Entry times, escape rates and smoothness of stationary measures

by

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Contents

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Declarations

I declare that the non referenced material contained in this thesis is original and my own work based on problems of my supervisor. The only exception is Theorem 4.2.6 whose statement and proof were entirely made by my supervisor. This thesis has not been submitted for a degree at any other university.

Abstract

In this thesis, we investigate three different phenomena in uniformly hyperbolic dynamics.

First, we study entry time statistics for ψ-mixing actions. More specifically, given a ψ-mixing dynamical system $(\mathcal{X}, T, \mathcal{B}_{\mathcal{X}}, \mu)$ we find conditions on a family of sets $\{\mathcal{H}_n \subset \mathcal{X} : n \in \mathbb{N}\}$ so that $\mu(\mathcal{H}_n)\tau_n$ tends in law to an exponential random variable, where τ_n is the entry time to $\mathcal{H}_n$. We apply this to hyperbolic toral automorphisms, and we obtain that $\mu(\mathcal{H}_n)\tau_n$ tends in law to an exponential random variable when $\{\mathcal{H}_n \subset \mathcal{X} : n \in \mathbb{N}\}$ are shrinking sets along the unstable direction.

Second, we prove escape rate results for special flows over subshifts of finite type, over conformal repellers and over Axiom A diffeomorphisms. Finally, we study escape rates for Axiom A flows. Our results are based on a discretisation of the flow and the application of the results in [39].

Third, we study the smoothness of the stationary measure with respect to smooth perturbations of the iterated function scheme and the weight functions that define it. Our main theorems relate the smoothness of the perturbation of: the iterated function scheme and the weight functions; to the smoothness of the perturbation of the stationary measure. The results depend on the smoothness of: the iterated function scheme and the weights functions; and the space on which the stationary measure acts as a linear operator.

Notation

Symbol	**Meaning**
$\mathbb{N}$	$\{1, 2, \ldots\}$.
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$.
$\mathbb{Z}$	$\{\ldots, -1, 0, 1, \ldots\}$.
$\mathbb{R}$	Real numbers.
$\mathbb{R}^{>p}$	$\{x \in \mathbb{R} : x > p\}$.
$\lvert x \rvert$	Absolute value of $x \in \mathbb{R}$.
$[x]$ or $\lfloor x \rfloor$	Integer part of $x \in \mathbb{R}$.
$\lceil x \rceil$	$\min\{n \in \mathbb{Z} : n \geq x\}, x \in \mathbb{R}$.
$\pmod 1$	$x \pmod 1 := x - [x], x \in \mathbb{R}$.
$\{x_n\}$	$\{x_n : n \in \mathbb{N}\}$.
$\delta \searrow 0$	$\delta > 0$ decreases to zero.
$\sim$	Homologous functions.
$\mathbb{1}_{\mathcal{X}}(\cdot)$	$\mathbb{1}_{\mathcal{X}}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{X}, \\ 0 & \text{if } x \notin \mathcal{X}. \end{cases}$
δ_x	Dirac measure supported on $\{x\}$.
$\lvert \mathcal{X} \rvert$	Cardinality of $\mathcal{X}$.
$\overline{\mathcal{X}}$	Closure of $\mathcal{X}$.
$\mathrm{int}(\mathcal{X})$	Interior of $\mathcal{X}$.
$\mathcal{B}_{\mathcal{X}}$	Borel algebra on $\mathcal{X}$.
$\mathcal{X}^*$	Dual space of $\mathcal{X}$.
$E(f \mid \mathcal{B}_{\mathcal{X}})$	Conditional expectation of f with respect to $\mathcal{B}_{\mathcal{X}}$.

Continued on next page

Symbol	**Meaning**
$\mathcal{L}^1(\mathcal{X}, \mu)$	$\{f : \mathcal{X} \to \mathbb{R}$ such that $\int \lvert f(x) \rvert d\mu(x) < \infty\}$.
μ_{Leb}	Lebesgue measure.
ds	Density of the Lebesgue measure on $\mathbb{R}$.
f^n	$f^1 = f$, $f^{n+1} = f \circ f^n$ for $n \in \mathbb{N}$.
$\bigvee_{i=1}^n$	$\bigvee_{i=1}^n \alpha_i := \{\cap_{i=1}^n \mathcal{A}_i : \mathcal{A}_i \in \alpha_i\}$.
$\gg$	$x \gg y$ iff x is much greater than y.
μ_φ^f	$(\mu_\varphi)^f$.
S_n^g	$S_n^g f := \sum_{k=0}^{n-1} f \circ g^k$.
$\sqcup$	Disjoint union.
$f \equiv 1$	$f(x) = 1$ for every x.

Chapter 1

Introduction

1.1 Motivation

This thesis on statistical and probabilistic properties of uniformly hyperbolic smooth dynamical systems is inspired by three different questions:

(a) Can we prove entry time results for different shrinking sets?

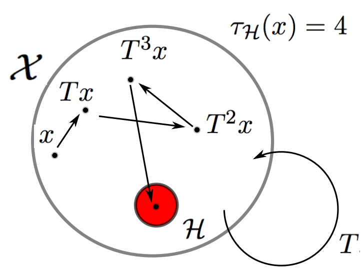
(b) Can we prove escape rate results for smooth flows?

(c) How smoothly does the stationary measure change under smooth perturbations of the parameters that define it?

The first two questions are about probabilistic properties and the third is a statistical property. In this thesis we restrict the three questions to a particular family of chaotic dynamical systems, called uniformly hyperbolic smooth dynamical systems, that were introduced by D.V. Anosov and S. Smale in the 1960's. In general terms, a dynamical system is said to be uniformly hyperbolic if the tangent space over the asymptotic part of the phase space splits into two complementary directions, one which is contracted and the other which is expanded under the action of the system, both at uniform rates. With absolute continuity, the study of hyperbolic dynamics started with H. Poincaré who studied homoclinic tangles, followed by the proof of ergodicity of geodesic flows on manifolds of constant negative curvature by G.A. Hedlund [47] and E. Hopf [50]. Beyond this assumption, Y.G. Sinai and A.N. Kolmogorov introduced the notion of metric entropy for Anosov diffeomorphims [58, 80]. In this case, it was necessary to work with a symbolic representation of the systems called a Markov partition [81, 82], generalising the work of R. L. Adler and B. Weiss [4]. After the results of Kolmogorov-Sinai, a crucial step was achieved by

R. Bowen, who used the machinery of thermodynamic formalism, that had been developed much earlier in physics by L. Carnot, S. Carnot, R. Clausius, L. Boltzmann and J. W. Gibbs, for a class of uniformly hyperbolic smooth dynamical systems, called Axiom A diffeomorphisms and Axiom A flows, that had been studied by S. Smale. He constucted Markov paritions for Axiom A flows in [11, 17] and studied thermodynamic properties in [12, 14, 15, 19] and [16]. At the same time, the work of D. Ruelle [78] built the basis of thermodynamic formalism. In mathematics, it has been successfully applied since then to systematically study uniformly hyperbolic smooth dynamical systems. This is indeed the main tool used in this thesis, and the reason why we have chosen to restrict our work to this particular family of dynamical systems. In order to explain how we addressed these questions, we state a precise problem for each one.

Can we prove entry time results for different shrinking sets? Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ be a measure preserving dynamical system, where $(\mathcal{X}, T)$ is a uniformly hyperbolic smooth dynamical systems and μ is an ergodic probability measure. Let $\mathcal{H} \subset \mathcal{X}$ be a Borel set and $\tau_{\mathcal{H}}$ be the first entry time function to the set $\mathcal{H}$, i.e. $\tau_{\mathcal{H}}(x) := \inf\{n \in \mathbb{N} : T^n x \in \mathcal{H}\}$. For sets $\{\mathcal{H}_n\}, \mathcal{H}_n \subset \mathcal{X}$ with the property that $\cap_{n\in\mathbb{N}}\mathcal{H}_n = \{x\}$, $x \in \mathcal{X}$, and under suitable hypotheses on the transformation T and the measure μ, one can prove that the sequence of random variables $X_n := \mu(\mathcal{H}_n)\tau_{\mathcal{H}_n}$ converges to an exponential random variable.



Problem 1.1.1. *For which sets $\{\mathcal{H}_n\}, \mathcal{H}_n \subset \mathcal{X}$, does the sequence of random variables X_n converge to an exponential random variable ?*

We consider a subshift of finite type $(\mathcal{X}, T)$ and a family of sets $\{\mathcal{H}_n\}, \mathcal{H}_n \subset \mathcal{X}$ shrinking to $\mathcal{H} \subset \mathcal{X}$. We find conditions for the sets $\{\mathcal{H}_n\}$ that depends on $\mathcal{X}$, μ and $\mathcal{H}$. This can be applied to study many different shrinking sets, for example, in the case of toral automorphims of the two torus and the measure of maximum entropy, we solve Problem 1.1.1 for families of sets shrinking to a segment along the unstable direction. In the same setting, our method leaves open some interesting cases, like horizontal shrinking strips.

Can we prove escape rate results for smooth flows? Escape rate results for smooth flows are a natural generalisation of known escape rates for discrete dynamical systems. Let us explain the discrete case first. Let $T : \mathcal{X} \to \mathcal{X}$ be a continuous map, where $\mathcal{X}$ is a compact metric space, μ be an ergodic probability measure and $\mathcal{H} \subset \mathcal{X}$ be a hole (a Borel set). Typical points fall into the hole after a finite number of iterates. Uniformly hyperbolic smooth dynamical systems have nice invariant probability measures μ (Gibbs), and we know that $\mu\{x \in \mathcal{X} : \tau_{\mathcal{H}}(x) \geq n\} \leq Ce^{-Rn}$, for $R = R(\mu, \mathcal{H}, \mathcal{X}) > 0$. We consider shrinking holes. For example, the ball of radius ϵ centred at $x_0 \in \mathcal{X}$, that we denote by $B(x_0, \epsilon)$. We are interested in studying the convergence of

$$\frac{R(\mu, B(x_0, \epsilon), \mathcal{X})}{\mu\left(B(x_0, \epsilon)\right)}$$

as ϵ tends to 0.

Example 1.1.2 (Theorem 4.0.9 in [22])**.** *For the map $Tx = 2x$ (mod 1) on the unit interval $[0, 1]$, we have that*

$$\lim_{\epsilon \to 0} \frac{R\left(\mu_{Leb}, B(x_0, \epsilon), [0, 1]\right)}{2\epsilon} = \begin{cases} 1 & \text{if } x_0 \in \mathcal{Y} \text{ is non periodic,} \\ 1 - \frac{1}{2^m} & \text{if } x_0 \text{ is periodic of period } m, \end{cases}$$

where μ_{Leb} is the Lebesgue measure on $[0, 1]$ and $\mathcal{Y}$ is a set of Lebesgue measure 1.

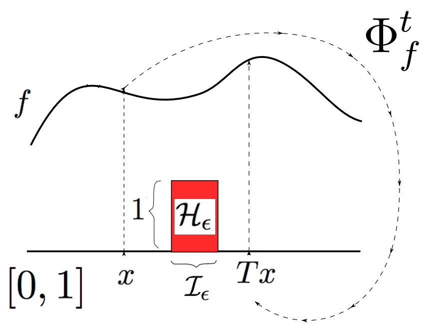


Figure 1.1

G. Keller and C. Liverani [54] proved a perturbation result that implies a similar formula for any expanding interval map. A. Ferguson and M. Pollicott obtained analogous results for Gibbs measures supported on conformal repellers [39]. A natural further step is to study the case of continuous dynamical systems.

Problem 1.1.3. *Given a uniformly hyperbolic smooth flow or smooth semi-flow $\Phi^t : \Lambda \to \Lambda$ with an ergodic probability measure μ on Λ. For the escape rate through a hole $\mathcal{H} \subset \Lambda$ defined by*

$$R(\mu, \mathcal{H}, \Lambda) := -\limsup_{t \to \infty} \frac{1}{t} \log \mu\{x \in \Lambda : \Phi^s x \notin \mathcal{H}, s \in [0, t]\},$$

can we describe the asymptotic behaviour of $R(\mu, \mathcal{H}, \Lambda)$ as $\mu(\mathcal{H})$ converges to 0?

In this thesis we are able to successfully answer this question on some particular cases. For example, let us assume that $\mathcal{X}$ is a smooth semi-flow or a conformal repeller and consider the special semi-flow (Λ_f, Φ^t) with roof function $f : \mathcal{X} \to \mathbb{R}^{>1}$ and phase space $\Lambda_f \subset \mathcal{X} \times \mathbb{R}$ (see Figure 1.1, in which we have drawn a special smooth semi-flow (Λ_f, Φ^t) with roof function $f : [0,1] \to \mathbb{R}^{>1}$ over the unit interval $[0,1]$ and a hole $\mathcal{H}_\epsilon = \mathcal{I}_\epsilon \times [0,1]$). We can prove in this case that

$$\begin{aligned}
&\lim_{\delta \searrow 0} \lim_{\epsilon \to 0} \frac{R\left(\nu, B(x_0,\epsilon) \times [0,\delta], \Lambda\right)}{\nu\left(B(x_0,\epsilon) \times [0,1]\right)} \\
&\qquad = \begin{cases} 1 & \text{if } (x_0,0) \text{ does not belong to any closed orbit,} \\ 1 - e^{\int_\tau \varphi(x_0,t)dt} & \text{if } (x_0,0) \in \tau \text{ and } \tau \text{ is a closed orbit,} \end{cases}
\end{aligned}$$

for every $x_0 \in \mathcal{X}$.

For special smooth flows over Axiom A diffeomorphisms, and more generally, for Axiom A flows, our results allow only to describe the asymptotic behaviour of the escape rate of a Gibbs measure through small sets that come from the projections of cylinder sets.

How smoothly does the stationary measure change under smooth perturbations of the parameter that define it? Assume that we have an iterated function scheme $\mathcal{T} = \{T_i\}_{i=1}^n$ with weight functions $\mathcal{G} = \{g_i\}_{i=1}^n$. Under suitable conditions on $\mathcal{G}$ there is a unique stationary measure $\mu_{\mathcal{T},\mathcal{G}}$. We are interested in studying how smooth are the changes $\mu_{\mathcal{T},\mathcal{G}}$ under smooth perturbations of $\mathcal{T}$ or $\mathcal{G}$ (or both). It is natural to consider the following perturbations

$$\begin{aligned}
\Lambda \ni \lambda \mapsto \mathcal{T}^{(\lambda)} &= \left\{T_i^{(\lambda)}\right\}_{i=1}^n \quad \text{and} \\
\Theta \ni \theta \mapsto \mathcal{G}^{(\theta)} &= \left\{g_i^{(\theta)}\right\}_{i=1}^n .
\end{aligned}$$

The problem that we consider is the following:

Problem 1.1.4. *Study the dependence of the stationary measure $\mu_{\mathcal{T}^{(\lambda)},\mathcal{G}^{(\theta)}}$ on $\lambda \in \Lambda$, $\theta \in \Theta$.*

We solve this problem in the particular case that $T : [0,1] \to [0,1]$ are contractions on the unit interval for the $\mathcal{C}^1$ norm. For fixed $\theta_0 \in \Theta$, we relate the smoothness of $\Lambda \ni \lambda \mapsto \mu = \mu_{\mathcal{T}^{(\lambda)},\mathcal{G}^{(\theta_0)}}$ to the smoothness of $\mathcal{T}^{(\lambda)}$ and $\mathcal{G}^{(\theta_0)}$; the regularity of $\Lambda \ni \lambda \mapsto \mathcal{T}^{(\lambda)}$ and the space on which μ acts as a linear operator. We similarly study the case when $\lambda = \lambda_0$ is fixed and θ is not, and finally the case when neither λ nor θ are fixed. We work on the spaces $C^{k+\alpha}$ of C^k functions with k-th derivative α-Hölder.

1.2 Basic definitions

This thesis is based on five main ingredients: subshifts of finite type (Chapters 2, 3 and 4), conformal repellers (Chapter 3), Axiom A diffeomorphisms (Chapters 2 and 3), Axiom A flows (Chapter 3) and stationary measures (Chapter 4). In order to separate well known results from new ones of the thesis, we state these basic definitions and important related results now and we only reference them from the main body.

1.2.1 Dynamical Systems and Ergodic Theory

A dynamical system is a smooth action of the reals or the integers on another object (usually a manifold). When the reals are acting, the system is called a continuous dynamical system, and when the integers are acting, the system is called a discrete dynamical system. A particular kind of continuous dynamical systems that we use are the smooth flows and semi-flows. These concepts require the definition of a homeomorphisms and a diffeomorphisms.

Definition 1.2.1 (Homeomorphisms and Diffeomorphisms). *1. A homeomorphism is a continuous map $f : \mathcal{X} \to \mathcal{Y}$ which is one-to-one and onto, and whose inverse $f^{-1} : \mathcal{Y} \to \mathcal{X}$ is also continuous.*

2. A diffeomorphism is a smooth homeomorphism with smooth inverse.

We can now define a smooth flow and semi-flow.

Definition 1.2.2 (Smooth flow and semi-flow). *1. A smooth flow $f^t : \mathcal{X} \to \mathcal{Y}$ is a family of diffeomorphisms depending smoothly on $t \in \mathbb{R}$ and satisfying $f^{s+t} = f^s \circ f^t$ for all $s, t \in \mathbb{R}$. In particular, f^0 is the identity map.*

2. A smooth semi-flow $f^t : \mathcal{X} \to \mathcal{Y}$ is a family of smooth maps depending smoothly on $t \in \mathbb{R}^{\geq 0}$ and satisfying $f^{s+t} = f^s \circ f^t$ for all $s, t \in \mathbb{R}^{\geq 0}$.

We will require to consider dynamical systems satisfying certain topological conditions.

Definition 1.2.3 (Transitivity and Mixing). *A continuous map $f : \mathcal{X} \to \mathcal{X}$ is said to be:*

1. Topologically transitive if, for every pair of non-empty open sets $\mathcal{Y}, \mathcal{W} \subset \mathcal{X}$ there exists $n \in \mathbb{N}$ such that $f^n(\mathcal{Y}) \cap \mathcal{W} \neq \emptyset$.

2. *Topologically mixing if, for every pair of non-empty open sets $\mathcal{Y}, \mathcal{W} \subset \mathcal{X}$, there exists $n \in \mathbb{N}$, such that, for all $k > n$, one has $f^k(\mathcal{Y}) \cap \mathcal{W} \neq \emptyset$.*

For flows it is similar.

From the measure theoretic point of view we work in this thesis with measure preserving dynamical systems. For this we need the concept of invariant probability measures.

Definition 1.2.4 (Invariant probability measure)**.** *A probability measure μ on a topological space $\mathcal{X}$ is invariant under a transformation $f : \mathcal{X} \to \mathcal{Y}$ if $\mu(\mathcal{A}) = \mu(f^{-1}\mathcal{A})$ for all measurable subsets $\mathcal{A}$, where $f^{-1}\mathcal{A}$ is the pre-image of $\mathcal{A}$ by f. We say μ is invariant under a flow $f^t : \mathcal{X} \to \mathcal{Y}$ if it is invariant under f^t for all t.*

Definition 1.2.5 (Measure preserving dynamical system)**.** *A measure preserving dynamical system is a system $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ where $\mathcal{X}$ is a topological space, $\mathcal{B}_{\mathcal{X}}$ the sigma-algebra over $\mathcal{X}$, $T : \mathcal{X} \to \mathcal{X}$ a measurable transformation and μ an invariant probability measure. It also refers to a system $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, f^t)$ where $\mathcal{X}$ is a topological space, $\mathcal{B}_{\mathcal{X}}$ the sigma-algebra over $\mathcal{X}$, $f^t : \mathcal{X} \to \mathcal{X}$ a smooth flow and μ an invariant probability measure.*

A particularly useful family of measure preserving dynamical systems is when the measure is an ergodic probability measure.

