Math 311
Spring 2018
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Day #11 Notes: Properties of Limits and Monotone Convergence

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Contents

1 Properties of Limits

[: *O*rder Limit Theorem] Suppose $a, b, c \in \mathbf{R}$ and $(a_n), (b_n)$ are sequences of real numbers so that $a_n \to a$ and $b_n \to b$. Then

1. If $a_n \geq 0 \; \forall n \in \mathbf{N}$, $a \geq 0$.
2. If $a_n \leq b_n \; \forall n \in \mathbf{N}$, $a \leq b$.
3. If $c \leq b_n \; \forall n \in \mathbf{N}$, $c \leq b$ and similarly, if $a_n \leq c \; \forall n \in \mathbf{N}$, then $a \leq c$.

Proof: 1) Let $a_n \geq 0$

Suppose $a < 0 \Rightarrow a/2 < 0$

then for $\varepsilon = -a/2 > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$|a_n - a| < -a/2$$

$$\Rightarrow a_n - a < -a/2$$

$$\Rightarrow a_n < -a/2 + a = a/2 < 0$$

contradiction since $a_n \geq 0$ for all $n \in \mathbb{N}$

(continued)

2) $a_n \leq b_n \implies b_n - a_n \geq 0$

so $\lim_{n\to\infty} b_n - a_n \geq 0$

$\implies \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n \geq 0 \implies b - a \geq 0 \implies b \geq a$

3) if $c \leq b_n \implies b_n - c \geq 0$

$\implies \lim_{n\to\infty} b_n - c \geq 0$

$\implies b - c \geq 0 \implies b \geq c.$

<u>Note</u>: strict inequalities doesn't hold under limits

eg: $a_n = \frac{1}{n} > 0$

but $\lim_{n\to\infty} a_n = 0.$

2 Monotonicity

Worksheet:

Definition: A sequence (x_n) is *convergent* if there exists some real number x so that (x_n) converges to x.

Definition: A sequence (x_n) is *bounded* if there exists a real number $M > 0$ such that $|x_n| < M$ for all $n \in \mathbf{N}$. ✓ we have seen this before.

Definition: A sequence (x_n) is *increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbf{N}$ and *decreasing* if $x_n \geq x_{n+1}$ for all $n \in \mathbf{N}$. A sequence is *monotone* if it is either increasing or decreasing.

1. [T/F] Every bounded sequence is convergent. **F**

   $$x_n = \begin{cases} 1 + \frac{1}{n} & \text{if } n \text{ is even} \\ -1 + \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases} \leftarrow \text{try to show this is not convergent.}$$

2. [T/F] Every convergent sequence is bounded. **T**

   We proved this before.

3. [T/F] Every monotone sequence is convergent. F

eg: $a_n = n$

4. [T/F] Every convergent sequence is monotone. F

eg: $a_n = \frac{(-1)^n}{n}$ $\quad a_n \xrightarrow[n\to\infty]{} 0$

but a_n is not monotone.

5. [T/F] If a sequence is monotone and bounded, then it is convergent. T

We will see the proof of this later.

Definition: Let x_n be a sequence. An *infinite series* is a formal expression of the form

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \cdots .$$

The corresponding *sequence of partial sums* (s_m) is given by $s_m = x_1 + x_2 + \cdots + x_m$, and we say the series $\sum_{n=1}^{\infty} x_n$ *converges to* S if the sequence (s_m) converges to S. In this case, we write $\sum_{n=1}^{\infty} x_n = S$.

6. [T/F] If (x_n) is a sequence of positive real numbers, then the partial sums for the series $\sum_{n=1}^{\infty} x_n$ form a bounded sequence. F

$x_n = n$ $\qquad \sum_{n=1}^{\infty} n$, then $S_N = \sum_{n=1}^{N} x_n$ is not bounded.

7. [T/F] If (x_n) is a sequence of positive real numbers, then the partial sums for the series $\sum_{n=1}^{\infty} x_n$ form a monotone sequence. T

$S_N = \sum_{n=1}^{N} x_n$, $\qquad S_{N+1} = \sum_{n=1}^{N+1} x_n = \sum_{n=1}^{N} x_n + x_{n+1}$

$$= S_N + x_{n+1}$$

$$\geqslant S_N$$

$\Rightarrow S_N$ is increasing.

8. [T/F] If $\sum_{n=1}^{\infty} x_n$ converges, then $(x_n) \to 0$. **T**

We will prove this next class.

9. [T/F] If $(x_n) \to 0$, then $\sum_{n=1}^{\infty} x_n$ converges. **F**

eg: $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \frac{1}{n} \longrightarrow 0$.

but as we will see next time

$\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

3 Conclusions

Today we learned about:

1. Properties of limits and ordering
2. Monotone Convergence
3. A start on Series

Wednesday we will learn about:

1. More on Monotonicity and Series
2. Subsequences

Upcoming Deadlines:

- This Wednesday: Homework #3
- This Wednesday: Homework #1 Rewrites

Questions?