

Day #6 Notes: More Consequences of Completeness

January 31, 2018

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1 Density of \mathbb{Q} in \mathbb{R}

Theorem 1 *If $x, y \in \mathbb{R}$ and $x < y$, then $\exists r \in \mathbb{Q}$ so that $x < r < y$.*

Proof:

2 Existence of Roots

Theorem 2 *There exists a real number whose square is 2.*

Proof: Consider the set $S = \{a \in \mathbb{R} \text{ s.t. } a^2 < 2\}$

$S \neq \emptyset$ since $1 \in S$, and S is bounded above by 2.

for any $a \in S$ $a < 2$ otherwise

if $a > 2 \Rightarrow a^2 > 4$. Contradiction.

Hence S has a sup. Let $s = \sup(S)$. claim: $s^2 = 2$, we will prove this by contradicting the fact $s^2 < 2$ and $s^2 > 2$.

suppose $s^2 < 2$, $\Rightarrow 2 - s^2 > 0$ $s > 1 \Rightarrow 2s+1 > 0$

find n s.t. $\frac{2-s^2}{2s+1} > \frac{1}{n}$

$$2 - s^2 > \frac{2s+1}{n} \Rightarrow 2 - s^2 - \frac{2s}{n} - \frac{1}{n} > 0$$

$$2 > s^2 + \frac{2s}{n} + \frac{1}{n}$$

$$> s^2 + \frac{2s}{n} + \frac{1}{n^2} = \left(s + \frac{1}{n}\right)^2$$

$$\Rightarrow s + \frac{1}{n} \in S$$

$$\text{but } s < s + \frac{1}{n} \text{ contradiction} \Rightarrow s^2 \geq 2$$

suppose that: $s^2 > 2$
we will find n s.t. $(s - \frac{1}{n})^2 > 2$

$$s^2 - \frac{2s}{n} + \frac{1}{n^2} > 2$$

$$s^2 - 2 > \frac{2s}{n} - \frac{1}{n^2}$$

$$s^2 - 2 > \frac{2s}{n}$$

$$\frac{s^2 - 2}{2s} > \frac{1}{n}$$

we need to find
 $n > 0$ s.t.
 $(s + \frac{1}{n})^2 < 2$

$$s^2 + \frac{2s}{n} + \frac{1}{n^2} < 2$$

$$s^2 + \frac{2s}{n} + \frac{1}{n} < 2$$

$$s^2 + \frac{2s+1}{n} < 2$$

$$\frac{2s+1}{n} < 2 - s^2$$

$$\frac{2s+1}{2-s^2} < n$$

$$\frac{2-s^2}{2s+1} > \frac{1}{n}$$

3 Cardinality

Complete the worksheet in class and we will go over it together:

Definition 1 Two sets A and B have the same cardinality if there exists a function $f : A \rightarrow B$ that is one-to-one and onto. In this case, we write $A \sim B$.

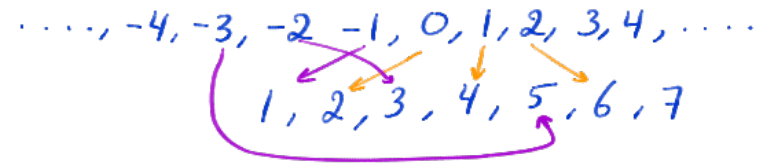
Definition 2 We say a set has cardinality n if $A \sim \{1, \dots, n\}$. If $\exists n \in \mathbb{N}$ so that A has cardinality n , we say that A is finite. We say a set A is countable if $A \sim \mathbb{N}$. If A is neither finite nor countable, then we say A is uncountable.

1. [T/F] If $A = \{1, 2, 3\}$ and $B = \{e, \pi, \sqrt{2}\}$ then $A \sim B$. **T**

define $f: A \rightarrow B$ st.
 $f(1) = e$
 $f(2) = \pi$
 $f(3) = \sqrt{2}$ } f is injective
and surjective.

2. [T/F] The even integers $2\mathbb{Z}$ have the same cardinality as the integers; that is, $2\mathbb{Z} \sim \mathbb{Z}$. **T**

define the function:
 $f: \mathbb{Z} \rightarrow 2\mathbb{Z}$
 $x \rightarrow 2x$
 $f(x) = 2x$.



3. [T/F] $\mathbb{Z} \sim \mathbb{N}$; that is, the integers are countable. **T**

define the function: $f: \mathbb{Z} \rightarrow \mathbb{N}$

$$f(z) = \begin{cases} 2z+2 & \text{if } z \geq 0 \\ -2z-1 & \text{if } z < 0 \end{cases}$$

4. [T/F] \mathbb{Q} is countable. **T**

creat the sets

$$A_n = \left\{ \frac{p}{q}, \text{ s.t. } p, q \in \mathbb{N}, p+q=n \text{ and, } p \text{ and } q \text{ are coprime} \right\}$$

A_n 's are all finite, they cover \mathbb{Q} , and $A_n \cap A_m = \emptyset \forall n \neq m$.

5. [T/F] \mathbb{R} is countable. **F**

4 Conclusions

Today we learned about:

1. Implications of the Axiom of Completeness, including:
 - (a) The Density of the Rationals in the Reals.
 - (b) The existence of roots.
 - (c) The cardinality of \mathbb{R} .

Friday we will learn about:

1. More on cardinality
2. A start on sequences and series

Upcoming Deadlines:

- Wednesday, Feb 7, 2018: Homework #2.

Questions?