Definition 1.2.6 (Ergodic probability measure)**.** *An invariant probability measure μ is ergodic if every invariant set $\mathcal{A}$ has either zero or full measure, i.e., for every set $\mathcal{A}$ such that $\mathcal{A} = f^{-1}\mathcal{A}$, $\mu(\mathcal{A})$ is equal to either 0 or 1.*

The most important theorem in Ergodic Theory is the Birkhoff ergodic theorem that we state in what follows.

Theorem 1.2.7 (Birkhoff ergodic theorem)**.** *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ be a measure preserving dynamical system. If $f \in \mathcal{L}^1(\mathcal{X}, \mu)$, then*

$$\lim_{n\to\infty} \frac{S_n^T f(x)}{n} = E(f|\mathcal{B}_{\mathcal{X}}) \text{ for } \mu\text{-a.e. } x \in \mathcal{X},$$

where by μ-a.e. we mean that there is a measurable set with full measure for which the property holds.

If the measure μ in Theorem 1.2.7 is also ergodic, then

$$\lim_{n\to\infty} \frac{S_n^T f(x)}{n} = \int f d\mu \text{ for } \mu\text{-a.e. } x \in \mathcal{X}.$$

Finally, we define the measure theoretic entropy.

Definition 1.2.8 (Measure theoretic entropy). *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ be a measure preserving dynamical system. Given a finite partition $\xi = \{\mathcal{W}_n\}$ of $\mathcal{X}$, where $\mathcal{W}_n \in \mathcal{B}_{\mathcal{X}}$. We define $H_\mu(\xi) = -\sum_{\mathcal{W} \in \xi} \mu(\mathcal{W}) \log \mu(\mathcal{W})$ and $h_\mu(T, \xi) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{j=0}^{n-1} T^{-j}\xi)$, where*

$$\bigvee_{j=0}^{n-1} T^{-j}\xi := \{\cap_{j=0}^{n-1} T^{-j}\mathcal{W}_j : \mathcal{W}_j \in \xi\}.$$

The measure theoretic entropy (or Sinai entropy) of T is denoted by $h_\mu(T)$ and defined by $h_\mu(T) = \sup h_\mu(T, \xi)$, where the supremum is taken over all finite or countable partitions ξ with $H_\mu(\xi) < \infty$.

Sinai theorem asserts that if we have a partition ξ such that forμ-a.e. $x, y \in \mathcal{X}$ with $x \neq y$, there exists $n \in \mathbb{N}$ such that x and y belong to different elements of the partition $\bigvee_{j=-n}^{n} T^{-j}\xi$, then $h_\mu(T) = h_\mu(T, \xi)$.

1.2.2 Subshifts of finite type

We will formally introduce the definition of subshift of finite type. Let A denote an irreducible and aperiodic $a \times a$ matrix of zeros and ones with $a > 2$, i.e. there exists $d \in \mathbb{N}$ for which $A^d > 0$ (all coordinates of A^d are strictly positive). We call the matrix A transition matrix. We define the non-invertibe subshift of finite type $\mathcal{X}^+ = \mathcal{X}_A^+ \subset \{1, \dots, a\}^{\mathbb{N}_0}$ such that

$$\mathcal{X}^+ := \{(x_n)_{n=0}^{\infty} : A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}_0\}$$

and the invertible subshift of finite type $\mathcal{X} = \mathcal{X}_A \subset \{1, \dots, a\}^{\mathbb{Z}}$ such that

$$\mathcal{X} := \{(x_n)_{n=-\infty}^{\infty} : A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\}.$$

On $\mathcal{X}^+$, the shift $\sigma : \mathcal{X}^+ \to \mathcal{X}^+$ is defined by $\sigma(x)_n = x_{n+1}$ for all $n \in \mathbb{N}_0$. On $\mathcal{X}$, the shift $\sigma : \mathcal{X} \to \mathcal{X}$ is defined by $\sigma(x)_n = x_{n+1}$ for all $n \in \mathbb{Z}$. Notice that with our definition $(\mathcal{X}, \sigma)$ and $(\mathcal{X}^+, \sigma)$ are topologically mixing (i.e. when $\mathcal{U}, \mathcal{V}$ are non-empty subsets of $\mathcal{X}$ or $\mathcal{X}^+$, there is an $n \in \mathbb{N}$ so that $\sigma^m \mathcal{U} \cap \mathcal{V} \neq \emptyset$ for all $m \in \mathbb{Z}, m > n$). For $x \in \mathcal{X}^+$ and $n \in \mathbb{N}$, we define the cylinder

$$[x]_n := \{y \in \mathcal{X}^+ : y_i = x_i \text{ for } i \in \{0, \dots, n-1\}\},$$

we denote by ξ_n the set of all the cylinders $[x]_n$ with $x \in \mathcal{X}^+$ and we call by $\mathcal{B}_{\mathcal{X}^+}$ the sigma-algebra generated by the closed sets of $\mathcal{X}^+$ (Borel algebra on $\mathcal{X}^+$). For

$x \in \mathcal{X}$ and $n, m \in \mathbb{Z}, m > n$ we define the cylinder

$$[x]_n^m := \{y \in \mathcal{X} : y_i = x_i, \text{ for } i \in \{n, \ldots, m-1\}\}$$

and also denote $x_{[n,m)} := x_n x_{n+1} \ldots x_{m-1} = (x_k)_{k=n}^{m-1}$, which corresponds to the concatenation of $m - n$ elements in $\{1, \ldots, a\}$. We denote by ξ_n^m the set of all the cylinders $[x]_n^m$ with $x \in \mathcal{X}$. In the particular, when $n = 0$ we denote $[x]_n^m = [x]_m$ and $\xi_n^m = \xi_m$. There is natural projection π from $\mathcal{X}$ to $\mathcal{X}^+$ defined by $\pi\left((x_n)_{n=-\infty}^{\infty}\right) = (x_n)_{n=0}^{\infty}$. In this way, a non-invertible space $\mathcal{X}^+$ can be always seen as the projection of an invertible shift space $\mathcal{X}$ by π. Denote by $\mathcal{M}_{\sigma,+}$ the space of σ invariant probability measure on $\mathcal{X}^+$ and by $\mathcal{M}_\sigma$ the corresponding space on $\mathcal{X}$. Given $\mu_{\mathcal{X}^+} \in \mathcal{M}_{\sigma,+}$ we can define $\mu_{\mathcal{X}} \in \mathcal{M}_\sigma$ by

$$\mu_{\mathcal{X}}([x]_n^m) = \mu_{\mathcal{X}^+}([\sigma^{-n} x]_{m-n})$$

where $n, m \in \mathbb{Z}, m > n$. This induced measure will be called the natural extension of an invariant probability measure from $\mathcal{X}^+$ to $\mathcal{X}$.

In the invertible case, for $\theta \in (0,1)$, we consider the metric $d_\theta(x,y) = \theta^m$ where $m = \inf\{n \in \mathbb{N}_0 : x_n \neq y_n \text{ or } x_{-n} \neq y_{-n}\}$ and $d(x,x) = 0$ for every $x \in \mathcal{X}$, we have in particular that $(\mathcal{X}, d_\theta)$ is a complete metric space. We say that $f : \mathcal{X} \to \mathbb{R}$ is continuous if it is continuous with respect to d_θ. We denote the space of continuous functions $f : \mathcal{X} \to \mathbb{R}$ by $\mathcal{C}^0(\mathcal{X}, \mathbb{R})$. Given $f : \mathcal{X} \to \mathbb{R}$ continuous and $m \in \mathbb{N}$ define

$$V_m(f) := \sup\{|f(x) - f(y)| : x, y \in \mathcal{X} \text{ and } x_i = y_i \ \forall i \in \{-m, \ldots, m\}\},$$

and the Lipschitz semi-norm

$$|f|_\theta := \sup\left\{\frac{V_m(f)}{\theta^m} : m \in \mathbb{N}\right\}.$$

Since constant functions all have Lipschitz semi-norm equal to zero, one needs to define the norm on the space of Lipschitz functions by

$$\|f\|_\theta := |f|_\theta + \|f\|_\infty,$$

where $\|f\|_\infty := \sup_{x \in \mathcal{X}}\{|f(x)|\}$. The space of continuous functions with finite Lipschitz norm is called the space of Lipschitz functions (or θ-Lipschitz functions). Recall that a continuous function is α-Hölder for d_θ if and only if it is Lipschitz for d_{θ^α}. We denote the space of α-Hölder maps by $\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$.

In the non-invertible case, for $\theta \in (0,1)$, we consider the metric on $\mathcal{X}^+$ given by $d_\theta(x,y) = \theta^m$, where $m = \inf\{n \in \mathbb{N} : x_n \neq y_n\}$ and $d(x,x) = 0$ for every $x \in \mathcal{X}^+$. Here $(\mathcal{X}^+, d_\theta)$ is a complete metric space. We say that $f : \mathcal{X}^+ \to \mathbb{R}$ is continuous if it is continuous with respect to d_θ. We denote the space of continuous functions $f : \mathcal{X}^+ \to \mathbb{R}$ by $\mathcal{C}^0(\mathcal{X}^+, \mathbb{R})$. Given $f : \mathcal{X}^+ \to \mathbb{R}$ continuous and $m \in \mathbb{N}$ define

$$V_m(f) := \sup_{z \in \mathcal{X}^+} \{|f(x) - f(y)|: x, y \in [z]_m\},$$

the Lipschitz semi-norm

$$|f|_\theta := \sup\left\{\frac{V_m(f)}{\theta^m} : m \in \mathbb{N}\right\}$$

and the Lipschitz norm

$$\|f\|_\theta := |f|_\theta + \|f\|_\infty,$$

where $\|f\|_\infty := \sup_{x \in \mathcal{X}^+}\{|f(x)|\}$. The space of continuous functions with finite Lipschitz norm is called the space of Lipschitz functions (or θ-Lipschitz functions). Again, a continuous function is α-Hölder for d_θ if and only if it is Lipschitz for d_{θ^α}. Hölder functions on $\mathcal{X}^+$ can be seen as a subclass of Hölder functions on $\mathcal{X}$ [16]. We denote the space of α-Hölder maps on $\mathcal{X}^+$ by $\mathcal{C}^\alpha(\mathcal{X}^+, \mathbb{R})$.

1.2.3 Thermodynamic Formalism and Gibbs measures

We introduce some results and definitions on thermodynamic formalism, in particular we define Gibbs measures, the pressure function P and the transfer operator. Along this section, let $\mathcal{X}$ be a topologically mixing subshift of finite type.

Definition 1.2.9. *Let $P : \mathcal{C}^0(\mathcal{X}, \mathbb{R}) \to \mathbb{R}$ denote the pressure defined by*

$$P(\varphi) := \lim_{n \to +\infty} \frac{1}{n} \log\left(\sum_{\sigma^n x = x} \exp\left(\sum_{k=0}^{n-1} \varphi(\sigma^k x)\right)\right)$$

where $\varphi \in \mathcal{C}^0(\mathcal{X}, \mathbb{R})$.

Remark 1.2.10. *The pressure function is well defined. Indeed, it is not hard to prove that if a sequence $\{u_n\}$ of non-negative reals satisfies that*

$$u_{n+p} \leq u_n + u_p \text{ for all } n, p \geq 1, \tag{1.1}$$

then the sequence $\left\{\frac{u_n}{n}\right\}$ is convergent (see for example Proposition 3.2 in [91]). The sequence $u_n = \log\left(\sum_{\sigma^n x = x} \exp\left(\sum_{k=0}^{n-1} \varphi(\sigma^k x)\right)\right)$ satisfies (1.1), then the pressure

$P(\varphi) = \lim_{n\to\infty} \frac{u_n}{n}$ *exists.*

The following result gives an alternative definition of the pressure.

Lemma 1.2.11 (Variational principle)**.** *We can write*

$$P(\varphi) = \sup\left\{h(\nu) + \int \varphi d\nu \ : \ \nu \textit{ is } \sigma\textit{-invariant probability measure}\right\},$$

where $h(\nu)$ is the measure theoretic entropy with respect to ν. Moreover, there is a unique σ-invariant probability measure μ_φ on $\mathcal{X}$ which satisfies

$$P(\varphi) = h(\mu_\varphi) + \int \varphi d\mu_\varphi.$$

Definition 1.2.12 (Gibbs measure)**.** *We say that a probability measure μ on $\mathcal{X}$ is a Gibbs measure (or an equilibrium state) of Hölder potential $\phi : \mathcal{X} \to \mathbb{R}$ if there is $c_1, c_2 > 0$ and $P \geq 0$ such that*

$$c_1 \leq \frac{\mu([x]_m)}{\exp\left(-Pm + S_m^\sigma \phi(x)\right)} \leq c_2$$

for every $x \in \mathcal{X}$ and $m \geq 0$, where $S_m^\sigma \phi(x) := \sum_{k=0}^{m-1} \phi(\sigma^k x)$.

Gibbs measures are related to the pressure function by the following proposition.

Proposition 1.2.13. *The probability measure μ on $\mathcal{X}$ is a Gibbs measure of Hölder potential $\phi : \mathcal{X} \to \mathbb{R}$ if and only if $\mu = \mu_\phi$.*

These basic properties can be found in [16], [69], [78] for example. Another property of the pressure function that we will use in Chapter 4 is the following.

Lemma 1.2.14. *The function $P : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathbb{R}$ is analytic. Moreover, the first and second derivatives are given by:*

1. $\frac{dP(\varphi + t\psi)}{dt}|_{t=0} = \int \psi d\mu_\varphi$*; and*
2. $\frac{\partial^2 P(\varphi + t_1\psi + t_2\xi)}{\partial t_1 \partial t_2}|_{(0,0)} = \sigma^2_{\mu_\varphi}(\psi, \xi)$ *where $\sigma^2_{\mu_\varphi}(\psi, \xi)$ is the variance of μ_φ*

and $\psi, \xi \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$.

This result can be found in [78] or [69]. For a proof including the details see [91], Propositions 6.12 and 6.13 in Section 6.6.

The key ingredient in the proof of Lemma 1.2.11, Proposition 1.2.13 and Lemma 1.2.14 is the transfer operator or Ruelle operator (also known as Perron-Frobenius-Ruelle operator) defined on the space $\mathcal{C}^0(\mathcal{X}^+, \mathbb{R})$ and Theorem 1.2.16.

Definition 1.2.15 (Transfer operator)**.** *Let $\phi : \mathcal{X}^+ \to \mathbb{R}$. The transfer operator is defined by*

$$\mathscr{L} f(x) := \sum_{y \in \sigma^{-1}(x)} e^{\phi(y)} f(y)$$

where $f \in \mathcal{C}^0(\mathcal{X}^+, \mathbb{R})$.

Theorem 1.2.16 (Perron-Frobenius-Ruelle)**.** *If $\phi \in \mathcal{C}^\alpha(\mathcal{X}^+, \mathbb{R})$ for some $\alpha > 0$, then*

(PFR 1) there is a simple eigenvalue $\beta > 0$ of $\mathscr{L}$ and an associated eigenvector $h > 0$ in $\mathcal{C}^0(\mathcal{X}^+, \mathbb{R})$,

(PFR 2) there is a unique probability measure μ in $\mathcal{X}^+$ such that $\int \mathscr{L} f d\mu = \beta \int f d\mu$ for every $f \in \mathcal{C}^0(\mathcal{X}^+, \mathbb{R})$ and for every function $v \in \mathcal{C}^0(\mathcal{X}^+, \mathbb{R})$, the sequence $\beta^{-n}\mathscr{L}^n v$ converges uniformly on $\mathcal{X}^+$ to $h\frac{\int v d\mu}{\int h d\mu}$, and

(PFR 3) the topological pressure of ϕ is $\beta = P(\phi)$.

1.2.4 Conformal repellers

In order to state our second theorem in Chapter 3 we need to define conformal repellers. We start with the definition of conformal linear maps.

Definition 1.2.17 (Conformal linear map)**.** *A non-constant linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is conformal (or conformal linear map) if and only if for every $x, y, z \in \mathbb{R}^n$*

$$\frac{\|Az - Ax\|}{\|Ay - Ax\|} = \frac{\|z - x\|}{\|y - x\|}. \tag{1.2}$$

In the case $n \geq 2$, Equation (1.2) is equivalent to $A = \lambda U$ where $\lambda \neq 0$ is a scalar and U is an orthogonal map.

We can now define conformal repellerss.

Definition 1.2.18 (Conformal repellers)**.** *Let $\mathcal{M}$ be a Riemmanian manifold, $f : \mathcal{M} \to \mathcal{M}$ a $\mathcal{C}^1$ map and $\mathcal{J} \subset \mathcal{M}$ a compact set such that $f\mathcal{J} = \mathcal{J}$. We say that $(\mathcal{J}, f)$ is a conformal repeller if:*

(a) $f|_{\mathcal{J}}$ is a conformal map, i.e. the differential of f at every point $p \in \mathcal{J}$ is a conformal linear map between the Euclidean spaces $T_p\mathcal{M}$ and $T_{f(p)}\mathcal{M}$;

(b) $\exists c > 0, \lambda \in \mathbb{R}^{>1} : \|df_x^n v\| \geq c\lambda^n \|v\|, \forall x \in \mathcal{J}, \forall v \in T_x\mathcal{M}, \forall n \in \mathbb{N};$

(c) f *is topologically mixing;*

(d) $\exists \mathcal{V} \supset \mathcal{J}$ *such that* $\mathcal{J} = \{x \in \mathcal{V} : f^n x \in \mathcal{V}, \forall n \in \mathbb{N}\}$.

An example of a conformal repeller is the Julia set of a hyperbolic rational map, i.e. the closure of the set of repelling periodic points of a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree greater or equal than two, where $\hat{\mathbb{C}}$ is the Riemann sphere.

1.2.5 Axiom A diffeomorphims

Suppose that $\mathcal{M}$ is a compact $\mathcal{C}^\infty$ Riemannian manifold, the tangent bundle of $\mathcal{M}$ is given by $T\mathcal{M} = \cup_{x\in\mathcal{M}} T_x\mathcal{M}$, where $T_x\mathcal{M}$ is the tangent space of $\mathcal{M}$ at x. Suppose also that $f : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism. First we need to define a hyperbolic set.

Definition 1.2.19. *A closed subset* $\Lambda \subset \mathcal{M}$ *is hyperbolic if* $f(\Lambda) = \Lambda$ *and each tangent space* $T_x\mathcal{M}$ *with* $x \in \Lambda$ *can be written as a direct sum*

$$T_x\mathcal{M} = \mathcal{E}_x^u \oplus \mathcal{E}_x^s$$

of subspaces such that

(a) $Df(\mathcal{E}_x^s) = \mathcal{E}_{f(x)}^s, Df(\mathcal{E}_x^u) = \mathcal{E}_{f(x)}^u;$

(b) there exist constants $c > 0$ *and* $\lambda \in (0, 1)$ *so that*

$$\|Df^n(v)\| \leq c\lambda^n \|v\| \text{ when } v \in \mathcal{E}_x^s, n \in \mathbb{N}_0 \tag{1.3}$$

and

$$\|Df^{-n}(v)\| \leq c\lambda^n \|v\| \text{ when } v \in \mathcal{E}_x^u, n \in \mathbb{N}_0; \tag{1.4}$$

(c) $\mathcal{E}_x^s, \mathcal{E}_x^u$ *vary continuously with* x.

We can now define Axiom A diffeomorphisms.

