Math 311
Spring 2018

Dr. Hussein Awala

Day #4 Notes: Consequences of Completeness

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Contents

Axiom 1 *Every nonempty subset of* $\mathbb{R}$ *that is bounded above has a least upper bound.*

What does this mean?

Definition 1 *For a set* $A \subset \mathbb{R}$*, we say*

- A *is bounded above if there exists a number* $b \in \mathbb{R}$ *such that* $a \leq b$ *for all* $a \in A$*. The number* b *is called an upper bound for* A*.*
- $a_0 = \max(A)$ *if* $a_0 \in A$ *and* $a_0 \geq a$ *for all* $a \in A$ *We might also say* a_0 *is the maximum of* A*.*
- *A real number* s *is the least upper bound (or sometimes, we use the Latin word supremum) for a set* $A \subset \mathbb{R}$ *if* s *is an upper bound for* A *and if for every upper bound* b *of* A*, we have* $s \leq b$*. If* s *exists, we use the notation* $s = \sup(A)$*.*

1 Characterization of Least Upper Bounds

Complete the worksheet and we will go over the answers:

In the problems below, assume s is an upper bound for A and that $A \neq \emptyset$.

1. [T/F] Then $s = \sup(A)$ if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.

2. [T/F] Then $s = \sup(A)$ only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.

3. [T/F] Then $s = \sup(A)$ if for every $\epsilon 0$, there exists $a \in A$ such that $s - \epsilon > a$.

4. [T/F] Then $s = \sup(A)$ only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon > a$.

5. [T/F] Then $s = \sup(A)$ if for every $\epsilon > 0$, there exists $a \in A$ such that $s + \epsilon > a$.

6. [T/F] Then $s = \sup(A)$ only if for every $\epsilon > 0$, there exists $a \in A$ such that $s + \epsilon > a$.

2 Consequences of Completeness

Theorem 1 (Nested Interval Property) *Suppose that* $(I_n)_{n\in\mathbb{N}}$ *is a nested sequence of closed intervals. That is,* $\forall n \in \mathbb{N}$, $I_n = [a_n, b_n]$ *and* $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. *Then* $\exists x \in \mathbb{R}$ *so that* $x \in \cap_{n=1}^{\infty} I_n$.

Proof:

(continued)

Theorem 2 (Archimedean Property) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ *so that* $n > x$, *and* $\forall y > 0$, $\exists m \in \mathbb{N}$ *so that* $y > \frac{1}{m}$.

Proof:

What does this mean? Why is it useful?

Density of $\mathbb{Q}$ in $\mathbb{R}$:

3 Conclusions

Today we learned about:

1. Implications of the Axiom of Completeness, including:
   (a) The Nested Interval Property
   (b) The Density of the Rationals in the Reals.

Monday we will learn about:

1. More Consequences of Completeness
2. Cardinality

Upcoming Deadlines:

Homework 1 due on Wednesday January 31, 2018