MST 112 - Exam 1 - Spring 2018

Calculus II

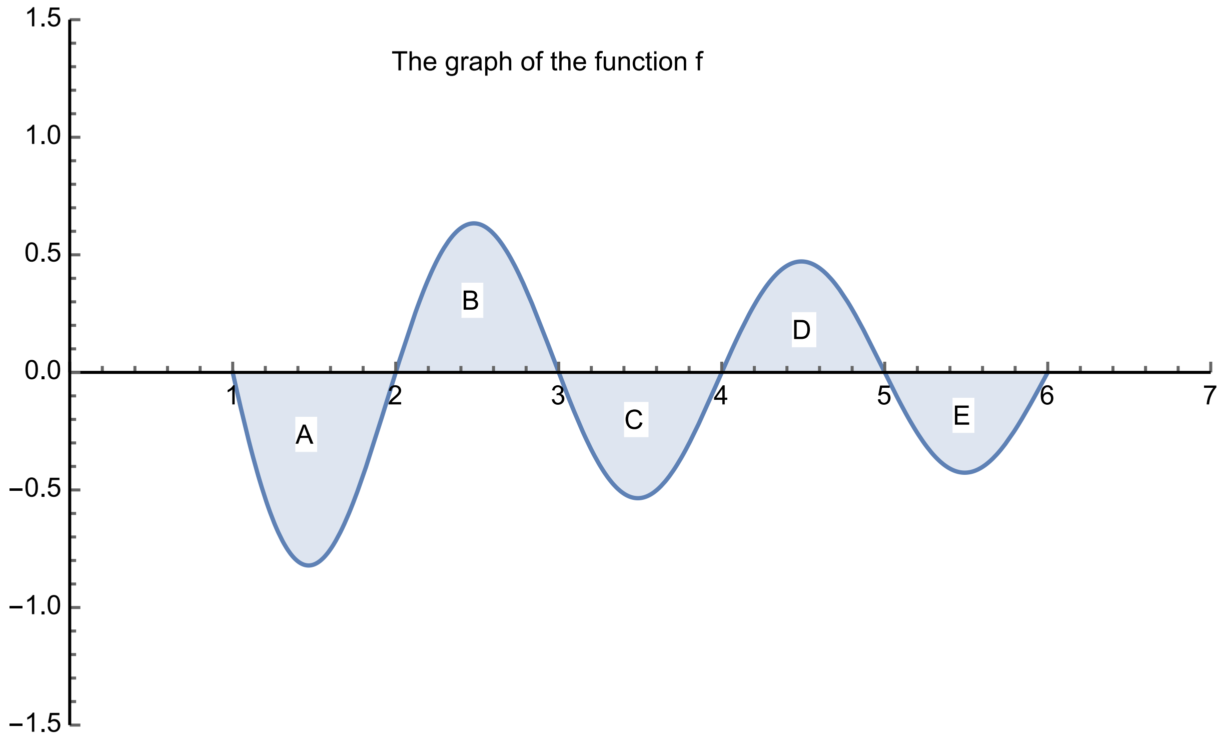
February 27, 2018

Instruction:

- You have 50 minutes to finish this exam.
- No Calculators, phones or laptops are allowed during this exam.
- It is expected that each student during this exam will conduct himself, herself or themselves within the guidelines of the WFU Honor Code. All academic work should be done with the high level of honesty and integrity that the university demands.

Name: Solution

(I) (12pts) Consider the function $g(x) = \int_1^x f(t)\,dt$ with the graph of the function f given below.



We are given the areas : $A = 10, \quad B = 8, \quad C = 6 \quad D = 5.$

a) Find $g(1)$, $g(3)$ and $g(5)$.

$g(1)=0 \qquad g(3)=-2 \qquad g(5)=-3$

b) Knowing that $g(6) = -7$ then find the measure of the area E.

$g(6)=-7 \Rightarrow \int_5^6 f(x)dx = -4 \Rightarrow E=4$

c) Find $g'(4)$.