Definition 1.2.20. *(a) We define* $\Omega = \Omega(f)$ *to be the set of all non wandering points, i.e., the set of points* $x \in \mathcal{M}$ *such that*

$$\mathcal{U} \cap \bigcup_{n\in\mathbb{N}} f^n\mathcal{U} \neq \emptyset,$$

for every neighbourhood $\mathcal{U}$ *of* x.

(b) f satisfies Axiom A if $\Omega(f)$ is hyperbolic and $\Omega(f) = \overline{\{x : x \text{ is periodic }\}}$.

Examples of Axiom A diffeomorphisms are the Anosov diffeomorphisms and Smale horseshoe maps.

Anosov diffeomorphisms

Let $\mathcal{M}$ be a compact $\mathcal{C}^\infty$ Riemannian manifold. An Anosov diffeomorphism is a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ such that $\mathcal{M}$ is hyperbolic. The simplest examples of compact $\mathcal{C}^\infty$ Riemannian manifold admiting Anosov diffeomorphisms are the tori $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, moreover, any Anosov diffeomorphism on the tori is topologically conjugate to one given by an isomorphisms of the tori with no eigenvalue of modulus 1 ([40]). In this thesis our basic tool, as we mentioned in Section 1.1, is thermodynamic formalism, on the other hand, as we will see later in this section, a crucial necessary condition is the existence of the so called Markov partitions. It happens that Markov partitions are hard to find for Anosov diffeomorphisms on tori $\mathbb{T}^n$ with $n > 2$, so in this thesis we will deal indeed only with Anosov diffeomorphisms on the two dimensional torus. An example is the Arnold's cat map, given by the diffeomorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \pmod 1 \\ x+2y \pmod 1 \end{pmatrix}.$$

An illustrative way to explain how this transformations acts on the two-torus is to represent the phase space by a square picture of a cat and see the stretching and folding process performed by the map T after we apply it to the original picture. In practical terms, we are representing $\mathbb{T}^2$ by an array of size $N \times N$ (for example $N = 1300$) choosing a colour for each coordinate and move the colour of the coordinate (i, j) to the coordinate $(i+j \pmod N, i+2j \pmod N)$. To give a concrete example of this, in Figure 1.2 we have chosen a picture of a cat to represent the phase space $\mathbb{T}^2$, and in Figure 1.3 we see how the map T stretched and folded the original picture.

Smale horseshoe map

The Smale horseshoe map is defined by the diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ in Figure 1.4 that acts on the unit square $[0, 1]^2 \subset \mathbb{R}^2$ (the unit square $[0, 1]^2$ is represented by the square abcd). It contracts in the horizontal direction, then expands in the vertical direction, folds the space and places it back over the unit square. In Figure 1.4 we show the action of f and the action of its inverse f^{-1}.

Figure 1.2: Representation of the phase space $\mathbb{T}^2$.

Figure 1.3: Representation of the phase space $\mathbb{T}^2$ stretched and folded by T.

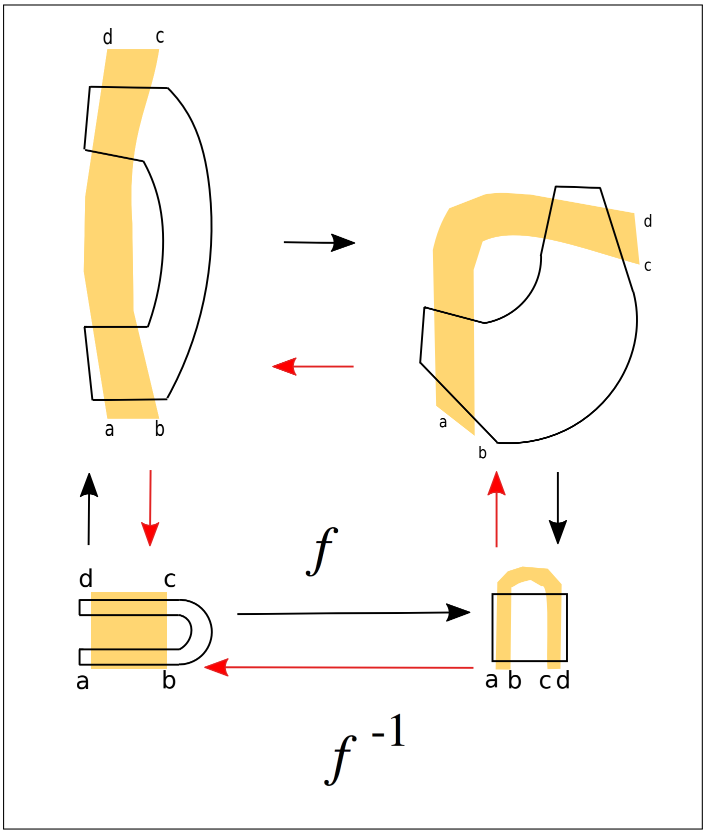


Figure 1.4: Horseshoe map.

Markov partitions, equilibrium states and semi-conjugacies

We start with some topological issues.

Lemma 1.2.21 (Lemma 3.1 in [16]). *Every Axiom A diffeomorphism has an adapted metric d, that is, $\Omega(f)$ is hyperbolic with respect to d with $c = 1$ in (1.3) and (1.4).*

A basic property is the following:

Proposition 1.2.22 (Spectral decomposition, Chapter 3, section B in [16]). *Suppose that f is an Axiom A diffeomorphisms, then one can write $\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$, where Ω_i are pairwise disjoint closed sets with*

(a) $f(\Omega_i) = \Omega_i$ and $f|_{\Omega_i}$ is topologically transitive;

(b) $\Omega_i = \mathcal{X}_{1,i} \cup \cdots \cup \mathcal{X}_{n,i}$ for $n = n(i)$, with the $\mathcal{X}_{j,i}$'s pairwise disjoint closed sets, $f(\mathcal{X}_{j,i}) = \mathcal{X}_{j+1,i}$ ($\mathcal{X}_{n+1,i} = X_{1,i}$) and $f^n|_{\mathcal{X}_{j,i}}$ topologically mixing.

Definition 1.2.23. *The sets Ω_i in the spectral decomposition of $\Omega(f)$ are called the basic sets of f.*

Let Ω_s be a basic set, given a Hölder function $\varphi : \Omega_s \to \mathbb{R}$, define $P = P(\varphi)$ by

$$P := \sup\left\{h_\mu(f) + \int \varphi d\mu : \mu \in \mathcal{M}_f(\Omega_s)\right\},$$

where $h_\mu(f)$ is the measure theoretic entropy (see Glossary or [16]) and $\mathcal{M}_f(\Omega_s)$ is the set of f invariant probability measures with support in Ω_s.

The next theorem gives an essential property of the basic sets, this is the existence of equilibrium states, i.e. of invariant probability measures μ on Ω_s such that

$$P = h_\mu(f) + \int \varphi d\mu,$$

where $h_{\mu_\varphi}(f)$ is the measure theoretic entropy.

Theorem 1.2.24 (Theorem 4.1 in [16]). *Let Ω_s be a basic set for an Axiom A diffeomorphism f and $\varphi : \Omega_s \to \mathbb{R}$ a Hölder continuous function. Then φ has a unique equilibrium state $\mu = \mu_\varphi$ (w.r.t. $f|_{\Omega_s}$). Furthermore, μ is an ergodic probability measure; μ is Bernoulli if $f|_{\Omega_s}$ is topologically mixing.*

In Chapter 3 we require some well known results that we include for completeness. These are the definition of Markov partitions, their existence for basic sets and the semi-conjugation of the basic sets with a subshift of finite type.

In order to define Markov partitions in a standard way we require some extra definitions and propositions. Indeed, Markov partitions are a particular set of *rectangles*, whose construction follows from the existence of the so called canonical coordinates. To define the canonical coordinates we need the following:

Definition 1.2.25. *For $x \in \mathcal{M}$ define*

$$\begin{aligned}
\mathcal{W}^s(x) &:= \{y \in \mathcal{M} : d(f^n x, f^n y) \to 0 \text{ as } n \to \infty\}, \\
\mathcal{W}^s_\epsilon(x) &:= \{y \in \mathcal{M} : d(f^n x, f^n y) \le \epsilon \text{ for all } n \in \mathbb{N}\}, \\
\mathcal{W}^u(x) &:= \{y \in \mathcal{M} : d(f^{-n} x, f^{-n} y) \to 0 \text{ as } n \to \infty\} \text{ and} \\
\mathcal{W}^u_\epsilon(x) &:= \{y \in \mathcal{M} : d(f^{-n} x, f^{-n} y) \le \epsilon \text{ for all } n \in \mathbb{N}\}.
\end{aligned}$$

Proposition 1.2.26 (Canonical coordinates, Proposition 2.3.33 in [16])**.** *Suppose that f satisfies Axiom A. For any small $\epsilon > 0$ there is a $\delta > 0$ so that $\mathcal{W}^s_\epsilon(x) \cap \mathcal{W}^u_\epsilon(y)$ consists of a single point $[x, y]$ whenever $x, y \in \Omega(f)$ and $d(x, y) \le \delta$. Furthermore $[x, y] \in \Omega(f)$ and the canonical coordinates function*

$$[\cdot, \cdot] : \{(x, y) \in \Omega(f) \times \Omega(f) : d(x, y) \le \delta\} \to \Omega(f)$$

is continuous.

We can define *rectangles* once having canonical coordinates.

Definition 1.2.27 (Chapter 3, section C in [16])**.** *A subset $\mathcal{R} \subset \Omega_s$ is called a* rectangle *if it has small diameter and*

$$[x, y] \in \mathcal{R} \text{ whenever } x, y \in \mathcal{R}.$$

A rectangle *$\mathcal{R}$ is called a* proper rectangle *if $\mathcal{R}$ is closed and $\mathcal{R} = \overline{int(\mathcal{R})}$. For $x \in \mathcal{R}$, we define*

$$\mathcal{W}^s(x, \mathcal{R}) := \mathcal{W}^s_\epsilon(x) \cap \mathcal{R} \text{ and } \mathcal{W}^u(x, \mathcal{R}) := \mathcal{W}^u_\epsilon(x) \cap \mathcal{R}.$$

The particular set of *rectangles* that define a Markov partition are specified in the next definition.

Definition 1.2.28 (Chapter 3, section C in [16])**.** *A Markov partition of Ω_s is a finite covering $\mathcal{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_m\}$ of Ω_s by* proper rectangles *with*

(a) $int(\mathcal{R}_i) \cap int(\mathcal{R}_j) = \emptyset$ for $i \neq j$,

(b) $f\mathcal{W}^u(x, \mathcal{R}_i) \supset \mathcal{W}^u(fx, \mathcal{R}_j)$ and
$f\mathcal{W}^s(x, \mathcal{R}_i) \subset \mathcal{W}^s(fx, \mathcal{R}_j)$ when $x \in int(\mathcal{R}_i), fx \in int(\mathcal{R}_j)$.

We have included the next lemma only because we require the definition of $\partial^s\mathcal{R}$ and $\partial^u\mathcal{R}$ in Chapter 3.

Lemma 1.2.29 (Lemma 3.11 in [16]). *Let $\mathcal{R}$ be a closed* rectangle. *As a subset of Ω_s, $\mathcal{R}$ has boundary*

$$\partial\mathcal{R} = \partial^s R \cup \partial^u\mathcal{R},$$

where

$$\partial^s\mathcal{R} = \{x \in \mathcal{R} : x \notin int(\mathcal{W}^u(x,\mathcal{R}))\}$$

$$\partial^u\mathcal{R} = \{x \in \mathcal{R} : x \notin int(\mathcal{W}^s(x,\mathcal{R}))\}.$$

Thermodynamic formalism is useful to study Axiom A diffeomorphisms essentially because of two theorems. The first is the existence of Markov partitions for the basic sets, and the second is that each basic sets Ω_s is semi-conjugate with a subshift of finite type, i.e. there exists a subshift of finite type $(\mathcal{X},\sigma)$ and continuous surjection $\pi : \mathcal{X} \to \Omega_s$ with $\pi \circ \sigma = f \circ \pi$.

Theorem 1.2.30 (Theorem 3.12 in [16]). *Let Ω_s be a basic set for an Axiom A diffeomorphism f. Then Ω_s has Markov partition $\mathcal{R}$ of arbitrarily small diameter.*

Finally, we can find a semi-conjugation of the basic sets with a subshift of finite type. Let Ω_s be a basic set for an Axiom A diffeomorphism f and $\mathcal{R} = \{\mathcal{R}_1, \ldots, \mathcal{R}_m\}$ denote a Markov partition of Ω_s. We define the transition matrix $A = A(\mathcal{R})$ by

$$A_{i,j} = \begin{cases} 1 & \text{if } \mathrm{int}(\mathcal{R}_i) \cap f^{-1}\mathrm{int}(\mathcal{R}_j) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

One can prove that $(\mathcal{X}_A,\sigma)$ is semi-conjugate with $(\Omega_s, f|_{\Omega_s})$, indeed we have the following theorem.

Theorem 1.2.31 (Theorem 3.18 in [16]). *For each $\underline{a} \in \mathcal{X}_A$ the set $\bigcap_{j\in\mathbb{Z}} f^{-j}\mathcal{R}_{a_j}$ consists of a single point, denoted $\pi(\underline{a})$. The map $\pi : \mathcal{X}_A \to \Omega_s$ is a semi-conjugation. Moreover, π is one-to-one on the residual set $\mathcal{Y} = \Omega_s \setminus \bigcup_{j\in\mathbb{Z}} f^j(\partial^s\mathcal{R} \cup \partial^u\mathcal{R})$.*

Let us give two concrete examples of applications of Theorem 1.2.30 and Theorem 1.2.31.

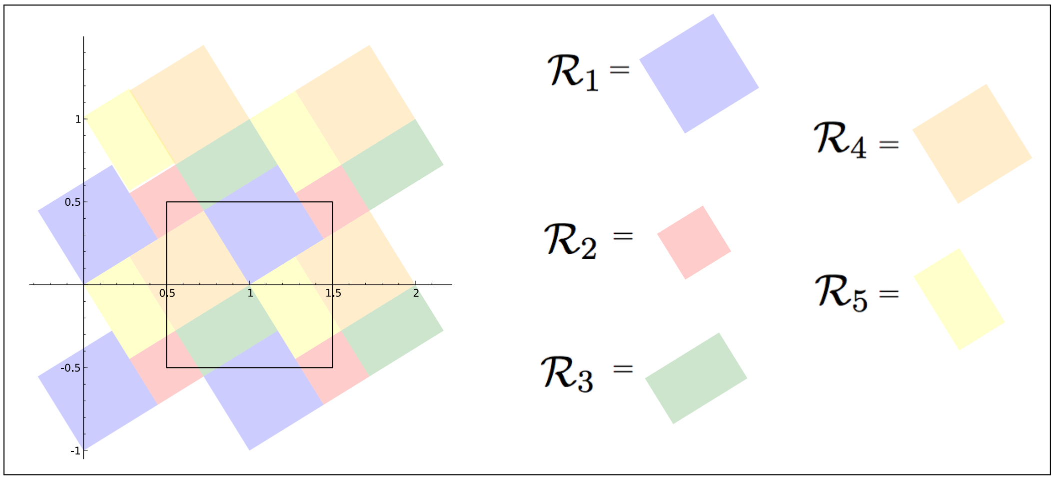


Figure 1.5: Markov partition.

Markov partition and semi-conjugacy for an Anosov diffeomorphisms on the two dimensional torus.

Let us consider the Anosov diffeomorphisms T on the two dimensional torus $\mathbb{T}^2$ given by

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can construct the Markov partition in Figure 1.5 with *rectangles* $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ and $\mathcal{R}_5$. Define the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the subshift of finite type $\mathcal{X}_A \subset \{1, \ldots, 5\}^{\mathbb{Z}}$ with transition matrix A. The map $\pi : \mathcal{X}_A \to \{0,1\}^{\mathbb{Z}}$, defined by $\pi(\underline{a}) = \cap_{i \in \mathbb{Z}} T^{-i} \mathcal{R}_{a_i} \in \mathbb{T}^2$ is a semi-conjugacy.

Markov partition and semi-conjugacy for the horseshoe map

Let us consider the Smale horseshoe map defined by the diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ in Figure 1.6, where we have marked each side of the horseshoe with a number 0 or 1. Call the unit square $\mathcal{U} = [0,1]^2 \subset \mathbb{R}^2$ and define the set $\mathcal{R}_0 \subset \mathbb{R}^2$ as the intersection of $\mathcal{U}$ with the side of the horseshoe marked with 0, and the set $\mathcal{R}_1 \subset \mathbb{R}^2$

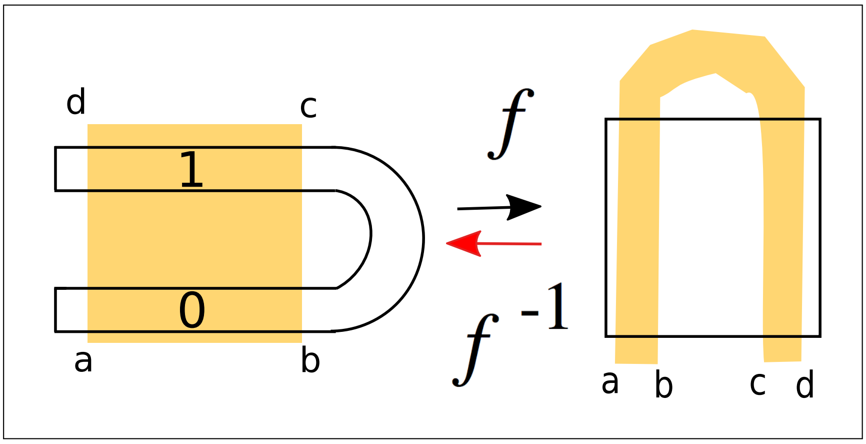


Figure 1.6: Horseshoe map.

as the intersection of $\mathcal{U}$ with the side of the horseshoe marked with 1. The invariant set for f and f^{-1} is the Cantor set

$$\Omega := \{x \in \mathcal{U} : f^k(x) \in \mathcal{U}, \forall k \in \mathbb{Z}\}.$$

This set Ω is conjugated to the subshift of finite type $\{0,1\}^{\mathbb{Z}}$, i.e. there is a continuous one-to-one map with continuous inverse $\pi : \Omega \to \{0,1\}^{\mathbb{Z}}$. In our case we can define $\pi(x) = \dots w_{-1}w_0w_1\dots$, where

$$w_k = \begin{cases} 0 & \text{if } f^k(x) \in \mathcal{R}_0 \\ 1 & \text{if } f^k(x) \in \mathcal{R}_1 \end{cases}$$

for $k \in \mathbb{Z}$, $f^0(x) = x$.

In Figure 1.7 we represent $\mathcal{U}$ by a black square. In figure 1.8 we represent the set $f^{-1}(\mathcal{U}) \cap \mathcal{U} \cap f(\mathcal{U})$ by four black squares and so on. An interesting feature of the pictures is that they allow to visualise the pre-images of the projections of cylinders sets. Indeed, In Figure 1.8 each of the four squares contain exactly one of the sets $\pi^{-1}([0.0]_{-1}^{1})$, $\pi^{-1}([0.1]_{-1}^{1})$, $\pi^{-1}([1.0]_{-1}^{1})$, $\pi^{-1}([1.1]_{-1}^{1})$, and so on.

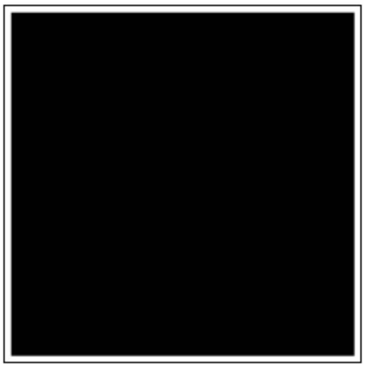


Figure 1.7: $\mathcal{U}$.

