Math 311
Spring 2018
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Day #19 Closed Sets and Limit point

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Contents

1 **Worksheet: Closed Sets** 2

2 **Limit points** 6

3 **Conclusions** 9

1 Worksheet: Closed Sets

Definition 1 *A set U is closed if it is the complement of an open set.*

16. *Is the empty set closed?*

Yes, because $\emptyset^c = \mathbb{R}$

and $\mathbb{R}$ is open.

17. *Is* $\mathbf{R}$ *closed?*

Yes, because $\mathbb{R}^c = \emptyset$

and $\emptyset$ is open

18. *Is* $\mathbf{R} - \{5\}$ *closed?*

No, because $(\mathbb{R} - \{5\})^c = \{5\}$

and $\{5\}$ is not open.

19. Is $\{5\}$ *closed?*

Yes, because $\{5\}^c = \mathbb{R} - \{5\}$

and $\mathbb{R} - \{5\}$ is open

20. Is $\mathbf{Q}$ *closed?*

No, because $\mathbb{Q}^c$ = Set of Irrationals

and the set of irrationals, just like $\mathbb{Q}$ is not open.

21. Is the interval $(2, 4)$ *closed?*

No, because $(2,4)^c = \underbrace{(-\infty,2]}_{\text{not open}} \cup [4,\infty)$

$\Rightarrow (2,4)$ is not closed.

22. Is the interval $[2, 4]$ *closed?*

Yes, because $[2,4]^c = \underbrace{(-\infty,2)}_{\text{open}} \cup \underbrace{(4,\infty)}_{\text{open}}$

$\Rightarrow [2,4]$ is closed.

23. *Is the interval* $[2,4)$ *closed?*

No, $[2,4)^c = (-\infty,2) \cup \underset{\uparrow \text{not open}}{[4,\infty)}$

$\Rightarrow [2,4)$ is not closed.

24. *Name a non-empty, bounded, closed set that is not of the form* $[a,b]$.

$(2,3) \cap (5,7)$

25. *[T/F] The union of two closed sets is a closed set.* T

Let A, B be two closed sets

$(A \cup B)^c = \underset{\uparrow \text{open}}{A^c} \cap \underset{\uparrow \text{open}}{B^c}$ ← open since intersection of two open sets is open.

26. *[T/F] The union of finitely many closed sets is a closed set.* T

Let $A_1, A_2, \dots, A_n$ be closed

$(A_1 \cup A_2 \cdots \cup A_n)^c = \underset{\uparrow \text{open}}{A_1^c} \cap \underset{\uparrow \text{open}}{A_2^c} \cdots \cap \underset{\uparrow \text{open}}{A_n^c}$ ← open

27. [T/F] *The union of infinitely many closed sets is a closed set.* F

Let $A_n = [\frac{1}{n}, \infty) \leftarrow$ closed

$\bigcup_{n=1}^{\infty} A_n = (0, \infty) \leftarrow$ open.

28. [T/F] *The intersection of two closed sets is a closed set.* T

A, B closed $\Longrightarrow (A \cap B)^c = \underset{\substack{\uparrow \\ \text{open}}}{A^c} \cup \underset{\substack{\uparrow \\ \text{open}}}{B^c} \leftarrow$ open

29. [T/F] *The intersection of finitely many closed sets is a closed set.* T

$A_1, A_2, \dots, A_n$ closed $\Longrightarrow (A_1 \cap A_2 \cdots \cap A_n)^c = \underset{\substack{\uparrow \\ \text{open}}}{A_1^c} \cup \underset{\substack{\uparrow \\ \text{open}}}{A_2^c} \cdots \cup \underset{\substack{\uparrow \\ \text{open}}}{A_n^c} \leftarrow$ open

30. [T/F] *The intersection of infinitely many closed sets is a closed set.* T

Let I be infinite and $\{A_\alpha\}_{\alpha \in I}$ closed sets.

$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} \underset{\substack{\uparrow \\ \text{open}}}{A_\alpha^c} \leftarrow$ open Union of open sets is always open.

2 Limit points

Definition 2 *A point $a \in \mathbf{R}$ is a limit point of the set $A \subset \mathbf{R}$ if, for every $\epsilon > 0$, the ϵ-ball around a intersects A in a point other than a; that is, if $(A \cap B_\epsilon(a)) - \{a\} \neq \emptyset$.*

eg: $A = (2,4)$ 2 is a limit point of A

for any $\varepsilon > 0$ $B_\varepsilon(2) \cap A - \{2\} = \begin{cases} (2, 2+\varepsilon) \neq \emptyset & \text{if } \varepsilon \leq 2 \\ (2,4) \neq \emptyset & \text{if } \varepsilon > 2 \end{cases}$

31. *[T/F] If a is a limit point of A, then $a \in A$.* F

2 in the above eg is a limit point of A but $2 \notin A$.

32. *[T/F] If a is a limit point of A, then there is a sequence $(a_n) \subset A - \{a\}$ with $a_n \to a$.* T

let a be a limit point of A then for each $n \in \mathbb{N}$ $B_{\frac{1}{n}}(a) \cap A - \{a\} \neq \emptyset$

Choose $a_n \in B_{\frac{1}{n}}(a) \cap A - \{a\}$

claim $\lim_{n \to \infty} a_n = a$, let $\varepsilon > 0$ choose N_0 s.t. $\frac{1}{N_0} < \varepsilon$

then for any $n \geq N_0$, $|a_n - a| < \frac{1}{n} \leq \frac{1}{N_0} < \varepsilon$.

33. *[T/F] If there is a sequence $(a_n) \subset A - \{a\}$ with $a_n \to a$, then a is a limit point of A.* T

for any $\varepsilon > 0$ $\exists$ $n \in \mathbb{N}$ s.t. $\underbrace{|a_n - a| < \varepsilon}_{\Downarrow\ a_n \in B_\varepsilon(a)}$ and $a_n \in A - \{a\}$

$\Longrightarrow$ $a_n \in B_\varepsilon(a) \cap A - \{a\}$ so $B_\varepsilon(a) \cap A - \{a\} \neq \emptyset$.

34. [T/F] If the set A is closed, then it contains all its limit points. T

Let $a \in A^c$, we want to show that a is not a limit point for A

since A is closed $\Rightarrow A^c$ is open

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $B_\varepsilon(a) \subset A^c$ $\Rightarrow B_\varepsilon(a) \cap A = \emptyset \Rightarrow B_\varepsilon(a) \cap A - \{a\} = \emptyset \Rightarrow a$ is not a limit point.

35. [T/F] If the set B is open, then it contains all its limit points. F

$(2,4)$ 2 is a limit point but $2 \notin (2,4)$.

36. [T/F] If the set C contains all its limit points, then it is closed. T

We want to show C^c is open. If $a \in C^c \Rightarrow a$ is not a limit point of C

So $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(a) \cap C - \{a\} = \emptyset$, since $a \notin C \Rightarrow B_\varepsilon(a) \cap C = \emptyset$

$\Rightarrow B_\varepsilon(a) \subset C^c$ so C^c is open $\Rightarrow C$ is closed.

37. [T/F] If the set D contains all its limit points, then it is open. F

$[2,4]$ contains all its limit points

but $[2,4]$ is not open

38. [T/F] If the set E contains all its limit points, then it is not open. F

$\mathbb{R}$ contains all its limit points
and $\mathbb{R}$ is open.

3 Conclusions

Today we learned about:

1. *Closed sets*
2. *Limit points*

Wednesday we will learn about:

1. *Isolated points*
2. *Compact sets*

Upcoming Deadlines:

- *This Friday: Homework #6*

Questions?