$g'(4)=0$

d) At what value in $[1, 6]$ does the function g attain its maximum. What is the maximum value?

max at $x=1$ and $g(1)=0$

e) On what interval(s) is the function g increasing?

on $(2,3)$ and $(4,5)$

f) On what interval(s) is the function g concave down?

on $(1, 1.5)$, $(2.5, 3.5)$ and $(4.5, 5.5)$

(II) (15pts, 5 each) Evaluate the following integrals:

a) $\int \cos^4(x)\,dx = \int (\cos^2 x)^2 dx = \int \left(\frac{\cos(2x)+1}{2}\right)^2 dx = \frac{1}{4}\int \cos^2(2x) + 2\cos(2x) + 1\,dx$

$= \frac{1}{4}\int \frac{\cos(4x)+1}{2} + 2\cos(2x) + 1\;dx$

$= \frac{1}{4}\int \frac{\cos(4x)}{2} + 2\cos(2x) + 3/2\;dx$

$= \frac{1}{4}\left(\frac{\sin(4x)}{8} + \sin(2x) + 3/2\,x\right) + C$

b) $\int_e^{e^2} \frac{\ln\left(\ln(x^2)\right)}{x}\,dx$

let $u = \ln(x^2) = 2\ln(x)$

$du = \frac{2}{x}dx$ or $\frac{dx}{x} = \frac{1}{2}du$

$x = e \Rightarrow u = 2$

$x = e^2 \Rightarrow u = 4$

$= \frac{1}{2}\int_2^4 \ln(u)\,du$

$w = \ln(u)$ $\quad w' = \frac{1}{u}$

$v' = 1$ $\quad v = u$

$= \frac{1}{2}\left(u\ln(u)\Big]_2^4 - \int_2^4 \frac{1}{u}u\,du\right)$

$= \frac{1}{2}\left(4\ln(4) - 2\ln(2) - u\Big]_2^4\right)$

$= \frac{1}{2}\left(8\ln(2) - 2\ln(2) - (4-2)\right)$

$= \frac{1}{2}\left(6\ln(2) - 2\right) = 3\ln(2) - 1$

c) $\int_0^1 xe^{x^2} \sin\left(e^{x^2}\right) dx$

$u = e^{x^2}$ $\quad$ $x=0 \Rightarrow u=1$

$du = 2xe^{x^2} dx$ $\quad$ $x=1 \Rightarrow u=e$

$\frac{du}{2} = xe^{x^2} dx$

$$\frac{1}{2}\int_1^e \sin(u)\,du$$

$$= \frac{1}{2}\left(-\cos(u)\Big]_1^e\right)$$

$$= \frac{1}{2}\left(-\cos(e) + \cos(1)\right)$$

$$= \frac{1}{2}\left(\cos(1) - \cos(e)\right).$$

(III) (5pts) Let $g(x) = \int_{x+1}^{\arcsin(x)} sin(t)\, dt$ for $0 < x < \pi/2$.

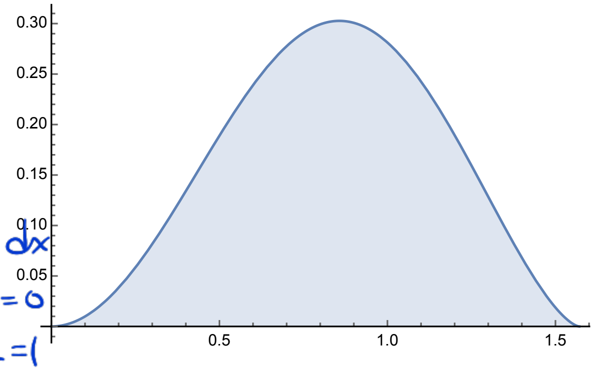
Use the Fundamental Theorem of Calculus to find the derivative $g'(x)$. Simplify your answer.

$$g'(x) = \sin(\arcsin(x))\,(\arcsin(x))' - \sin(x+1) \cdot (x+1)'$$

$$= x \cdot \frac{1}{\sqrt{x^2-1}} - \sin(x+1)$$

$$= \frac{x}{\sqrt{x^2-1}} - \sin(x+1).$$

(IV) (6pts) Consider the shaded area shown below, bounded by the function $f(x) = \sin^2(x)\sqrt{\cos^3(x)}$ and the $x-axis$ between $x = 0$ and $x = \pi/2$. Find the volume that is generated by rotating the shaded area around the $x-axis$.



$$V = \pi \int_0^{\pi/2} \left(\sin^2(x)\sqrt{\cos^3 x}\right)^2 dx$$

$$= \pi \int_0^{\pi/2} \sin^4 x \underbrace{\cos^3 x}_{\substack{\cos^2 x \;\; \cos x \\ (1-\sin^2 x)\cos x}} dx$$