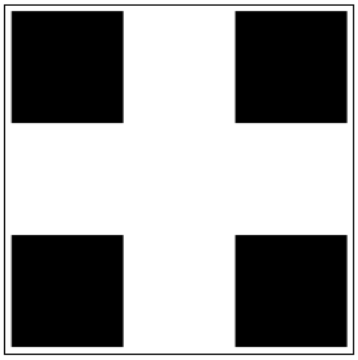


Figure 1.8: $f^{-1}(\mathcal{U}) \cap \mathcal{U} \cap f(\mathcal{U})$.

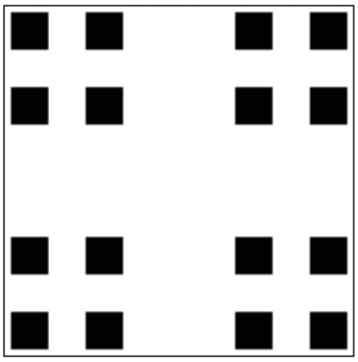


Figure 1.9: $f^{-2}(\mathcal{U}) \cap f^{-1}(\mathcal{U}) \cap \mathcal{U} \cap f(\mathcal{U}) \cap f^{2}(\mathcal{U})$.

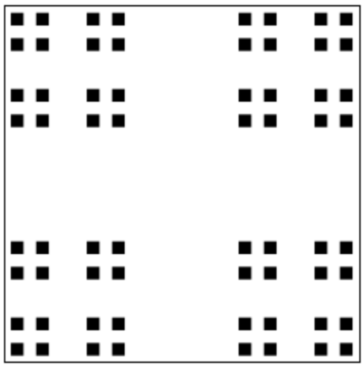


Figure 1.10: $f^{-3}(\mathcal{U}) \cap f^{-2}(\mathcal{U}) \cap f^{-1}(\mathcal{U}) \cap \mathcal{U} \cap f(\mathcal{U}) \cap f^{2}(\mathcal{U}) \cap f^{3}(\mathcal{U})$.

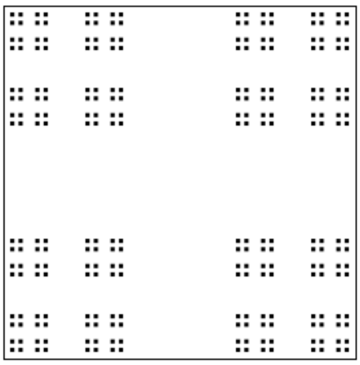


Figure 1.11: $f^{-4}(\mathcal{U}) \cap f^{-3}(\mathcal{U}) \cap f^{-2}(\mathcal{U}) \cap f^{-1}(\mathcal{U}) \cap \mathcal{U} \cap f(\mathcal{U}) \cap f^{2}(\mathcal{U}) \cap f^{3}(\mathcal{U}) \cap f^{4}(\mathcal{U})$.

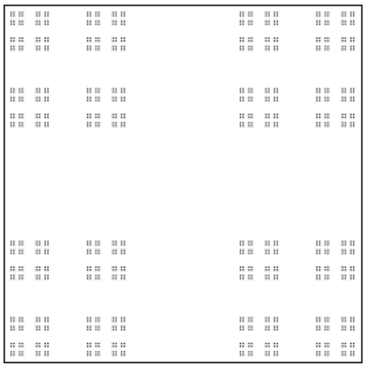


Figure 1.12: $f^{-5}(\mathcal{U}) \cap f^{-4}(\mathcal{U}) \cap f^{-3}(\mathcal{U}) \cap f^{-2}(\mathcal{U}) \cap f^{-1}(\mathcal{U}) \cap \mathcal{U} \cap f(\mathcal{U}) \cap f^{2}(\mathcal{U}) \cap f^{3}(\mathcal{U}) \cap f^{4}(\mathcal{U}) \cap f^{5}(\mathcal{U})$.

1.2.6 Axiom A flows

Let $\mathcal{M}$ be a compact $\mathcal{C}^\infty$ Riemannian manifold and $\Phi^t : \mathcal{M} \to \mathcal{M}$ be a differentiable flow. Denote $\Phi = \{\Phi^t\}$. A closed Φ invariant set $\Lambda \subset \mathcal{M}$ containing no fixed points is hyperbolic if the tangent bundle restricted to Λ can be written as

$$T_\Lambda = \mathcal{E} + \mathcal{E}^s + \mathcal{E}^u,$$

where $\mathcal{E}$ is the one-dimensional bundle tangent to the flow, and there are constants $c, \lambda > 0$ so that

(A) $\|D\Phi^t(v)\| \leq ce^{-\lambda t}\|v\|$ for $v \in \mathcal{E}^s, t > 0$ and

(B) $\|D\Phi^{-t}(v)\| \leq ce^{-\lambda t}\|v\|$ for $v \in \mathcal{E}^u, t > 0$.

A closed invariant set Λ is a basic hyperbolic set if

(a) Λ contains no fixed points and is hyperbolic;

(b) the periodic orbits of $\Phi^t|_\Lambda$ are dense in Λ;

(c) $\Phi^t|\Lambda$ is a topologically transitive flow; and

(d) there is an open set $\mathcal{U} \supset \Lambda$ with $\Lambda = \bigcap_{t\in\mathbb{R}} \Phi^t\mathcal{U}$.

We always consider basic hyperbolic sets that are neither a point, nor a single closed orbit.

We can now define Axiom A flows.

Definition 1.2.32. *(i) The non wandering set Ω is defined by*

$$\Omega := \{z \in \mathcal{M} : \textit{ for every neighbourhood } \mathcal{V} \textit{ of } z \textit{ and every } t_0 > 0, \textit{ there is a } t \in \mathbb{R}^{>t_0} \textit{ with } \Phi^t(\mathcal{V}) \cap \mathcal{V} \neq \emptyset\}.$$

(ii) Φ is said to be Axiom A flow if $\Omega = \Omega' \sqcup \{x_1, \dots, x_n\}$ is the disjoint union of a set Ω' satisfying (a) and (b) and the set $\{x_1, \dots, x_n\}$ is a finite set ($n \in \mathbb{N}$), where x_i are hyperbolic fixed points.

Let Λ be a basic hyperbolic set for an Axiom A flows Φ and let $\varphi : \Lambda \to \mathbb{R}$ be a Hölder function. We define $P = P(\Phi|_\Lambda, \varphi)$ by

$$P := \sup\left\{h_\mu(\Phi^1) + \int \varphi d\mu : \mu \text{ is invariant for } \Phi|_\Lambda\right\},$$

where $h_\mu(\Phi^1)$ is the measure theoretic entropy. An equilibrium state of φ is a probability measure μ on Λ such that attains the supremum, i.e.

$$P = h_\mu(\Phi^1) + \int \varphi d\mu.$$

For $z \in \Lambda, t > 0$ and $\epsilon > 0$ small, let

$$B_{z,\Phi|\Lambda}(\epsilon, t) := \{y \in \Lambda : d(\Phi^s y, \Phi^s z) \leq \epsilon \text{ for all } s \in [0, t]\}.$$

We have the following theorem:

Theorem 1.2.33 (Theorem 3.3 in [20]). *Assume that Λ is a basic hyperbolic set for Φ and that $\varphi : \Lambda \to \mathbb{R}$ is a Hölder continuous function. Then φ has a unique equilibrium state μ_φ. Furthermore, μ_φ is an ergodic probability measure and positive on non-empty open sets of Λ, and for any $\epsilon > 0$ there is $C_\epsilon > 0$ so that*

$$\mu_\varphi(B_{z,\Phi|\Lambda}(\epsilon, t)) \geq C_\epsilon \exp\left(-P(\Phi|_\Lambda, \varphi)t + \int_0^t \varphi(\Phi^s z) ds\right)$$

for all $z \in \Lambda, t > 0$.

Assume for the rest of the subsection that Λ is a basic hyperbolic set for an Axiom A flow Φ, $\varphi : \Lambda \to \mathbb{R}$ is a Hölder function and μ is the unique equilibrium state of φ with support in Λ. In order to state our result we require an important result by R. Bowen.

Theorem 1.2.34 (Main theorem in [13]). *There is a special flow $(\tilde{\Lambda}_f, \tilde{\Phi}_f)$ with Lipschitz roof function f and a finite to one continuous surjection $\rho : \tilde{\Lambda}_f \to \Lambda$ so that*

$$\rho \tilde{\Phi}_f^t = \Phi^t \rho$$

and for $z \in \tilde{\Lambda}_f$, the Φ^t-orbit of $\rho(z)$ is periodic, transitive, strongly recurrent, or almost periodic if and only if the $\tilde{\Phi}_f^t$-orbit of z is.

Using the map ρ in Theorem 1.2.34 there is a natural way to identify the probability measure μ_φ in Theorem 1.2.33 with an equilibrium state for a special flow. For this, recall that in ergodic theory two measure preserving transformations $(\mathcal{X}_i, \mathcal{B}_{\mathcal{X}_i}, m_i, \phi_i)$ are said to be isomorphic if there exist measure preserving maps $\psi_1 : \mathcal{X}_1 \to \mathcal{X}_2$, $\psi_2 : \mathcal{X}_2 \to \mathcal{X}_1$ such that $\psi_1\psi_2$ (respectively $\psi_2\psi_1$) is the identity on $\mathcal{X}_2$ (respectively $\mathcal{X}_1$) and $\phi_2\psi_1(x) = \psi_1\phi_1(x)$ for m_1-a.e. $x \in \mathcal{X}_1$, $\phi_1\psi_2(x) = \psi_2\phi_2(x)$ for m_2-a.e. $x \in \mathcal{X}_2$.

Proposition 1.2.35 ([20]). *The map ρ in Theorem 1.2.34 is a measure theoretic isomorphism between (Φ, Λ, μ) and $(\tilde{\Phi}_f, \tilde{\Lambda}_f, \tilde{\mu})$, where $\tilde{\mu}$ is the equilibrium state of $\varphi \circ \rho$.*

We finish by showing a concrete example of an Axiom A flow.

Example of an Axiom A flow

We consider the Smale horseshoe map defined by the diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ in Figure 1.6 that acts on the unit square $\mathcal{U} = [0,1] \times [0,1] \subset \mathbb{R}^2$. Recall the subsets $\mathcal{R}_0, \mathcal{R}_1$, the Cantor set Ω, and the conjugacy $\pi : \Omega \to \{0,1\}^{\mathbb{Z}}$ defined in Subsection 1.2.5, "Markov partition and semi-conjugacy for the horseshoe map". For every $m \in \mathbb{N}_0$ and $x \in \Omega$ denote the composition of m times f by $f^m(x) := f \circ \cdots f(x)$ (where f^0 is the identity map) and define

$$S^f_{m+1} g(x) := \sum_{i=0}^{m} g \circ f^i(x).$$

Let us consider the continuous function $g : \Omega \to \{1, 2\}$ defined by

$$g(x) = \begin{cases} 2 & \text{if } x \in \mathcal{R}_1, \\ 1 & \text{if } x \in \mathcal{R}_0. \end{cases}$$

We define the continuous action Φ^t_g on $\Lambda_g := \{(x,t) : x \in \Omega, 0 \leq t < g(x)\}$ onto itself defined by

$$\Phi^t_g(x, s) := \Big(f^m(x), s + t - S^f_m g(x)\Big) \text{ for } S^f_m g(x) \leq s + t < S^f_{m+1} g(x),$$

where $m \in \mathbb{N}_0$. We have that (Λ_g, Φ^t_g) is an Axiom A flow. In Figure 1.13 we "draw" the flow: the Cantor set Ω has been represented by the set

$$\tilde{\Omega} = f^{-4}(\mathcal{U}) \cap f^{-3}(\mathcal{U}) \cap f^{-2}(\mathcal{U}) \cap f^{-1}(\mathcal{U}) \cap \mathcal{U} \cap f(\mathcal{U}) \cap f^2(\mathcal{U}) \cap f^3(\mathcal{U}) \cap f^4(\mathcal{U})$$

that we drew in Figure 1.11. The set Λ is represented by the green and red parallelepipeds on $\tilde{\Omega} \subset \mathcal{U}$. The direction of the flow is upward in the picture. A point in Λ flows along the vertical direction up to reaching the top of the parallelepiped to which belongs to, then it goes down to the set $\Omega \times \{0\}$, according to the Smale horseshoe map.

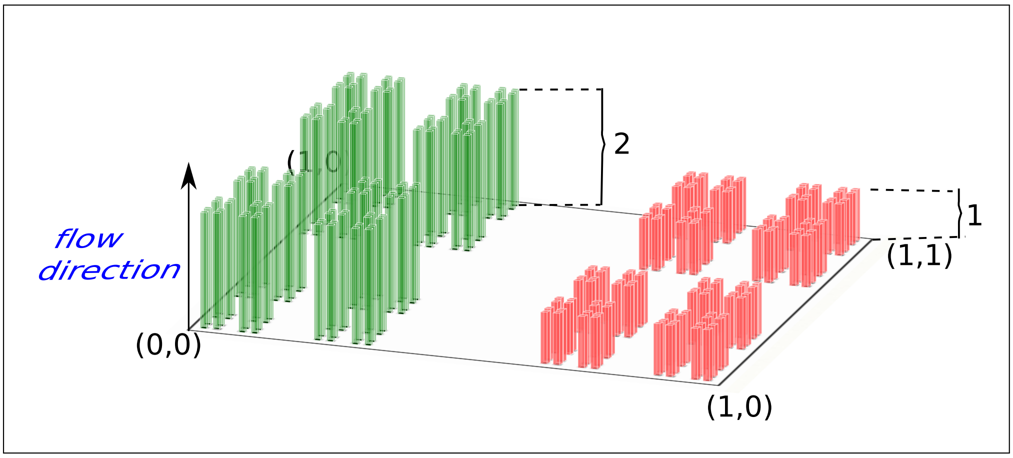


Figure 1.13: Example of Axiom A flow.

1.2.7 Stationary measures

In this thesis we are only concerned with the study of stationary measures for iterated function schemes. To make this precise, consider two complete metric spaces $(\mathcal{M}, d)$ and $(\mathcal{N}, \tilde{d})$. Define the Lipschitz semi norm Lip of $A : \mathcal{M} \to \mathcal{N}$ by

$$\mathrm{Lip}(A) = \sup_{x \neq y} \frac{\tilde{d}(A(x), A(y))}{d(x, y)}.$$

Definition 1.2.36 (Iterated function scheme)**.** *An iterated function scheme is a finite family of contractions with respect to* Lip*, i.e. a family of maps $\mathcal{T} = \{T_i\}_{i=1}^n$ where $n \in \mathbb{N}$, $T_i : \mathcal{M} \to \mathcal{M}$ and*

$$\max_{i=1,\ldots,n} \mathrm{Lip}(T_i) < 1.$$

Given an iterated function scheme $\mathcal{T}$, it follows that there exists a unique closed bounded set $\mathcal{K} \subset \mathcal{M}$ such that

$$\mathcal{K} = \cup_{i=1}^n T_i \mathcal{K}.$$

We call $\mathcal{K}$ the limit set of $\mathcal{T}$. A basic example to keep in mind is the case of $(\mathcal{M}, d) = ([0, 1], |\ |)$, for the unit interval $[0, 1]$ and the absolute value $|\ |$ on $\mathbb{R}$, and $T_1(x) = \frac{x}{3}$, $T_2(x) = \frac{x}{3} + \frac{2}{3}$. The limit set in this example is the famous middle third Cantor set.

Associated to the iterated function scheme $\mathcal{T}$, we can consider a family of weight functions $\mathcal{G} = \{g_i\}_{i=1}^n$, $g_i : \mathcal{M} \to (0,1)$ such that

$$\sum_{i=1}^{n} g_i \equiv 1 \text{ and} \tag{1.5}$$

$$\sum_{i=1}^{n} \|g_i\| \mathrm{Lip}(T_i) < 1, \tag{1.6}$$

where $\|g\| = \sup\{g(x) : x \in \mathcal{M}\}$.

Definition 1.2.37 (Stationary measure)**.** *Given an iterated function scheme $\mathcal{T}$ with weight functions $\mathcal{G}$, let $\mathcal{P}(\mathcal{M})$ be the set of Borel regular probability measures having bounded support. A stationary measure $\mu \in \mathcal{P}(\mathcal{M})$ is a fixed point for the operator $\mathscr{S} = \mathscr{S}_{\mathcal{T},\mathcal{G}} : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ defined by*

$$\mathscr{S}(\nu)(f) := \sum_{i=1}^{n} \int g_i(x) f(T_i(x)) d\nu(x),$$

where $\nu \in \mathcal{P}(\mathcal{M})$ and $f : \mathcal{M} \to \mathbb{R}$ is a continuous compactly supported function.

Remark 1.2.38. *Two direct but important facts from the definition of stationary measure are the following:*

(i) A stationary measure for $(\mathcal{T}, \mathcal{G})$ is supported on the limit set of $\mathcal{T}$ (a proof is given in [51], Section 4.4).

(ii) A probability measure $\mu \in \mathcal{P}(\mathcal{M})$ is a fixed point of $\mathscr{S}$ if and only if

$$\mathscr{S}(\mu)(f) = \int f(x) d\mu(x)$$

for every continuous compactly supported function $f : \mathcal{M} \to \mathbb{R}$.

We have the following well known theorem:

Theorem 1.2.39. *Suppose that $\mathcal{M}$ is a compact metric space. An iterated function scheme $\mathcal{T}$ with weight functions $\mathcal{G}$ satisfying (1.5) and (1.6) has a unique stationary measure.*

A proof of this theorem can be found in [51] for constant weight functions, using the contractive mapping principle. A small modification of the same argument can be applied here. Recall also that the existence of a stationary measure is a classic result [42] (Lemma 1.2).

Proof. The space $\mathcal{P}(\mathcal{M})$ can be equipped with the Kantorovich-Rubinshtein norm [8]

$$|||\mu||| = \sup\left\{\int f d\mu : f : \mathcal{M} \to \mathbb{R}, \mathrm{Lip}(f) \leq 1\right\}.$$

The operator $\mathscr{S}$ is a contraction on the space $(\mathcal{P}(\mathcal{M}), |||\ |||)$. Indeed, for $\mu, \nu \in \mathcal{P}(\mathcal{M})$ and a function $f : \mathcal{M} \to \mathbb{R}$, we have that

$$\mathscr{S}(\mu)(f) - \mathscr{S}(\nu)(f) = \int \sum_{i=1}^{n} g_i(x) f(T_i(x))(d\mu - d\nu)(x). \tag{1.7}$$

If $g : \mathcal{M} \to (0, 1)$ and $T : \mathcal{M} \to \mathcal{M}$ with $\mathrm{Lip}(T) < \infty$, then

$$\begin{aligned} &\sup\left\{\int f(T(x))g(x)d\mu(x) : f : \mathcal{M} \to \mathbb{R}, \mathrm{Lip}(f) \leq 1\right\} \\ &\leq \mathrm{Lip}(T)\|g\|\sup\left\{\int f d\mu(x) : f : \mathcal{M} \to \mathbb{R}, \mathrm{Lip}(f) \leq 1\right\}. \end{aligned} \tag{1.8}$$

From equations (1.7) and (1.8) we conclude that

$$|||\mathscr{S}(\mu) - \mathscr{S}(\nu)||| \leq \left(\sum_{i=1}^{n} \mathrm{Lip}(T_i)\|g_i\|\right) |||\mu - \nu||| = L|||\mu - \nu|||,$$

where $L = \sum_{i=1}^{n} \mathrm{Lip}(T_i)\|g_i\| < 1$ by hypothesis, and thus $\mathscr{S}$ is a contraction. On the other hand, $\mathcal{P}(\mathcal{M})$ with the metric $|||\,|||$ is a complete metric space. It follows that $\mathscr{S}$ has a unique fixed point on $\mathcal{P}(\mathcal{M})$ by the contraction mapping principle. □

Remark 1.2.40. *A proof that $\mathcal{P}(\mathcal{M})$ with the metric $|||\,|||$ is a complete metric space can be found in [53], Chapter 8, §4, where it is proved that $(\mathcal{P}(\mathcal{M}), |||\,|||)$ is a compact metric space. A more general result can be found in [59], Theorem 4.2. On the other hand, it is also possible to prove the completeness of $\mathcal{P}(\mathcal{M})$ with the metric $|||\,|||$ by using similar arguments to those in [66].*

First example of stationary measures

Our first example appears in [71] (Theorem 2.1), it is quite simple and allows to get an useful insight of how a stationary measure looks like.

