Math 311
Spring 2018
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Day #10 Notes: Properties of Limits

February 9, 2018

Contents

1 Properties of Limits

Definition 1 *A sequence is* bounded *if* $\exists M > 0$ *so that* $\forall n \in \mathbb{N},\ |a_n| < M$.

Proposition 1 *Every convergent sequence is bounded.*

proof: Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence $\Rightarrow$ $\exists\ a \in \mathbb{R}$ s.t. $\lim_{n\to\infty} a_n = a$.

so fix $\varepsilon = 1$, then $\exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$ $|a_n - a| < 1$

Next, let $M = \max\{|a_1|, \dots, |a_{N-1}|, |a|+1\}$

claim : $|a_n| \leq M$ for all $n \in \mathbb{N}$.

if $n \in \{1, \dots, N-1\} \Rightarrow$ by def $|a_n| \leq M$

if $n \geq N \Rightarrow |a_n - a| < 1$

$\Rightarrow |a_n| - |a| < 1$

$\Rightarrow |a_n| < |a| + 1 \leq M$

to prove that

$|a-b| > |a| - |b|$

start with

$|a| = |a - b + b|$

$\leq |a-b| + |b|$

$\Rightarrow |a| - |b| \leq |a-b|$

Theorem 1 (Algebraic Limit Theorem) *Suppose that (a_n) and (b_n) are sequences and $a, b, c \in \mathbb{R}$. Suppose $a_n \to a$ and $b_n \to b$. Then:*

1. $(ca_n) \to ca$
2. $(a_n + b_n) \to a + b$
3. $(a_n b_n) \to ab$
4. *If* $b \neq 0$, $(\frac{a_n}{b_n} \to \frac{a}{b})$.

Proof: 1 Let $\varepsilon > 0$ since $a_n \xrightarrow[n\to\infty]{} a$ and for $\varepsilon' = \varepsilon/c > 0$

$\Rightarrow \exists\, N \in \mathbb{N}$ s.t. for any $n \geqslant N$ we have $|a_n - a| < \varepsilon' < \varepsilon/c$

then for the same $N \in \mathbb{N}$ for any $n \geqslant N$ $|ca_n - ca| = c|a_n - a| < c\,\varepsilon/c = \varepsilon$.

thus $\lim_{n\to\infty} c a_n = c a$.

2 $a_n + b_n \longrightarrow a + b$

Let $\varepsilon > 0 \Rightarrow \varepsilon/2 > 0$

$\to$ There exist $N_1 > 0$ s.t. $\forall\, n \geqslant N_1$, $|a_n - a| < \varepsilon/2$

$\to$ There exists $N_2 > 0$ s.t. $\forall\, n \geqslant N_2$, $|b_n - b| < \varepsilon/2$

then take $N = \max\{N_1, N_2\}$

then for any $n \geqslant N \Rightarrow |a_n - b_n - a - b| \leqslant |a_n - a| + |b_n - b| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$.

(continued)

(3) *Hint:*

$$|a_n b_n - ab| = |a_n b_n - a b_n + a b_n - ab|$$

$$\leq |b_n||a_n - a| + |a||b_n - b|$$

and use the fact that $(b_n)_{n \in \mathbb{N}}$ is bounded.

(continued)

Theorem 2 (Order Limit Theorem) *Suppose* $a, b, c \in \mathbb{R}$ *and* $(a_n), (b_n)$ *are sequences of real numbers so that* $a_n \to a$ *and* $b_n \to b$*. Then*

1. *If* $a_n \geq 0 \ \forall n \in \mathbb{N}$, $a \geq 0$.

2. *If* $a_n \leq b_n \ \forall n \in \mathbb{N}$, $a \leq b$.

3. *If* $c \leq b_n \ \forall n \in \mathbb{N}$, $c \leq b$ *and similarly, if* $a_n \leq c \ \forall n \in \mathbb{N}$, *then* $a \leq c$.

Proof:

(continued)

2 Conclusions

Today we learned about:

1. Properties of limits

Monday we will learn about:

1. The Monotone Convergence Theorem

2. Infinite series

Upcoming Deadlines:

- Wednesday February 14, 2018: Homework #3