$\cos x \, dx = du$

$u = \sin x$

$du = \cos x \, dx$

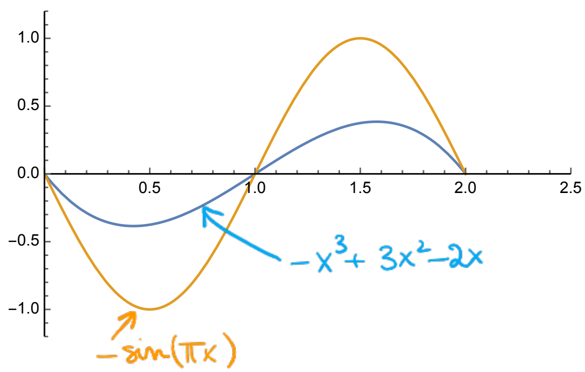
$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{2} \Rightarrow u = 1$

$$= \pi \int_0^1 u^4(1-u^2)\,du = \pi \int_0^1 u^4 - u^6\,du$$

$$= \pi\left(\frac{u^5}{5} - \frac{u^7}{7}\Big]_0^1\right) = \pi\left(\frac{1}{5} - \frac{1}{7}\right)$$

(V) (5pts) Find the area bounded between the graph of the function $f(x) = -x^3 + 3x^2 - 2x$ and the function $g(x) = -\sin(\pi x)$ for $0 \le x \le 2$. (See the graphs of the functions in the figure).



$$= \int_0^1 (-x^3 + 3x^2 - 2x) - (-\sin(\pi x))\,dx$$

$$+ \int_1^2 (-\sin(\pi x) - (-x^3 + 3x^2 - 2x))\,dx$$

$$= \left(-\frac{x^4}{4} + x^3 - x^2 - \frac{\cos(\pi x)}{\pi}\right)\Big]_0^1 + \left(\frac{\cos(\pi x)}{\pi} + \frac{x^4}{4} - x^3 + x^2\right)\Big]_1^2$$

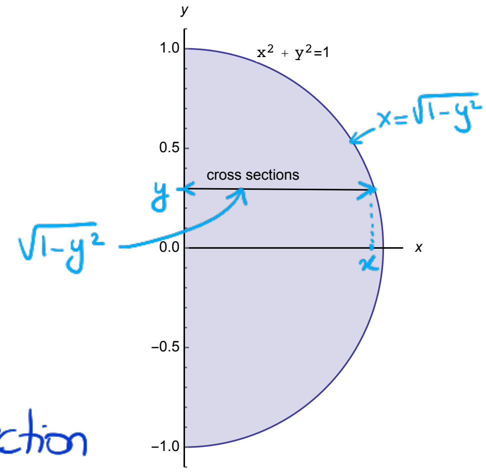
$$= \left(-\frac{1}{4} + \cancel{1} - \cancel{1} - \frac{\cos(\pi)}{\pi}\right) - \left(-\frac{\cos(0)}{\pi}\right) + \left(\frac{\cos(2\pi)}{\pi} + \overset{4}{\cancel{\tfrac{16}{4}}} - \cancel{8} + 4\right) - \left(\frac{\cos(\pi)}{\pi} + \frac{1}{4} - \cancel{1} + \cancel{1}\right)$$

$$-\frac{1}{4} + \frac{1}{\pi} + \frac{1}{\pi} \quad + \quad \frac{1}{\pi} - \left(-\frac{1}{\pi} + \frac{1}{4}\right)$$

$$-\frac{1}{4} + \frac{2}{\pi} + \frac{2}{\pi} - \frac{1}{4} = \frac{4}{\pi} - \frac{1}{2}$$

(VI) (7pts) We want to find the volume of a solid whose base is half a circle (see the shaded area), and the cross sections perpendicular to the $y-axis$ are squares.

a) Find the area of each individual cross section. (your answer might be in terms of x or y)



at each y the side of the square has length $\sqrt{1-y^2}$ $\Longrightarrow$ Area of the cross section

$A = 1-y^2$.

b) Use part (a) to find the volume of this solid.

To find the volume we only need to integrate from -1 to 1, the area of the cross section.

$$\int_{-1}^{1} 1-y^2\, dy = y - \frac{y^3}{3}\Big]_{-1}^{1}$$

$$= 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)$$

$$= \frac{4}{3}.$$