Theorem 1.2.41 (Pincus)**.** *Let $T_i : [0, 1] \to [0, 1]$ be 1-1 Lipschitz transformations, $i = 1, 2$ with weight functions $\{g_1 \equiv p, g_2 \equiv 1 - p =: q\}$ for some $p \in (0, 1)$. Suppose additionally that:*

1. $T_1(0) = 0$ *and* $T_2(1) = 1$;

2. $\mathrm{Lip}(T_1) = \alpha$ *and* $\mathrm{Lip}(T_2) = \beta$*, where* $\alpha, \beta \in (0,1)$*;*

3. $T_1[0,1] \cap T_2[0,1] \neq \emptyset$*; and*

4. $q > \frac{\beta}{\alpha+\beta}$ *and* $\left(\frac{\beta}{q}\right)^q \left(\frac{\alpha}{p}\right)^p < 1$.

Then there is a unique stationary measure μ *on* $[0,1]$*, moreover:*

1. μ *is nonatomic (has no atoms);*

2. μ *is singular with respect to the Lebesgue measure on* $[0,1]$*; and*

3. the support of μ *is the unit interval* $[0,1]$.

Second example of stationary measures

We present in what follows an interesting example that relates iterated function schemes and stationary measures with the product of random matrices and Lyapunov exponents. More general examples built on similar ideas can be found in [73]. Let us consider $A_0, A_1 \in SL(2, \mathbb{R}^{>0})$ and denote by $\| \ \|$ a norm on the space $\mathbb{R}^{2\times 2}$. For $i = 0,1$ we define the map $\hat{A}_i : [0,1] \to [0,1]$ such that for $x \in [0,1]$

$$\frac{A_i(x, \sqrt{1-x^2})}{\|A_i(x, \sqrt{1-x^2})\|} = (x', y') = (\hat{A}_i(x), y') \in \mathbb{R}^2.$$

The iterated function scheme $\{\hat{A}_0, \hat{A}_1\}$ with weight functions $\{g_0 \equiv p, g_1 \equiv 1-p\}$ for some $p \in (0,1)$, has a unique stationary measure ν on $[0,1]$. Moreover, $\nu = \mu(\pi^{-1})$ where $\pi : \mathcal{X} := \{0,1\}^{\mathbb{N}_0} \to [0,1]$ is the projection map

$$\pi(i_0, i_1, \ldots) = \sum_{k=0}^{\infty} \frac{i_k}{2^{k+1}}$$

and μ is the Bernoulli measure of parameters $(p, 1-p)$ on $\mathcal{X}$. The Lyapunov exponent is defined for μ-a.e. $\underline{i} = (i_0, i_1, \ldots) \in \mathcal{X}$ by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \int \log\|A_{i_0} \cdots A_{i_{n-1}}\| d\mu(\underline{i}).$$

Using thermodynamic formalism in [73] (Lemma 3.2) or using the Furstenberg measure in [42], it is possible to prove that

$$\lambda = g_0 \int \log \left| \frac{d\hat{A}_0}{dx}(x) \right| d\nu(x) + g_1 \int \log \left| \frac{d\hat{A}_1}{dx}(x) \right| d\nu(x) \tag{1.9}$$

The relationship that we mentioned in the first paragraph comes from equation (1.9) and the application of the following theorem.

Theorem 1.2.42 (Pointwise version of Furstenberg and Kesten Theorem)**.** *Let $\{A_1, \ldots, A_k\}$ be a finite set of non-singular $d \times d$ real matrices with $d \geq 2$. Let $(p_1, \ldots, p_k)$ be a probability vector, and $\mu = (p_1, \ldots, p_k)^{\mathbb{N}_0}$ be the associated Bernoulli measure on the space of sequences $\mathcal{X} := \{1, \ldots, k\}^{\mathbb{N}_0}$. Then, for μ-a.e. $\underline{i} = (i_0, i_1, \ldots) \in \mathcal{X}$ one has*

$$\lim_{n\to\infty} \frac{1}{n} \int \log\|A_{i_0} \cdots A_{i_{n-1}}\| d\mu(\underline{i}) = \lim_{n\to\infty} \frac{1}{n} \|A_{i_0} \cdots A_{i_{n-1}}\|.$$

Remark 1.2.43. *Theorem 1.2.42 is a direct consequence of Theorem 2 in [43] and the application of the pointwise Birkhoff ergodic theorem for ergodic probability measures. A short and complete proof of Theorem 1.2.42 can be given using only Kingman's subadditive ergodic theorem. For this, see [87], Chapter 3, § 3.1 , Theorem 3.3, § 3.1.4 , Corollary 3.1 and § 3.2 , Theorem 3.12. Finally use that $\mu = (p_1, \ldots, p_k)^{\mathbb{N}_0}$ is ergodic.*

Iterated function schemes with stationary measures and Lyapunov exponents for the product of random matrices are two different subjects that have been independently studied. One may expect to use iterated function schemes with stationary measures to study the Hausdorff dimension of measures, because the Lyapunov exponent has been used to study it [10, 30].

1.3 Organisation

The main body of this thesis is divided into three main chapters. Chapter 2 contains our work related with the question: can we prove entry time results for different shrinking sets? Chapter 3 contains our work related with the question: can we prove escape rate results for smooth flows? Finally, Chapter 4 contains our answer of the question: how smoothly does the stationary measure change under smooth perturbations of the parameters that define it?

Chapter 2

Entry time statistics

2.1 Introduction

We develop further results about higher order entry times. That is the rate at which points enter to small sets.

Consider a measure preserving dynamical system $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ where μ is a finite, invariant and ergodic measure together with a sequence of Borel sets $\{\mathcal{U}_n\} = \{\mathcal{U}_n\}_{n\in\mathbb{N}}$ with $\mathcal{U}_n \subset \mathcal{X}, \mu(U_n) > 0$ and for which the sequence $\{\mathcal{U}_n\}$ shrinks to a point. Under an appropriate mixing assumption, one can generally show that for

$$\tau_n(x) = \tau_{\mathcal{U}_n}(x) := \inf\{k \geq 1 : T^k(x) \in \mathcal{U}_n\},$$

the sequence of random variables $X_n := \mu(\mathcal{U}_n)\tau_n$, called (rescaled) *entry time* or *first hitting time*, converges in law to an exponential random variable. The first paper related to this result is [36] for continued fractions. More recently such convergence results have been obtained for examples in which U_n are balls or cylinders shrinking to a point: for continuous time Markov chains see [5, 6], for expanding maps of the interval see [32, 33, 31], for general ϕ-mixing processes (consequently also for ψ-mixing processes) see [44], for α-mixing processes see [1, 3], for Axiom A diffeomorphisms see [48, 49], for Gibbs measures on shift spaces see [72], for unimodal maps see [21], for partially hyperbolic systems see [37, 24]. Some extensions of the classic Poisson limit theorem can be found in [56] and further related reviews in [2, 28, 45]. It is not clear however that the limit distribution of the sequence of random variables X_n is still valid if we remove the condition that $\{\mathcal{U}_n\}$ are shrinking to a single point.

In this chapter we consider the problem of finding conditions on the pair

$(\{\mathcal{U}_n\}, \mu)$ so that X_n converges in law to an exponential random variable when $\mu(\mathcal{U}_n) \to 0$ as $n \to \infty$, but the sequence $\{\mathcal{U}_n\}$ does not necessarily shrink to a single point (or finitely many points). The conditions we impose on $\mathcal{U}_n$ and μ are also used in [26, 46]. In Section 2.4 we discuss the relationship between the results in this chapter and those of [26, 46].

In our main theorem we obtain the convergence of X_n to an exponential random variable for general families of sets $\{\mathcal{U}_n\}$ under some conditions that depend on the return time to $\mathcal{U}_n$ defined by

$$\eta_n := \inf\{\tau_n(x) : x \in \mathcal{U}_n\}.$$

Indeed, let $(\mathcal{X}, \sigma)$ be a topologically mixing subshift of finite type with

$$\mathcal{X} \subset \prod_{-\infty}^{\infty}\{1, \ldots, a\} = \{1, \ldots, a\}^{\mathbb{Z}}.$$

We consider the pair $(\{\mathcal{U}_n\}, \mu)$, where $\mathcal{U}_n \subset \mathcal{X}$ is in the sigma-algebra generated by $\prod_{-n}^{n-1}\{1, \ldots, a\}$ and $\mu = \mu_{\text{Parry}}$ is the probability measure of maximum entropy or Parry measure.

Theorem 2.1.1 (Main Theorem)**.** *The sequence of random variables X_n converges in law to an exponential random variable if*

I. $\sigma^{-(n+1)}\mathcal{U}_{n+1} \subset \sigma^{-n}\mathcal{U}_n$ *for every* $n > 0$ *and* $n\mu\left(\mathcal{U}_{\lfloor \eta_n/2 \rfloor}\right) \to 0$ *as* $n \to \infty$,

II. the return times are given by $\eta_n = n + k(n) + 1$, *where* $k : \mathbb{N} \to \mathbb{N}$ *is a non decreasing function and* $n\mu\left(\mathcal{U}_{k(n)}\right) \to 0$ *as* $n \to \infty$, *or*

III. there exists a sequence $\{\mathcal{V}_n\}$ *with* $\{n\mu(\mathcal{V}_n)\} \to 0$ *as* $n \to \infty$, *where* $\mathcal{V}_n \supset \mathcal{U}_n$ *is a set in the sigma-algebra generated by* $\prod_{-n}^{-n+\lfloor \eta_n/2 \rfloor}\{1, \ldots, a\}$ *or* $\prod_{n-\lfloor \eta_n/2 \rfloor}^{n-1}\{1, \ldots, a\}$ *for every* $n \geq 1$.

An application of our method is to the study of Gibbs measures and sets which do not necessarily shrink to single points. Indeed, suppose that we have two subshifts of finite type $\mathcal{X}$ and $\mathcal{Y} \subset \mathcal{X}$, and the family of sets $\{\mathcal{U}_n\}, \mathcal{U}_n \subset \mathcal{X}$ satisfies $\cap_{n=1}^{\infty}\mathcal{U}_n = \{xy : y \in \mathcal{Y}\}$, where x is a forbidden sequence in $\mathcal{Y}$. We prove that for μ a Gibbs measure of Hölder potential, and under suitable conditions, that depend on μ and the topological entropy of $(\mathcal{X}, \sigma)$ and $(\mathcal{Y}, \sigma)$, the sequence X_n converges in law to an exponential random variable.

We can also apply our theorem to toral automorphisms. In particular we obtain that if $\{\mathcal{U}_n\}$ is a sequence of strips converging to a segment along the unstable direction, then X_n converges in law to an exponential random variable.

2.2 Motivation and basic definitions

The motivation to the study of the sequence of random variables X_n goes back to the Poincaré recurrence theorem.

Theorem 2.2.1 (Poincaré recurrence theorem)**.** *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ be a measure preserving dynamical system with finite measure. Then for any measurable set $\mathcal{A}$*

$$\mu(\mathcal{A}) = \mu\{x \in \mathcal{A} : T^n(x) \in \mathcal{A} \textit{ for infinitely many } n\}.$$

Definition 2.2.2 (Entrance time)**.** *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ be a measure preserving dynamical system. We define the entrance time (or hitting time) into a measurable set $\mathcal{A} \subset \mathcal{X}$ by*

$$\tau_{\mathcal{A}}(x) := \inf\{k \geq 1 : T^k(x) \in \mathcal{A}\},$$

where $x \in \mathcal{X}$. If $x \in \mathcal{A}$, the map $\tau_{\mathcal{A}}(x)$ is called return time (to $\mathcal{A}$).

In terms of the return time map, the Poincaré recurrence theorem says that if $\mu(\mathcal{A}) > 0$, then

$$\mu\{x \in \mathcal{A} : \tau_{\mathcal{A}}(x) < \infty\} = 1,$$

i.e. μ almost every point of $\mathcal{A}$ returns to $\mathcal{A}$. If the measure μ is also ergodic, then

$$\mu\{x \in \mathcal{X} : \tau_{\mathcal{A}}(x) < \infty\} = 1.$$

This in particular implies that the sequence $X_n := \mu(\mathcal{U}_n)\tau_{\mathcal{U}_n}$ is well defined whenever the measure μ is ergodic and $\mu(\mathcal{U}_n) > 0$.

Entry and return times are related by the following observation.

Remark 2.2.3. *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ be a measure preserving dynamical system. Then, for any measurable set $\mathcal{A} \subset \mathcal{X}$, we have that*

$$\mu(\{x \in \mathcal{X} : \tau_{\mathcal{A}}(x) = k\}) = \mu(\{x \in \mathcal{A} : \tau_{\mathcal{A}}(x) \geq k\}).$$

Proof. By definition we have for every $k \geq 1$

$$\begin{aligned}\{x \in \mathcal{X} : \tau_{\mathcal{A}}(x) > k\} &= \{x \in \mathcal{X} : T(x) \notin \mathcal{A}, \tau_{\mathcal{A}}(x) > k-1\} \\ &= T^{-1}(\mathcal{A} \cap \{x \in \mathcal{X} : \tau_{\mathcal{A}}(x) > k-1\}).\end{aligned}$$

By the invariance of the measure μ we have

$$\mu(\{x \in \mathcal{X} : \tau_{\mathcal{A}}(x) > k\}) = \mu(\{x \in \mathcal{X} : \tau_{\mathcal{A}}(x) > k-1\}) - \mu(\mathcal{A} \cap \{x \in \mathcal{X} : \tau_{\mathcal{A}}(x) > k-1\}),$$

and this concludes the proof. □

An important theorem known as the Kac's lemma gives more information about the return times than the Poincaré recurrence theorem, but only in the case that the measure is ergodic. Let us start with a definition.

Definition 2.2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, i.e. Ω is a set, $\mathcal{F}$ a sigma-algebra on Ω and $\mathbb{P}$ is a probability measure on $(\Omega, \mathbb{F})$. The expectation with respect to $\mathbb{P}$ of a measurable function $f : \Omega \to \mathbb{R}$ is defined by*

$$\mathbb{E}(f) = \mathbb{E}_{\mathbb{P}}(f) := \int_{\Omega} f d\mathbb{P}.$$

Theorem 2.2.5 (Kac's lemma [52])**.** *If $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ is a measure preserving dynamical system with finite and ergodic measure. Then for any measurable set $\mathcal{A} \subset \mathcal{X}$ with strictly positive measure*

$$\mathbb{E}_{\mu_{\mathcal{A}}}(\mu(\mathcal{A})\tau_{\mathcal{A}}) = 1,$$

where $\mu_{\mathcal{A}}(\cdot) := \frac{\mu(\cdot \cap \mathcal{A})}{\mu(\mathcal{A})}$.

This theorem motivates the following problem:

Problem 2.2.6. *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ be a measure preserving dynamical system with finite and ergodic measure. If we consider a sequence of measurable sets $\mathcal{U}_n \subset \mathcal{X}$ with strictly positive measure. What can we say about the limit of the sequence of random variables $X_n := \mu(\mathcal{U}_n)\tau_{\mathcal{U}_n}$?*

Remark 2.2.7. *The Kac's lemma motivates also other problems that we do not study in this thesis, for example: what are the statistics properties of $\mu(\mathcal{A})\tau_{\mathcal{A}}$? The study of this problem requires additionally mixing conditions on the measure. See for example [23].*

To study Problem 2.2.6 it is necessary an additional mixing condition on the measure μ. In particular, we will require ψ-mixing. See other kind of mixing conditions in [45].

Definition 2.2.8. *Let $(\mathcal{X}, \sigma)$ be a subshift of finite type with $\mathcal{X} \subset \{1, \ldots, a\}^{\mathbb{Z}}$ for some integer $a > 1$. The measure preserving dynamical system $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma)$ is ψ-mixing if for every $k \in \mathbb{N}$, $\mathcal{U}$ in the σ-algebra generated by $\prod_0^{k-1}\{1, \ldots, a\}$ and $\mathcal{V}$ in the σ-algebra generated by $\{1, \ldots, a\}^* := \cup_{n \in \mathbb{N}_0} \prod_0^n \{1, \ldots, a\}$ as the 'gap' $\triangle \to \infty$:*

$$\sup_n \sup_{\mathcal{U}, \mathcal{V}} \left| \frac{\mu(\mathcal{U} \cap \sigma^{-\triangle - k}\mathcal{V})}{\mu(\mathcal{U})\mu(\mathcal{V})} - 1 \right| = \psi_{\triangle} \to 0.$$

Along this chapter we investigate Problem 2.2.6 from the view point of probability theory. Let us recall some basic definitions that we will use. First, the exponential random variables.

Definition 2.2.9 (Exponential random variable)**.** *A random variable $X : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be an exponential random variable with parameter λ if it has cumulative distribution*

$$F_X(t) := \mathbb{P}\{w \in \Omega : X(w) \le t\} = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \ge 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Second, the convergence in law.

Definition 2.2.10 (Convergence in law)**.** *A sequence of random variables $\{X_n\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to converge in law to an exponential random variable of parameter 1 if for every $t \in \mathbb{R}$, its cumulative distribution converges to the cumulative distribution $F_X(t)$, of an exponential random variable X of parameter $\lambda = 1$. In other words,*

$$\lim_{n \to \infty} |\mathbb{P}\{w \in \Omega : X_n(x) \ge t\} - e^{-t}| = 0 \text{ for every } t > 0. \tag{2.1}$$

Our specific setting to study Problem 2.2.6 is the following.

Definition 2.2.11 (M-systems)**.** *We define a M-system as any system*

$$(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T, \{\mathcal{U}_n\}, \{X_n\})$$

where

(i) the system $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, T)$ is a measure preserving dynamical system and μ is an ergodic probability measure;

(ii) the sequence $\{\mathcal{U}_n\}$ is a sequence of Borel sets $\mathcal{U}_n \subset \mathcal{X}$ such that $\mu(\mathcal{U}_n) > 0$ and $\mu(\mathcal{U}_n) \to 0$; and

(iii) the sequence $\{X_n\}$ is a sequence of random variables $X_n : \mathcal{X} \to \mathbb{R}$ defined by $X_n(x) := \mu(\mathcal{U}_n)\tau_n(x)$.

Definition 2.2.12 (M_a-systems)**.** *We define a M_a-system as any M-system*

$$(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma, \{\mathcal{U}_n\}, \{X_n\})$$

where $(\mathcal{X}, \sigma)$ is a topologically mixing subshift of finite type with $\mathcal{X} \subset \{1, \ldots, a\}^{\mathbb{Z}}$ for some integer $a > 1$ and the system $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma)$ is a ψ-mixing measure preserving dynamical system where the sequence $\{\psi_{\triangle}\}_{\triangle \in \mathbb{N}}$ is bounded.

