Math 311
Spring 2018
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Day #6 Notes: More Consequences of Completeness

January 31, 2018

Contents

1 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1 *If* $x, y \in \mathbb{R}$ *and* $x < y$*, then* $\exists\, r \in \mathbb{Q}$ *so that* $x < r < y$.

Proof:

2 Existence of Roots

Theorem 2 *There exists a real number whose square is 2.*

Proof: Consider the set $S = \{a \in \mathbb{R} \text{ s.t. } a^2 < 2\}$

$S \neq \phi$ since $1 \in S$, and S is bounded above by 2.

for any $a \in S$ $a < 2$ otherwise

if $a > 2 \Rightarrow a^2 > 4$. Contradiction.

Hence S has a sup. Let $s = \sup(S)$. claim: $s^2 = 2$, we will prove this by contradicting the fact $s^2 < 2$ and $s^2 > 2$.

suppose $s^2 < 2$, $\Rightarrow$ $2 - s^2 > 0$ $\boxed{s > 1 \Rightarrow 2s+1 > 0}$

find n s.t. $\frac{2-s^2}{2s+1} > \frac{1}{n}$

$$2 - s^2 > \frac{2s+1}{n} \Rightarrow 2 - s^2 - \frac{2s}{n} - \frac{1}{n} > 0$$

$$2 > s^2 + \frac{2s}{n} + \frac{1}{n}$$

$$> s^2 + \frac{2s}{n} + \frac{1}{n^2} = \left(s + \frac{1}{n}\right)^2$$

$\Rightarrow$ $s + \frac{1}{n} \in S$

but $s < s + \frac{1}{n}$ contradiction $\Rightarrow$ $s^2 \geqslant \sqrt{2}$

we need to find $n > 0$ s.t. $(s + \frac{1}{n})^2 < 2$

$s^2 + \frac{2s}{n} + \frac{1}{n^2} < 2$

$s^2 + \frac{2s}{n} + \frac{1}{n} < 2$

$s^2 + \frac{2s+1}{n} < 2$

$\frac{2s+1}{n} < 2 - s^2$

$\frac{2s+1}{2-s^2} < n$

$\frac{2-s^2}{2s+1} > \frac{1}{n}$

Suppose that: $s^2 > 2$

we will find n s.t. $(s - \frac{1}{n})^2 > 2$

$s^2 - \frac{2s}{n} + \frac{1}{n^2} > 2$

$s^2 - 2 > \frac{2s}{n} - \frac{1}{n^2}$

$s^2 - 2 > \frac{2s}{n}$

$\frac{s^2-2}{2s} > \frac{1}{n}$.

3 Cardinality

Complete the worksheet in class and we will go over it together:

Definition 1 *Two sets A and B have the same cardinality if there exists a function $f : A \to B$ that is one-to-one and onto. In this case, we write $A \sim B$.*

Definition 2 *We say a set has cardinality n if $A \sim \{1, \ldots, n\}$. If $\exists n \in \mathbb{N}$ so that A has cardinality n, we say that A is finite. We say a set A is countable if $A \sim \mathbb{N}$. If A is neither finite nor countable, then we say A is uncountable.*

1. [T/F] If $A = \{1, 2, 3\}$ and $B = \{e, \pi, \sqrt{2}\}$ then $A \sim B$. T

   define $f : A \longrightarrow B$ s.t.

   $\left.\begin{array}{l} f(1) = e \\ f(2) = \pi \\ f(3) = \sqrt{2} \end{array}\right\}$ f is injective and surjective.

2. [T/F] The even integers $2\mathbb{Z}$ have the same cardinality as the integers; that is, $2\mathbb{Z} \sim \mathbb{Z}$. T

   define the function:

   $$f : \mathbb{Z} \longrightarrow 2\mathbb{Z}$$
   $$x \longrightarrow 2x$$

   $f(x) = 2x$.

$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$

$1, 2, 3, 4, 5, 6, 7$

3. [T/F] $\mathbb{Z} \sim \mathbb{N}$; that is, the integers are countable. T

define the function: $f: \mathbb{Z} \longrightarrow \mathbb{N}$

$$f(z) = \begin{cases} 2z+2 & \text{if } z \geq 0 \\ -2z-1 & \text{if } z < 0 \end{cases}$$

4. [T/F] $\mathbb{Q}$ is countable. T

creat the sets

$$A_n = \left\{ \frac{p}{q}, \text{s.t. } p, q \in \mathbb{N},\ p+q=n \text{ and, } p \text{ and } q \text{ are coprime} \right\}$$

A_n's are all finite, they cover $\mathbb{Q}$, and $A_n \cap A_m = \emptyset \ \forall\, n \neq m$.

5. [T/F] $\mathbb{R}$ is countable. F

4 Conclusions

Today we learned about:

1. Implications of the Axiom of Completeness, including:
   (a) The Density of the Rationals in the Reals.
   (b) The existence of roots.
   (c) The cardinality of $\mathbb{R}$.

Friday we will learn about:

1. More on cardinality
2. A start on sequences and series

Upcoming Deadlines:

- Wednesday, Feb 7, 2018: Homework #2.

Questions?