscr{S}f+c$ for the function $f := (-\frac{1}{2}I+K^*)^{-1}\left(\frac{\partial u}{\partial\nu}\right)\in L^p(\Omega)$. This finishes the proof of the direct implication in (164) and completes the proof of the proposition.

We are now ready to present the proof of Theorem 5.

Proof. Fix $p\in(1,\infty)$ satisfying (163). The existence of a solution for $(MBVP_p)$ follows immediately from Theorem 3.

Turning our attention to proving uniqueness, assume that u is a solution of the homogeneous mixed value problem $(MBVP_p)$ (i.e. $f_D=0$ on D and $f_N=0$ on N). In particular

$$\Delta u = 0 \text{ in } \Omega \text{ and } \mathscr{N}(\nabla u)\in L^p(\partial\Omega)\,. \tag{181}$$

Our goal is to show that $u\equiv 0$ in Ω.

However, if Ω is a sector of aperture $\theta\in(0,2\pi)$ the operators $\pm\frac{1}{2}I+K^*$ are invertible on $L^p(\partial\Omega)$ whenever $p\in(1,\infty)$ is such that

$$p\neq\begin{cases}\frac{2\pi-\theta}{\pi-\theta} \text{ if } 0<\theta<\pi\,,\\ \frac{\theta}{\theta-\pi} \text{ if } \pi<\theta<2\pi\,,\end{cases} \tag{182}$$

(see e.g., [38]). Since (163) guarantees that (182) holds, it follows that the hypotheses of Proposition 1 are satisfied by the sector Ω and the integrability exponent p. Consequently, using (181) and Proposition 1 we obtain

$$u=\mathscr{S}f+c \text{ for some function } f\in L^p(\partial\Omega) \text{ and some constant } c\,. \tag{183}$$

Thus $0=u\big|_D^{n.t.}=(Sf)|_D+c$ and $0=\frac{\partial u}{\partial\nu}\Big|_N=(-\frac{1}{2}I+K^*)f=0$. In particular this ultimately yields $T(f_D,f_N)=(0,0)\in L^p(D)\oplus L^p(N)$. Finally, since p is as in (163), Theorem 3 applies and gives that T is invertible on $L^p(D)\oplus L^p(N)$. In particular $f=0$ on $\partial\Omega$ and thus u is a constant in Ω. Since $u\big|_D^{n.t.}=0$ this ultimately allows us to conclude that $u\equiv 0$ in Ω, as desired. This completes the proof of the theorem.

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