We investigate the convergence in law of X_n in a M_a-system. A useful trick that we learned from [31] is to consider

$$\lim_{n\to\infty} \left| \mu\left\{x \in \mathcal{X} : \tau_n(x) > \lfloor t/\mu(\mathcal{U}_n) \rfloor\right\} - (1 - \mu(\mathcal{U}_n))^{\lfloor t/\mu(\mathcal{U}_n) \rfloor} \right| = 0 \text{ for every } t > 0, \tag{2.2}$$

that is equivalent to (2.1) in the case that $\mu(\mathcal{U}_n) \to 0$.

2.3 Auxiliary results

The purpose of this section is understanding Theorem 2.1.1 with the help of a proposition (Main proposition) and its corollary (First corollary), that are interesting by themselves. The proofs of these auxiliary results will be used to prove Theorem 2.1.1.

Main proposition

We present an important proposition, indeed a few improvements of it will prove Theorem 2.1.1.

Proposition 2.3.1 (Main proposition)**.** *Suppose that $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma, \{\mathcal{U}_n\}, \{X_n\})$ is a M_a-system where $\mathcal{U}_n$ is in the sigma algebra generated by $\prod_0^{n-1}\{1, \ldots, a\}$ for every $n \geq 1$. If*

$$\eta_n = n \tag{2.3}$$

and

$$n\mu(\mathcal{U}_n) \to 0 \text{ as } n \to \infty \tag{2.4}$$

then the sequence of random variables X_n converges in law to an exponential random variable of parameter 1.

Notice that the condition $\eta_n = n$ is equivalent to $\eta_n \geq n$, because of the definition of the sequence of sets $\{\mathcal{U}_n\}$. As an application of the proposition we can give the following example.

Example 2.3.2. *Suppose that $\mathcal{X} = \{0, 1, 2\}^{\mathbb{Z}}$ and*

$$\mathcal{U}_n = \sqcup_{x_1,\ldots,x_{x_{n-1}} \in \{1,2\}} [0, x_1, x_2, \ldots, x_{n-1}]_n.$$

Let μ be a Bernoulli probability measure on $\mathcal{X}$ defined by a probability vector $(p_0, p_1, p_2) \in (0,1)^3$. Then $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma, \{\mathcal{U}_n\}, \{X_n\})$ is a M_a-system and (2.3), (2.4) are satisfied, therefore Proposition 2.3.1 applies.

Another application is to Gibbs measures, because they are automatically ψ-mixing. We have the following remark.

Remark 2.3.3 (Gibbs measures are ψ-mixing)**.** *The explicit formula for ψ is written in [16]: Theorem 1.7, Lemma 1.8, Lemma 1.9, Lemma 1.10, Lemma 1.12, Lemma 1.3 and Proposition 1.14. Suppose that we have a measure preserving dynamical system $(\mathcal{X}_A, \mathcal{B}_{\mathcal{X}_A}, \mu, \sigma)$, where $(\mathcal{X}_A, \sigma)$ is a subshift of finite type and μ is the Gibbs measure of Holder potential $\phi : \mathcal{X}_A \to \mathbb{R}$. We require here some extra definitions in order to define the Perron Frobenius-Ruelle operator (or transfer operator) that will be used to obtain the formula of ψ. We define the one-sided shift of finite type $\mathcal{X}^+ = \mathcal{X}_A^+$ by $\mathcal{X}^+ := \{(x_n)_{n=0}^{\infty} : A(x_n, x_{n+1}) = 1$ for all $n \in \mathbb{N}_0\}$. We can define the shift action σ on $\mathcal{X}^+$ by $\sigma(x)_n = x_{n+1}$ for all $n \in \mathbb{N}_0$. Two functions $\phi, \varphi \in \mathcal{C}$ are said to be homologous if there is $u \in \mathcal{C}$ so that $\phi(x) = \varphi(x) - u(x) + u(\sigma x)$ for all $x \in \mathcal{X}$. Given a function $\phi \in \mathcal{F}$, there exists an homologous function $\varphi \in \mathcal{F}$ such that $\varphi(x) = \varphi(y)$ if and only if $x_i = y_i$ for every $i \in \mathbb{N}_0$ ([16]). We can define from φ a function $\hat{\phi} : \mathcal{X}^+ \to \mathbb{R}$ by taking $\hat{\phi}(\{x_i\}_{i=0}^{\infty}) = \varphi(\{y_i\}_{i=-\infty}^{\infty})$ for $\{y_i\}_{i=0}^{\infty} = \{x_i\}_{i=0}^{\infty}$ and some election of $\{y_i\}_{i=\infty}^{-1}$ so that $\{y_i\}_{i=-\infty}^{\infty} \in \mathcal{X}$. The Perron Frobenius operator*

$$\mathscr{L} f(x) = \sum_{y \in \sigma^{-1}(x)} e^{\hat{\phi}(y)} f(y)$$

acts on the space of continuous functions $f : \mathcal{X}_A^+ \to \mathbb{R}$. Theorem 1.7 in [16] proves that there exist $\lambda > 0$, a continuous function $h : \mathcal{X}_A^+ \to \mathbb{R}^{>0}$ and $\nu \in \mathcal{M}_\sigma$ such that $\mathscr{L} h = \lambda h$, $\mathscr{L}^* \nu = \lambda \nu$, and $\int h(x) d\nu(x) = 1$. We fix this h.

- *Let $M > 0$ such that $A^M > 0$.*

- *Let* $\alpha, b \in (0,1)$ *such that* $\sup\{|\phi(x) - \phi(y)|\colon x_i = y_i \text{ for } |i|< k\} \le b\alpha^k$ *for all* $k \in \mathbb{N}_0$.
- *Define* $B_m := \exp(\sum_{k=m+1}^{\infty} 2b\alpha^k)$ *for* $m \in \mathbb{N}_0$.
- *Define* $K := \lambda^M e^{M\|\phi\|} B_0$.
- *Take* $L > 0$ *such that* $B_m, B_m^{-1}, B_{m+1}e^{b\alpha^m} \in [-L, L]$ *for all* $m \in \mathbb{N}_0$.
- *Find a pair of constants* $0 < u_1, u_2$ *such that* $u_1(x-y) \le e^x - e^y \le u_2(x-y)$ *for all* $x, y \in [-L, L]$, $x > y$.
- *Define* $\eta := u_2(1-\alpha)(4\alpha u_1 \|h\| K)^{-1}$.
- *Define* $\beta := \sqrt[M]{1-\eta}$.
- *Define* $\kappa := (1-\eta)^{-1}(\|h\| + K) \sup_{0\le r<M} \|\lambda^{-r}\mathscr{L}^r\|$.

Then $\psi_\triangle = \kappa K \beta^\triangle$.

Before giving an example in the case of Gibbs measures let us mention that Proposition 2.3.1 gives a condition that depends on the pressure.

Remark 2.3.4. *Let* $(\mathcal{X}, \sigma)$ *be a subshift of finite type and* $\mathcal{U}_n = [x^1]_n \sqcup \cdots \sqcup [x^{m_n}]_n$, *where* $x^1, \ldots, x^{m_n} \in \mathcal{X}$, *for every* $n \ge 1$. *If* μ *is a Gibbs measure on* $\mathcal{X}$ *of Hölder potential* $\phi : \mathcal{X} \to \mathbb{R}$, *then*

$$0 \le \mu(\mathcal{U}_n) \le m_n \exp\left(-Pn + \|S_n^\sigma \phi\|\right), \tag{2.5}$$

for some constants $c > 0$. *In particular, we deduce that the hypothesis*

$$nm_n \exp\left(-Pn + \|S_n^\sigma \phi\|\right) \to 0 \text{ as } n \to \infty$$

is enough in order to satisfy the hypothesis (2.4).

Example 2.3.5. *Let* A *be an irreducible and aperiodic matrix with entries* 0 *and* 1 *of size* $a \times a$ *with* $a > 2$ *and let* B *be a submatrix of* A *of size* $b \times b$ *with* $b \in \{2, \ldots, a-1\}$. *Denote by* λ_A *and* λ_B *the Perron eigenvalues of* A *and* B, *respectively. Consider the subshift of finite type* $\mathcal{X}_A \subset \{1, \ldots, a\}^{\mathbb{Z}}$ *and* $\mathcal{X}_B \subset \mathcal{X}_A$. *Suppose without lost of generality that* $\mathcal{X}_B \subset \{1, \ldots, b\}^{\mathbb{Z}}$. *Define for* $n \in \mathbb{N}$,

$$\mathcal{U}_n = \sqcup_{x_1, \ldots, x_{n-1} \in \{1,2,\ldots b\}} [a x_1 \cdots x_{n-1}]_n,$$

then the sequence of random variables $X_n := \mu(\mathcal{U}_n)\tau_n$ converges in law to an exponential random variable of parameter 1 *for any equilibrium state with Hölder potential ϕ such that* $3\,\|\phi\| < \lambda_A - \lambda_B$. *As (2.4) is clearly satisfied, then Proposition 2.3.1 applies.*

We extend Proposition 2.3.1 in order to allow short return times. In this case we do need to care about considering shrinking sets, unlike in previous construction.

First corollary

Let us start with a definition.

Definition 2.3.6. *We say that a sequence of sets $\{\mathcal{U}_n\}$ is a sequence of shrinking sets if $\mathcal{U}_n \supset \mathcal{U}_{n+1}$ for every $n \in \mathbb{N}$.*

We can now state our first corollary.

Corollary 2.3.7 (First corollary)**.** *Suppose that $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma, \{\mathcal{U}_n\}, \{X_n\})$ is a M_a-system where $\mathcal{U}_n$ is a Borel set in the sigma algebra generated by $\prod_0^{n-1}\{1, \ldots, a\}$ for every $n \geq 1$. If the sequence of sets $\{\mathcal{U}_n\}$ is shrinking and it satisfies*

$$n\mu\left(\mathcal{U}_{\lfloor \eta_n/2 \rfloor}\right) \to 0 \text{ as } n \to \infty, \tag{2.6}$$

then the sequence of random variables X_n converges in law to an exponential random variable of parameter 1.

Example 2.3.8. *Let us choose $x \in \mathcal{X}$ and consider $\{\mathcal{U}_n\} = \{[x]_n : n \in \mathbb{N}\}$. If $n\mu\left([x]_{\lfloor \eta_n/2 \rfloor}\right) \to 0$, then the sequence of random variables $X_n := \mu([x]_n)\tau_n$ converges in law to an exponential random variable of parameter* 1.

Example 2.3.9. *In particular, for the measure of maximum entropy μ_{Parry}. If $\{\mathcal{U}_n\}$ is a shrinking sequence and $n\mu_{Parry}(\mathcal{U}_{\lfloor \eta_n/2 \rfloor}) \to 0$, then $X_n := \mu_{Parry}(\mathcal{U}_n)\tau_n$ converges in law to an exponential random variable of parameter* 1. *This result is not sharp, but to have certain control on η_n is necessary.*

Example 2.3.10. *Suppose that $(\mathcal{X}, \sigma)$ is the full shift in two symbols, i.e its transition matrix A is a two by two matrix with* 1 *in each coordinate, and suppose that μ is the uniform probability measure, i.e. $\mu([x]_n) = 2^{-n}$ for every $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Then $X_: = \mu(\mathcal{U}_n)\tau_n$ converges in law to an exponential random variable of parameter* 1 *for any sequence of sets $\{\mathcal{U}_n\}$ such that $\eta_n = \log_2(n) + k_n$, where $\{k_n\} \subset \mathbb{N}$ is a divergent sequence.*

2.4 Related results

In this section we compare the results in [46] and [26] with the original results of this chapter. We define in what follows the first hitting time distribution and the first return time distribution, both will be related by Theorem 2.4.3.

Definition 2.4.1 (From [46])**.** *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma, \{\mathcal{U}_n\}, \{X_n\})$ be a M-system.*

(i) The first hitting time distribution is defined by

$$F_{X_n}(t) := \mu(X_n < t)$$

for $t > 0$.

(ii) The first return time distribution is defined by

$$\tilde{F}_{X_n}(t) := \frac{1}{\mu(\mathcal{U}_n)} \mu(\mathcal{U}_n \cap \{X_n < t\})$$

for $t > 0$.

Remark 2.4.2. *Notice that the first hitting time distribution coincides with the cumulative distribution we defined in Subsection 2.3.*

The main theorem in [46] is the following.

Theorem 2.4.3 (Main theorem in [46])**.** *There exists a distribution function F such that F_{X_n} converges weakly to F if and only if there exists a distribution function $\tilde{F}$ such that $\tilde{F}_{X_n}$ converges weakly to $\tilde{F}$, moreover, $F(t) = \int_0^t (1 - \tilde{F}(s))ds$.*

In our results we consider a M_a-system, therefore Theorem 2.4.3 will be always available, because a M_a-system is a ψ-mixing M-system. In particular, this theorem shows that the exponential distribution is the only distribution which can be asymptotic to both F_{X_n} and $\tilde{F}_{X_n}$. However, if the measure μ is ψ-mixing, we can provide a direct proof of the fact that the exponential distribution is asymptotic to both F_{X_n} and $\tilde{F}_{X_n}$. This is the content of the next corollary.

Corollary 2.4.4 (Second corollary)**.** *Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma, \{\mathcal{U}_n\}, \{X_n\})$ be a M_a-system where $\mathcal{U}_n$ is in the sigma-algebra generated by $\prod_0^{n-1}\{1, \dots, a\}$ for every $n \geq 1$. If*

$$\eta_n = n \tag{2.7}$$

and

$$n\mu(\mathcal{U}_n) \to 0 \text{ as } n \to \infty \tag{2.8}$$

then

$$\lim_{n\to\infty}\left|\frac{\mu\{x\in\mathcal{U}_n : \tau_n > \lfloor t/\mu(\mathcal{U}_n)\rfloor\}}{\mu(\mathcal{U}_n)} - (1-\mu(\mathcal{U}_n))^{\lfloor t/\mu(\mathcal{U}_n)\rfloor}\right| = 0, \tag{2.9}$$

for every $t > 0$.

The proof of Corollary 2.4.4 is a direct consequence of Proposition 2.3.1 and Theorem 2.4.3. We include in Subsection 2.5.4 a direct proof that only uses Proposition 2.3.1.

We now move to study the relationship of our results with [26]. For this, let us introduce two definitions.

Definition 2.4.5. *Let* Ω *be a non empty set and* $\mathcal{U}\subset\Omega$ *a subset. We define the map* $\mathbb{1}_{\mathcal{U}}:\Omega\to\{0,1\}$ *by*

$$\mathbb{1}_{\mathcal{U}}(x) := \begin{cases} 1 & \textit{if } x\in\mathcal{U}, \\ 0 & \textit{if } x\notin\mathcal{U}. \end{cases}$$

Definition 2.4.6 (From [26])**.** *Let* $(\mathcal{X},\mathcal{B}_{\mathcal{X}},\mu,T,\{\mathcal{U}_n\},\{X_n\})$ *be a* M*-system. We define the sequence of hitting times as the sequence of random variables* $Y_n:\mathcal{X}\to\mathcal{M}$ *with*

$$Y_n(x) := \sum_{k>0}\mathbb{1}_{\mathcal{U}_n}(T^k x)\delta_{kc_n},$$

where $\mathcal{M}$ *is the set of Borel sigma-finite measures on* $[0,\infty)$, δ_t *is the Dirac measure at* $t>0$ *and* $\{c_n\}$ *is a chosen sequence such that* $c_n\to 0$ *as* $n\to\infty$.

The main difference between the results in this chapter and the ones in [26] is that, whereas we study the statistics of first hitting times, [26] studies the statistics of hitting times. On the other hand, in [26] it is necessary to impose two conditions on the measure μ and the sequence of sets $\{\mathcal{U}_n\}$, the first in particular implies that the limit $\lim_{n\to\infty} c_n^{-1}\mu(\mathcal{U}_n)$ exists and the second is a kind of ψ-mixing condition. In our results we require similar conditions (M_a-systems).

Regarding the demonstrations, in [26], the proof relies on the application of the Laplace Transforms Technique in a similar way than in [29] and the study of the Pianigiani-Yorke measure in [34]. Our proofs instead follows an idea in [31], Theorem 8.11, which refers the idea back to [9].

2.5 Proofs

We have separated the proofs of our results into four subsections. In §2.5.1 we introduce the notation that will be used along the section and we prove Proposition 2.3.1. In §2.5.2 we prove Corollary 2.3.7. In §2.5.3 we combine what we did in §2.5.1 and §2.5.2 in order to prove Theorem 2.1.1. Finally, in §2.5.4 we prove Corollary 2.4.4 using only ideas from the first subsection.

2.5.1 Main proposition

The goal of this subsection is to prove Proposition 2.3.1. In what follows we suppose that $(\mathcal{X}, \sigma, \mu)$ is a subshift of finite type where μ is a probability measure on $\mathcal{B}_{\mathcal{X}}$ and that we have a sequence $\{\mathcal{U}_n\}$ of Borel sets $\mathcal{U}_n \subset \mathcal{X}$ such that $\mu(\mathcal{U}_n) \to 0$. Instead of (2.1) we consider the equivalent condition (2.2). We prove the assertion (2.2) in several steps. The first is to replace (2.2) by a limit involving the sum of N terms.

Lemma 2.5.1. *For $n \in \mathbb{N}$, $\epsilon_n = \mu(\mathcal{U}_n)$ and $N = [t/\epsilon_n]$ with $t > 0$ we have that*

$$\mu\{\tau_n > N\} - (1-\epsilon_n)^N = \sum_{q=0}^{N-1} (1-\epsilon_n)^{N-q-1}(\mu\{\tau_n > q+1\} - (1-\epsilon_n)\mu\{\tau_n > q\}). \tag{2.10}$$

The proof of Lemma 2.5.1 consists in noticing that most of the term in the sum of the right hand side of the equation (2.10) cancel when summed with other term, the unique terms that do not cancel are indeed the ones on the left hand side of the equation.

For what follows we require additionally that μ is an invariant probability measure on $\mathcal{B}_{\mathcal{X}}$. For $p, q \in \mathbb{N}_0$ we use the notation $\mu\{\tau_p > q\}$ instead of $\mu\{x \in \mathcal{X} : \tau_p(x) > q\}$ and for every $n \in \mathbb{N}$ we define $\epsilon_n := \mu(\mathcal{U}_n)$ and $N := [t/\epsilon_n]$, where $t > 0$.

For $q \in \{0, \dots, N-1\}$ define

$$p_q(n) := (1-\epsilon_n)^{N-q-1}(\mu\{\tau_n > q+1\} - (1-\epsilon_n)\mu\{\tau_n > q\}),$$

$S_1(N) := \sum_{q=0}^{n-1} p_q(n)$ and $S_2(N) := \sum_{q=n}^{N-1} p_q(n)$. To obtain (2.2) we will show with the help of a few lemmas that

$$\mu\{\tau_n > N\} - (1-\epsilon_n)^N = S_1(N) + S_2(N) \to 0 \text{ as } n \text{ tends to infinity.}$$

The second step of our proof is to bound the term $S_1(N)$. It will be useful to denote by $\mathbb{E}$ the expectation with respect to μ.

We have the following lemma.

Lemma 2.5.2. *For all $n \in \mathbb{N}$, we have that $S_1(N) \leq n\epsilon_n$.*

The proof is a direct consequence of a useful identity in the next lemma that requires the measure μ to be invariant.

Lemma 2.5.3. *For all $n \in \mathbb{N}$, we have that for all $q \in \{0, \dots, N-1\}$*

$$\mu\{\tau_n > q+1\} - (1-\epsilon_n)\mu\{\tau_n > q\} = \epsilon_n \mu\{\tau_n > q\} - \mu\{x \in \mathcal{U}_n : \tau_n(x) > q\}.$$

Proof. The result is direct from the definitions of the sets $\{\tau_n > q+1\}$ and $\{\tau_n > q\}$. Indeed, we can write the following identities:

$$\begin{aligned}
&\mu\{\tau_n > q+1\} - (1-\epsilon_n)\mu\{\tau_n > q\} \\
&= \mathbb{E}\left(\prod_{i=1}^{q+1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - (1-\epsilon_n)\mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{q+1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - (1-\epsilon_n)\mathbb{E}\left((\mathbb{1}_{\mathcal{U}_n^c} + \mathbb{1}_{\mathcal{U}_n})\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{q+1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - (1-\epsilon_n)\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n^c}\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - (1-\epsilon_n)\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{q+1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n^c}\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \\
&\quad + \epsilon_n\mathbb{E}\left(\mathbb{1}_{U_n^c}\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - (1-\epsilon_n)\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)
\end{aligned}$$

$$\begin{aligned}
&= \epsilon_n\mathbb{E}\left((1-\mathbb{1}_{\mathcal{U}_n})\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - (1-\epsilon_n)\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \\
&= \epsilon_n\mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\mu\{\tau_n > q+1\} - (1-\epsilon_n)\mu\{\tau_n > q\} \\
&\quad = \epsilon_n\mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right),
\end{aligned} \tag{2.11}$$

and this is enough to conclude the proof, because

$$\epsilon_n \mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) = \epsilon_n \mu\{\tau_n > q\} - \mu\{x \in \mathcal{U}_n : \tau_n(x) > q\}.$$

□

In the third step, and most difficult one we bound $S_2(N)$. This step requires additionally that μ is ψ-mixing. Or equivalently, that $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma, \{\mathcal{U}_n\}, \{X_n\})$ is a M_a-system. In order to obtain an upper bound for $S_2(N)$ we will state a lemma with some intermediate bounds.

Lemma 2.5.4. *For all $n, k \in \mathbb{N}$, we have*

$$\mathbb{E}(\mathbb{1}_{\mathcal{U}_n} \mathbb{1}_{\mathcal{U}_n} \circ \sigma^{n+k}) \leq \epsilon_n^2 (1 + \psi_k), \tag{2.12}$$

and for $q \in \{2n+1, \ldots, N-1\}$ we have

$$\left| \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \right| \leq \epsilon_n^2 (n + \sum_{i=0}^{n-1} \psi_i), \tag{2.13}$$

$$\left| \mathbb{E}\left(\prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \right| \leq n\epsilon_n \tag{2.14}$$

and

$$\left| \epsilon_n \mathbb{E}\left(\prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \right| \leq \epsilon_n \psi_n. \tag{2.15}$$

Proof. Inequality (2.12) requires the ψ-mixing condition and inequality (2.13) is a consequence of (2.12).

Proof of (2.12) We can use the ψ-mixing condition to conclude that

$$\left|\mathbb{E}(\mathbb{1}_{\mathcal{U}_n} \mathbb{1}_{\mathcal{U}_n} \circ \sigma^{n+k}) - \epsilon_n^2\right| = \left|\mathbb{E}(\mathbb{1}_{\mathcal{U}_n} \mathbb{1}_{\mathcal{U}_n} \circ \sigma^{n+k}) - \mu(\mathcal{U}_n)^2\right| \leq \epsilon_n^2 \psi_k.$$

Then trivially

$$\mathbb{E}(\mathbb{1}_{\mathcal{U}_n} \mathbb{1}_{\mathcal{U}_n} \circ \sigma^{n+k}) \leq \epsilon_n^2 \psi_k + \epsilon_n^2,$$

as required.

Proof of (2.13) Let $n, k \in \mathbb{N}$ and $q \in \{2n+1, \dots, N-1\}$. We have

$$\begin{aligned}
&\left| \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \right| \\
&= \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \cdot \left(1 - \prod_{i=n}^{2n-1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \right) \\
&\leq \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \left(1 - \prod_{i=n}^{2n-1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \right) \\
&\leq \sum_{i=n}^{2n-1} \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i \right) \leq \epsilon_n^2 \sum_{i=0}^{n-1} \psi_i + n\epsilon_n^2,
\end{aligned}$$

where we have used (2.12) in the last inequality.

Proof of (2.14) Let $n, k \in \mathbb{N}$ and $q \in \{2n+1, \dots, N-1\}$. We have

$$\begin{aligned}
&\left| \mathbb{E}\left(\prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) - \mathbb{E}\left(\prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \right| \\
&= \mathbb{E}\left(\prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \cdot \left(1 - \prod_{i=n}^{2n-1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \right) \\
&\leq \mathbb{E}\left(1 - \prod_{i=n}^{2n-1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \leq \sum_{i=n}^{2n-1} \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \circ \sigma^i \right) = n\epsilon_n.
\end{aligned}$$

Proof of (2.15) Let $n, k \in \mathbb{N}$ and $q \in \{2n+1, \dots, N-1\}$. We have

$$\begin{aligned}
&\left| \epsilon_n \mathbb{E}\left(\prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \right| \\
&\leq \psi_n \epsilon_n \mathbb{E}\left(\prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \leq \psi_n \epsilon_n,
\end{aligned}$$

because of the condition of ψ-mixing.

□

We can now bound $S_2(N)$.

Lemma 2.5.5. *For all $n \in \mathbb{N}$, we have that*

$$S_2(N) \leq 4(n+1)(t+1)\epsilon_n + (t+1)\psi_n + N\epsilon_n^2 \sum_{i=0}^{n-1} \psi_i.$$

Proof. Recall that $S_2(N) = \sum_{q=n}^{N-1} p_q(n)$. From (2.11) we have that

$$\begin{aligned} p_q(n) &= (1-\epsilon_n)^{N-q-1}\left(\epsilon_n \mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right) \\ &\le \epsilon_n \mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \end{aligned}$$

for every $q \in \{n, \ldots, N-1\}$.

For a fixed q we bound $p_q(n)$ by the sum of two terms:

$$\begin{aligned} &\epsilon_n \mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \\ &\le \left|\epsilon_n \mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \epsilon_n \mathbb{E}\left(\prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\ &\quad + \left|\epsilon_n \mathbb{E}\left(\prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| =: S_{21}(N,q) + S_{22}(N,q). \end{aligned}$$

Notice that we have used (2.3) to obtain that

$$\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) = \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right).$$

Our goal now is to bound $S_{21}(N,q)$ and $S_{22}(N,q)$. For the first term we have the simple inequality $S_{21}(N,q) \le n\epsilon_n^2$, because

$$\begin{aligned} S_{21}(N,q) &\le \epsilon_n \mathbb{E}\left(\left|\prod_{i=1}^{n-1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i - 1\right| \cdot \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \\ &\le \epsilon_n \mathbb{E}\left(1 - \prod_{i=1}^{n-1} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) \le \epsilon_n \mathbb{E}\left(\sum_{i=1}^{n-1} \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) = n\epsilon_n^2. \end{aligned}$$

To bound $S_{22}(N,q)$ we use Lemma 2.5.4. Suppose that $q > 2n$, then

$$\begin{aligned}
S_{22}(N,q) &= \left|\epsilon_n \mathbb{E}\left(\prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\
&\le \left|\epsilon_n \mathbb{E}\left(\prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\
&\quad + \epsilon_n \left|\mathbb{E}\left(\prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\
&\quad + \left|\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=2n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\
&=: I_1(N,q) + I_2(N,q) + I_3(N,q).
\end{aligned}$$

We have that $I_1(N,q) \le \epsilon_n \psi_n$ by (2.15), $I_2(N,q) \le n\epsilon_n^2$ by (2.14) and $I_3(N,q) \le \epsilon_n^2 \sum_{i=0}^{n-1} \psi_i + n\epsilon_n^2$ by (2.13). In the case $n \le q \le 2n$ we can use that $S_{22}(N,q) \le \epsilon_n$. Finally,

$$\begin{aligned}
S_2(N) &\le \sum_{q=n}^{N} (S_{21}(N,q) + S_{22}(N,q)) \\
&= \sum_{q=n}^{N} S_{21}(N,q) + \sum_{q=n}^{2n} S_{22}(N,q) + \sum_{q=2n+1}^{N} S_{22}(N,q) \\
&\le Nn\epsilon_n^2 + (n+1)\epsilon_n + N(I_1(N,q) + I_2(N,q) + I_3(N,q)) \\
&\le \epsilon_n(n(t+1) + (n+1)) + (t+1)\psi_n + 2(t+1)n\epsilon_n + N\epsilon_n^2 \sum_{i=0}^{n-1} \psi_i,
\end{aligned}$$

which concludes the proof. □

2.5.2 First corollary

The proof of the corollary is very similar to the one of Proposition 2.3.1, but we need to modify some details. To complete the proof we require the following lemma, where $n \in \mathbb{N}$ and $q > n$.

Lemma 2.5.6. *If $\eta_n = n$ then*

$$\left|\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=\eta_n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| = 0$$

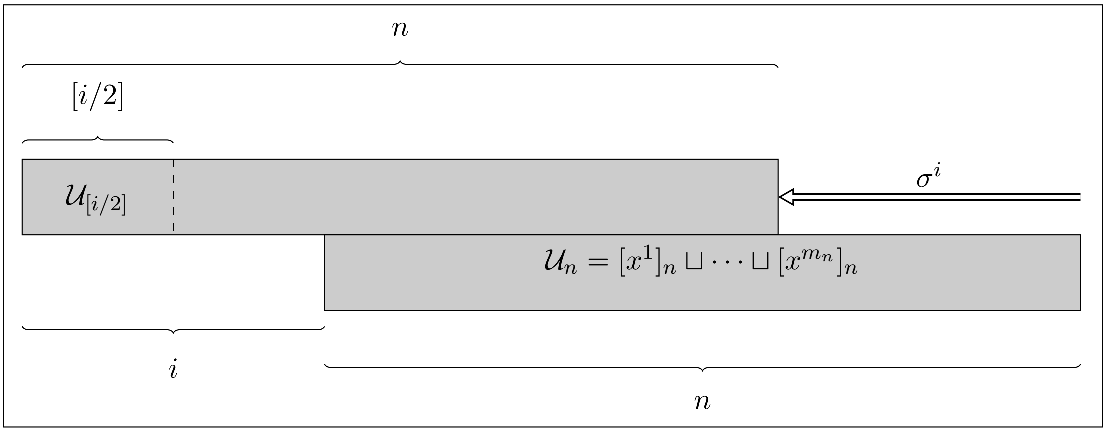


Figure 2.1: Proof of the inequality (2.16).

and if $\eta_n < n$ then

$$\left| \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=\eta_n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \right|$$
$$\leq n\epsilon_n \epsilon_{\lfloor \eta_n/2 \rfloor}(1 + \psi_{\lfloor \eta_n/2 \rfloor}).$$

Proof. The case $\eta_n = n$ is trivial, so suppose that $\eta_n < n$. It is clear that

$$\left| \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=\eta_n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i \right) \right| \leq \sum_{i=\eta_n}^{n-1} \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i \right).$$

In Figure 2.1 we have chosen $i \in \{\eta_n, \ldots, n-1\}$ and we represent the set $\mathcal{U}_n = [x^1]_n \sqcup \cdots \sqcup [x^{m_n}]_n$ for some $x^1, \ldots, x^{m_n} \in \mathcal{X}$, $m_n \in \mathbb{N}$ and the set $\sigma^i \mathcal{U}_n$. We have also draw a representation of the set $\mathcal{U}_{[i/2]}$. We can see that the action of the shift moved the rectangle at the bottom to the left, the result is the rectangle that we draw at the top. It is clear that

$$\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i \right) \leq \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_{[i/2]}} \circ \sigma^i \right) = \mu\left(\mathcal{U}_n \cap \sigma^{-i} \mathcal{U}_{[i/2]} \right).$$

From the ψ-mixing condition of the measure μ we obtain that

$$|\mu(\mathcal{U}_n \cap \sigma^{-i} \mathcal{U}_{[i/2]}) - \epsilon_n \epsilon_{[i/2]}| \leq \psi_{[i/2]} \epsilon_{[i/2]} \epsilon_n,$$

and therefore

$$\sum_{i=\eta_n}^{n-1} \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) \leq \sum_{i=\eta_n}^{n-1} \left(\psi_{[i/2]}\epsilon_{[i/2]}\epsilon_n + \epsilon_n\epsilon_{[i/2]}\right) \tag{2.16}$$
$$\leq n\epsilon_n\epsilon_{\lfloor \eta_n/2 \rfloor}(1 + \psi_{\lfloor \eta_n/2 \rfloor}).$$

□

The proof of Corollary 2.3.7 comes from the observation that for $q \in \{n, \ldots, N-1\}$ we have

$$\begin{aligned}
&\left|\epsilon_n\mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\
&\leq \left|\epsilon_n\mathbb{E}\left(\prod_{i=1}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \epsilon_n\mathbb{E}\left(\prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\
&\quad + \left|\epsilon_n\mathbb{E}\left(\prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\
&\quad + \left|\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=\eta_n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \\
&:= S_{21}(N,q) + S_{22}(N,q) + S_{\text{extra}}(N,q).
\end{aligned}$$

We can use Lemma 2.5.6 to bound $S_{\text{extra}}(N,q)$ and the proof of Corollary 2.3.7 follows directly from the proof of Proposition 2.3.1.

2.5.3 Theorem 2.1.1

We can use the same notation and definitions used in the previous proofs and write

$$\mu(\tau_n > N) - (1-\epsilon_n)^N = S_1(N) + S_2(N),$$

where

$$S_2(N) \leq \sum_{q=n}^{N-1} S_{21}(N,q) + S_{22}(N,q) + S_{\text{extra}}(N,q) \text{ and}$$

$$S_{\text{extra}}(N,q) := \left|\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \prod_{i=\eta_n}^{q} \mathbb{1}_{\mathcal{U}_n^c} \circ \sigma^i\right)\right| \leq \sum_{i=\eta_n}^{2n-1} \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right).$$

Notice that the case I. is exactly the condition required in Corollary 2.3.7. Therefore, we only need to prove the theorem in the cases II. and III. The unique difference in these cases and the one of First corollary is that we need to find a new bound for $\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right)$ when $i \in \{\eta_n, \ldots, 2n-1\}$.

Lemma 2.5.7. *If $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma, \{\mathcal{U}_n\}, \{X_n\})$ is a M_a-system where $\mathcal{U}_n$ is in the sigma algebra generated by $\prod_{-n}^{n-1}\{1, \ldots, a\}$ for every $n \geq 1$. Then, for every $n \geq 1$*

$$\begin{aligned}
&\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) \\
&\leq \begin{cases} \mu(\mathcal{U}_n)\mu\left(\mathcal{U}_k\right) + \psi_{\triangle}\epsilon_k\epsilon_n & \text{if } \eta_n = n + k + \triangle \text{ with } k + \triangle \geq 1, \\ \mu(\mathcal{V}_n)\mu(\mathcal{U}_n) + \psi_{\lfloor \eta_n/2 \rfloor}\mu(\mathcal{V}_n)\epsilon_n & \text{if } \eta_n < n+1, \end{cases}
\end{aligned}$$

for all $i \geq \eta_n$.

Proof. Let us fix $n \in \mathbb{N}$. We have two cases: $\eta_n = n + k + \triangle$ with $k + \triangle \geq 1$ or $\eta_n < n + 1$. Suppose first that $\eta_n = n + k + \triangle$ with $k + \triangle \geq 1$ and that $i \in \{\eta_n, \ldots, 2n-1\}$, then

$$\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) \leq \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_k} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) = \mu\left(\mathcal{U}_k \cap \sigma^{-i}\mathcal{U}_n\right)$$

and

$$|\mu\left(\mathcal{U}_k \cap \sigma^{-i}\mathcal{U}_n\right) - \mu(\mathcal{U}_k)\mu(\mathcal{U}_n)| \leq \psi_{\triangle}\epsilon_k\epsilon_n. \tag{2.17}$$

Therefore

$$\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) \leq \mu(\mathcal{U}_k)\mu(\mathcal{U}_n) + \psi_{\triangle}\epsilon_k\epsilon_n.$$

In Figure 2.2 we represented the sets $\mathcal{U}_n \cap \sigma^{-i}\mathcal{U}_n$ and $\mathcal{U}_k \cap \sigma^{-i}\mathcal{U}_n$. The light grey rectangle at the top represents the set $\mathcal{U}_n$ and the one at the bottom the set $\sigma^{-i}\mathcal{U}_n$. The darker grey rectangle represents the set $\mathcal{U}_k \supset \mathcal{U}_n$. The "gap" $\triangle$ between the coordinates fixed by the sets $\mathcal{U}_k$ and $\sigma^{-i}\mathcal{U}_n$ allows to use the ψ-mixing condition of the measure μ to conclude inequality (2.17).

Suppose now that $\eta_n < n + 1$ and that $\mathcal{V}_n$ has coordinates fixed only in $\{-n, \ldots, -n + \lfloor \eta_n/2 \rfloor\}$ (the case that $\mathcal{V}_n$ has coordinates fixed only in $\{n - \lfloor \eta_n/2 \rfloor, \ldots, n-1\}$ is similar), then

$$\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) \leq \mathbb{E}\left(\mathbb{1}_{\mathcal{V}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) = \mu\left(\mathcal{V}_n \cap \sigma^{-i}\mathcal{U}_n\right)$$

and

$$|\mu\left(\mathcal{V}_n \cap \sigma^{-i}\mathcal{U}_n\right) - \mu(\mathcal{V}_n)\mu(\mathcal{U}_n)| \leq \psi_{\lfloor \eta_n/2 \rfloor}\mu(\mathcal{V}_n)\epsilon_n. \tag{2.18}$$

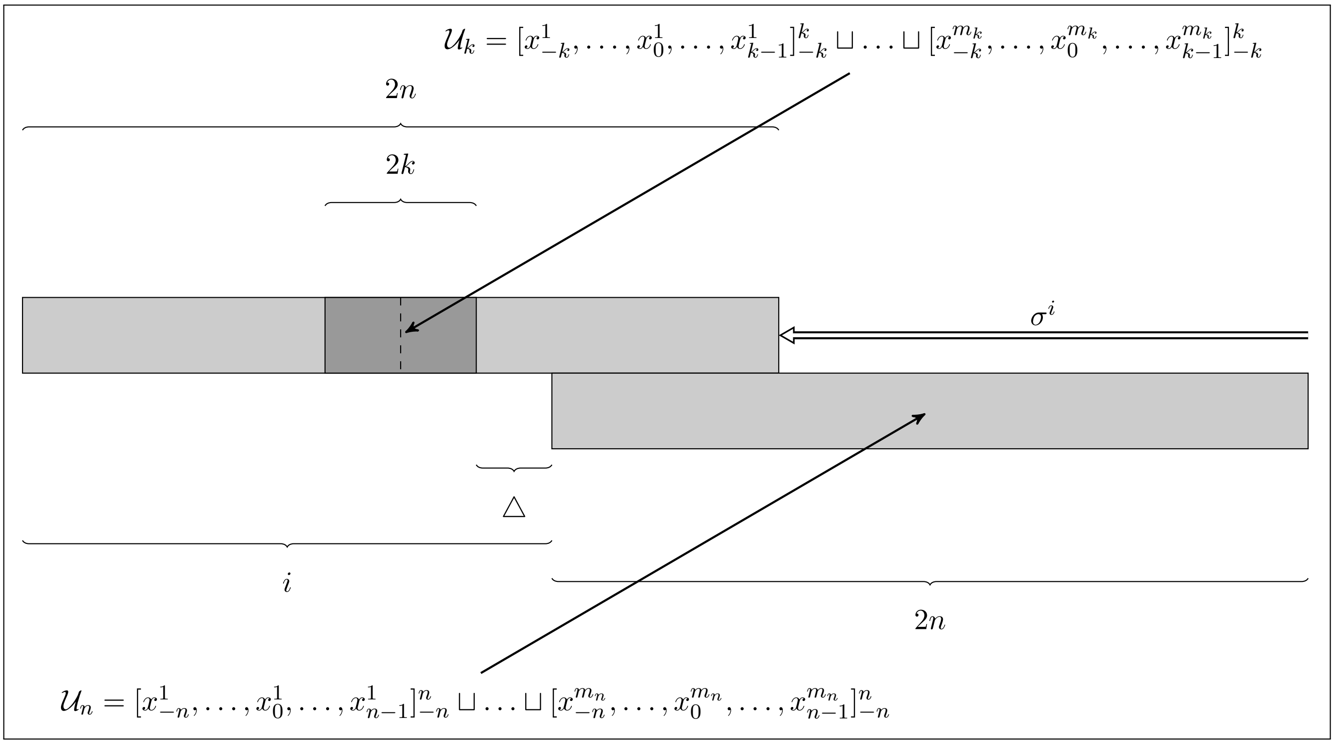


Figure 2.2: Proof of the inequality (2.17).

Therefore

$$\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n} \cdot \mathbb{1}_{\mathcal{U}_n} \circ \sigma^i\right) \leq \mu(\mathcal{V}_n)\mu(\mathcal{U}_n) + \psi_{\lfloor \eta_n/2 \rfloor}\mu(\mathcal{V}_n)\epsilon_n.$$

In Figure 2.3 we represented the sets $\mathcal{U}_n \cap \sigma^{-i}\mathcal{U}_n$ and $\mathcal{V}_n \cap \sigma^{-i}\mathcal{U}_n$. The light grey rectangle at the top represents the set $\mathcal{U}_n$ and the one at the bottom the set $\sigma^{-i}\mathcal{U}_n$. The darker grey rectangle represents a set $\mathcal{U}_k \supset \mathcal{U}_n$, for some $k < n$. The "gap" $\lfloor \eta_n/2 \rfloor$ between the coordinates fixed by the sets $\mathcal{V}_n$ and $\sigma^{-i}\mathcal{U}_n$ allows to use the ψ-mixing condition of the measure μ to conclude inequality (2.18).

$\square$

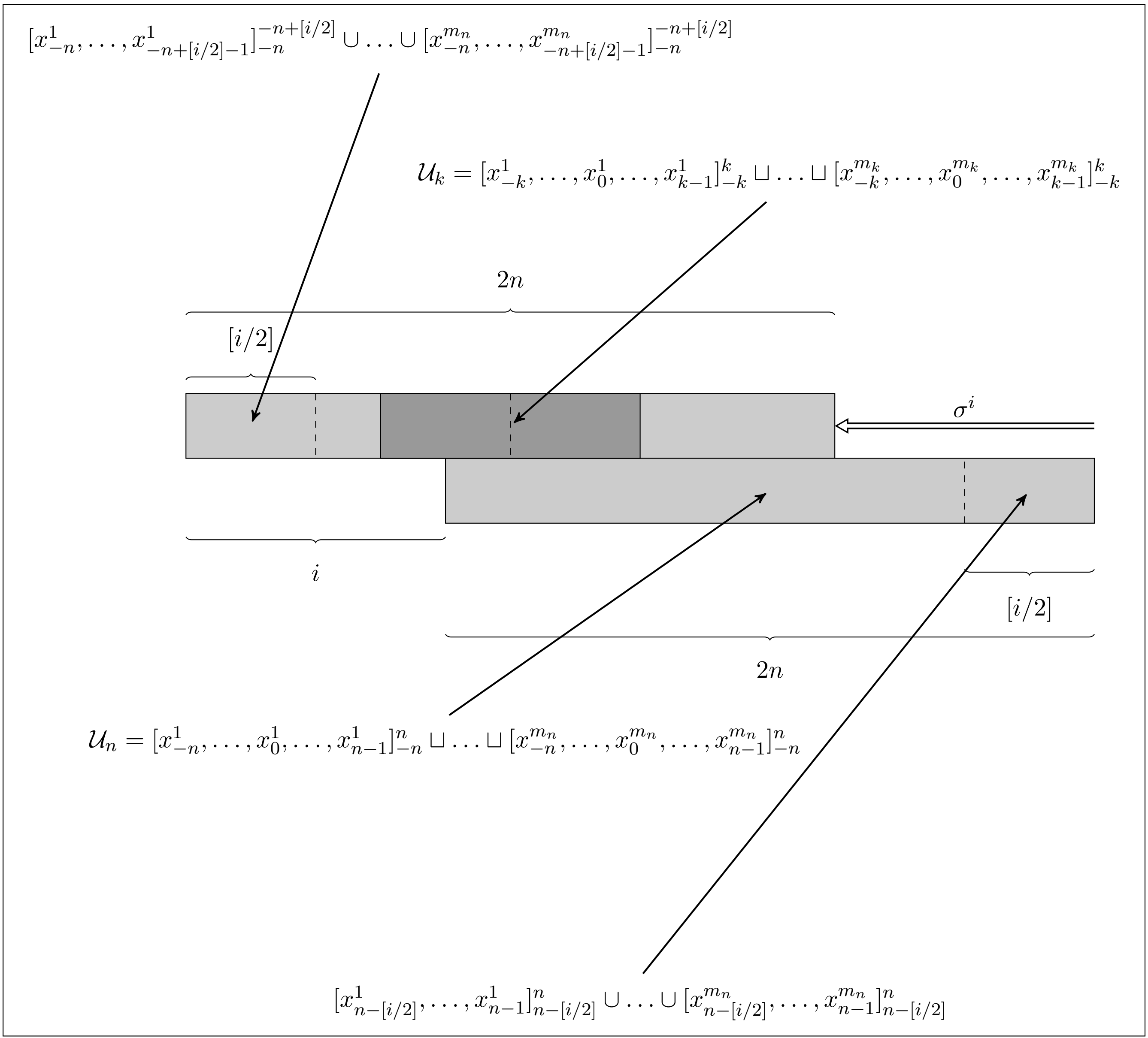


Figure 2.3: Proof of the inequality (2.18).

2.5.4 Second corollary

Proof. Recall that for every $n \in \mathbb{N}$, $\epsilon_n = \mu(\mathcal{U}_n)$ and $N = [t/\epsilon_n]$. The result reduces to the following inequalities

$$
\begin{aligned}
&|\mu\{x \in \mathcal{U}_n : \tau_n > N\} - \mu\{\tau_n > N\}\epsilon_n| \\
&= \left|\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=1}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right) - \epsilon_n\mathbb{E}\left(\prod_{i=1}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right)\right| \\
&= \left|\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=n}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right) - \epsilon_n\mathbb{E}\left(\prod_{i=1}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right)\right| \\
&\leq \left|\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=n}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right) - \mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=2n}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right)\right| \\
&\quad + \left|\mathbb{E}\left(\mathbb{1}_{\mathcal{U}_n}\prod_{i=2n}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right) - \epsilon_n\mathbb{E}\left(\prod_{i=2n}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right)\right| \\
&\quad + \epsilon_n\left|\mathbb{E}\left(\prod_{i=2n}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right) - \mathbb{E}\left(\prod_{i=n}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right)\right| \\
&\quad + \epsilon_n\left|\mathbb{E}\left(\prod_{i=n}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right) - \mathbb{E}\left(\prod_{i=1}^{N}\mathbb{1}_{\mathcal{U}_n^c}\circ\sigma^i\right)\right| \\
&\leq \epsilon_n^2\sum_{i=0}^{n-1}\psi_i + n\epsilon_n^2 + \epsilon_n\psi_n + 2n\epsilon_n^2,
\end{aligned}
$$

where the last one follows from Lemma 2.5.4. Dividing by ϵ_n we conclude the proof. □

2.6 Other results

2.6.1 An application

A continuous ergodic automorphism of the two dimensional torus or also called hyperbolic toral automorphism of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, is a transformation $T : \mathbb{T}^2 \to \mathbb{T}^2$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \pmod 1 \\ cx + dy \pmod 1 \end{pmatrix}$, where $a, b, c, d \in \mathbb{R}$ and the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ does not have eigenvalues of modulus 1. In 1967, Adler and Weiss in [4] proved that for any hyperbolic matrix (real matrix whose eigenvalues all have nonzero real parts) with integer coefficients and determinant ± 1 there exists a symbolic coding with a subshift of finite type. We suppose that the matrix M has two real and strictly positive eigenvalues, one strictly bigger than 1 and the other strictly smaller than 1. We call the eigenvalues by $\lambda = \lambda_u > 1 > \lambda_s > 0$. In [4] is considered the partition of $\mathbb{T}^2$ into two parallelograms $\mathcal{R}_1$ and $\mathcal{R}_2$ whose sides consist in two connected segments through the origin in each of the characteristic directions of M. One can take the partition into r parallelograms $\mathcal{P}_1, \ldots, \mathcal{P}_r$ determined by $\mathcal{R}_i \cap T\mathcal{R}_j$ for $i, j = 1, 2$ and consider the associated transition matrix A. The symbolic coding is then given by $(\mathcal{X}_A, \sigma)$. In [4] is proved that the matrix A is irreducible and aperiodic, then by the theorem of Perron-Frobenius there is a unique measure of maximum entropy or Parry measure μ_{Parry} on $\mathcal{B}_{\mathcal{X}_A}$. It is also proved that the Perron-eigenvalue of A is λ, so in particular $h_{\text{top}}(\sigma) = \log(\lambda)$ (this is a well known result that can be found in [62]). One can consider the map $\pi : \mathbb{T}^2 \to \mathcal{X}_A$, when $x \in \mathbb{T}^2$, $\pi(x)_n$ corresponds to the element $i \in \{1, \ldots, r\}$ such that $T^n(x) \in \mathcal{P}_i$. By definition $\sigma \circ \pi = \pi \circ T$ and $h_{\pi^*\mu_{\text{Haar}}}(T) = h_{\mu_{\text{Haar}}}(\sigma) = \log(\lambda)$, where $\pi^*\mu_{\text{Haar}} = \mu(\pi^{-1})$ is a probability measure on $\mathcal{B}_{\mathcal{X}_A}$. This implies that $\pi^*\mu_{\text{Haar}} = \mu_{\text{Parry}}$, and so π is a conjugacy between $(\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, \mu_{\text{Haar}}, T)$ and $(\mathcal{X}_A, \mathcal{B}_{\mathcal{X}_A}, \mu_{\text{Parry}}, \sigma)$. Because of the variational principle we know that for $\phi \in \mathcal{F}$ (recall that we defined $\mathcal{F}$ to be the space of Hölder functions on $\mathcal{X}_A$) the pressure $P = P(\phi) = \sup\{h_\mu(\sigma) + \int \phi d\mu : \mu \in \mathcal{M}_\sigma\}$. So in particular, the probability measure μ_{Parry} satisfies the inequality

$$c_1 \leq \frac{\mu_{\text{Parry}}([x]_m)}{\exp(-m h_{\text{top}}(\sigma))} = \frac{\mu_{\text{Parry}}([x]_m)}{\lambda^{-m}} \leq c_2 \tag{2.19}$$

for every $x \in \mathcal{X}_A$, $m \in \mathbb{N}$ and some fixed constants $c_1, c_2 > 0$.

Shrinking strip along the unstable direction

In this subsection we find strips $\mathcal{S}_n$ converging to a segment on $\mathbb{T}^2$ as n tends to infinity, for which

$$\lim_{n\to\infty} \mu_{\text{Leb}} \left\{ z \in \mathbb{T}^2 : \mu_{\text{Leb}}(\mathcal{S}_n) \tau_{\mathcal{S}_n}(z) > t \right\} = e^{-t}$$

for the Lebesgue measure μ_{Leb} on $\mathbb{T}^2$.

To make the exposition simpler, we restrict us to the hyperbolic toral automorphism T given by $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

The Lebesgue measure μ_{Leb} on $\mathbb{T}^2$ is preserved by T, because M has determinant -1. We construct the Markov partition in Figure 1.5 with rectangles $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ and $\mathcal{R}_5$.

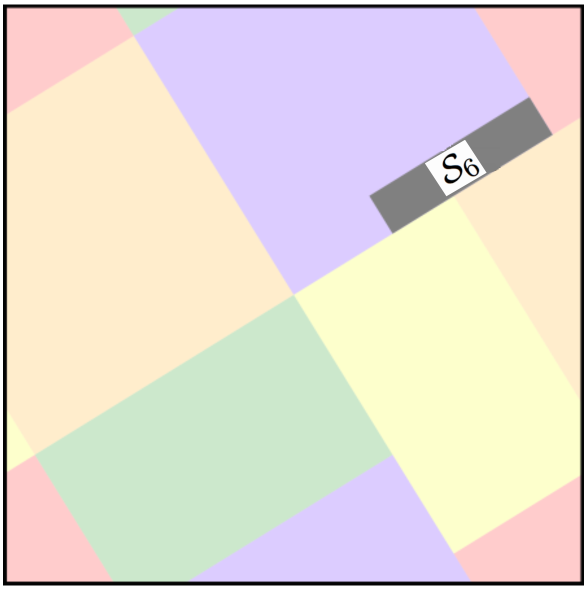


Figure 2.4: Strip along the unstable direction.

Definition 2.6.1 (Shrinking strips and cylinders)**.** *For each* $k \in \mathbb{N}$, *we define for* $n = 2(k+1)$ *the strip* $\mathcal{S}_n := \cap_{i=-2k}^{1} T^{-i}\mathcal{R}_{x_i}$ *and the cylinder* $\mathcal{A}_n = [1515\ldots1514]^2_{-n+2}$, *where* $x_{-2k} = 1, x_{-2k+1} = 5, x_{-2k+2} = 1, x_{-2k+3} = 5, \ldots, x_{-2} = 1, x_{-1} = 5, x_0 = 1, x_1 = 4$.

Definition 2.6.2 (Entry time to the strip)**.** *For each* $n \in \mathbb{N}$, *define the entry time to the strip* $\mathcal{S}_n$ *by* $\tau_{\mathcal{S}_n} : \mathbb{T}^2 \to \mathbb{N}$, $\mathbb{T}^2 \ni z \mapsto \inf\{k \in \mathbb{N} : T^k(z) \in \mathcal{S}_n\} \in \mathbb{N}$.

We consider

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the continuous surjection[1] $\pi : \mathcal{X}_A \to \mathbb{T}^2$, $\mathcal{X}_A \ni x \mapsto \cap_{i\in\mathbb{Z}} T^{-i}\mathcal{R}_{x_i} \in \mathbb{T}^2$.

The Lebesgue measure μ_{Leb} on $\mathbb{T}^2$ corresponds to the equilibrium state for the function 0 on $\mathbb{T}^2$. Calling μ_{Parry} the Parry measure on $\mathcal{X}_A$, we have that $(\mathbb{T}^2, T, \mu_{\text{Leb}})$ and $(\mathcal{X}_A, \sigma, \mu_{\text{Parry}})$ are conjugate[2].

This implies that for any number $t > 0$ and any $k \in \mathbb{N}$

$$\mu_{\text{Leb}}\left\{z \in \mathbb{T}^2 : \mu_{\text{Leb}}(\mathcal{S}_n)\tau_{\mathcal{S}_n}(z) > t\right\} = \mu_{\text{Parry}}\left\{x \in \mathcal{X}_A : \mu_{\text{Parry}}(\mathcal{A}_n)\tau_n(x) > t\right\},$$

[1]Theorem 3.18 in [16].
[2]Proof of 4.1 in [16].

where $n = 2(k+1)$ and τ_n is the entry time to $\mathcal{A}_n$. We also have:

Proposition 2.6.3. *For any number $t > 0$ and $n = 2(k+1)$ with $k \in \mathbb{N}$, we have that*

$$\lim_{k\to\infty} \mu_{Leb}\left\{z \in \mathbb{T}^2 : \mu_{Leb}(\mathcal{S}_n)\tau_{\mathcal{S}_n}(z) > t\right\} = e^{-t}.$$

Let $\lambda_s < 0 < 1 < \lambda_u$ be the eigenvalues of M. Call by v_s, v_u their respective normalised eigenvectors. Define $l = \frac{1}{\sqrt{1+(1-\lambda_s)^2}}$.

Definition 2.6.4 (Shrinking strips with arbitrary width and its entry time function)**.** *For each $\epsilon > 0$. We define the strip $\tilde{S}_\epsilon$ to be the rectangle with vertices $l(\lambda_s - 2)M^{-1}v_s$, lv_u, $lv_u - \epsilon l v_s$ and $l(\lambda_s - 2)M^{-1}v_s - \epsilon l v_s$. Define the entry time to $\tilde{S}_\epsilon$, $\tau_{\tilde{S}_\epsilon}(z) := \inf\{k \in \mathbb{N} : T^k(z) \in \tilde{S}_\epsilon\}$ for $z \in \mathbb{T}^2$.*

Corollary 2.6.5. *For any number $t > 0$, for every $\delta > 0$, there exists $\epsilon > 0$ such that for any $0 < \epsilon' < \epsilon$*

$$e^{-t/\lambda_s^2} - \delta \le \mu_{Leb}\left\{z \in \mathbb{T}^2 : \mu_{Leb}(\tilde{S}_{\epsilon'})\tau_{\tilde{S}_{\epsilon'}}(z) > t\right\} \le e^{-t\lambda_s^2} + \delta.$$

Proof. Fix $t > 0$ and $\delta > 0$. For each $n = 2(k+1)$, let $\{\mathcal{A}_n\}$ be as in Definition 2.6.1. For any $\epsilon > 0$, there exists $k_\epsilon \in \mathbb{N}$ such that $l\lambda_s^{2k_\epsilon} > \epsilon > l\lambda_s^{2k_\epsilon+2}$. Define $n_\epsilon = 2(k_\epsilon + 1)$. We have

$$\begin{aligned}\mu_{\mathrm{Leb}}\left\{z \in \mathbb{T}^2 : \mu_{\mathrm{Leb}}(\tilde{S}_\epsilon)\tau_{\tilde{S}_\epsilon}(z) > t\right\} &\le \mu_{\mathrm{Leb}}\left\{z \in \mathbb{T}^2 : \tau_{n_\epsilon+2}(z)\mu_{\mathrm{Leb}}(S_{n_\epsilon+2}) > t\frac{\mu_{\mathrm{Leb}}(S_{n_{\epsilon+2}})}{\mu_{\mathrm{Leb}}(S_{n_\epsilon})}\right\} \\ &= \mu_{\mathrm{Leb}}\left\{z \in \mathbb{T}^2 : \tau_{n_\epsilon+2}(z)\mu_{\mathrm{Leb}}(S_{n_\epsilon+2}) > t\lambda_s^2\right\}.\end{aligned}$$

We know that for $n = 2(k+1)$

$$\lim_{k\to\infty} \mu_{\mathrm{Leb}}\left\{z \in \mathbb{T}^2 : \tau_{n+2}(z)\mu_{\mathrm{Leb}}(S_{n+2}) > t\lambda_s^2\right\} = e^{-t\lambda_s^2},$$

then for ϵ sufficiently small choose k_ϵ such that n_ϵ is big enough to satisfy

$$\mu_{\mathrm{Leb}}\left\{z \in \mathbb{T}^2 : \tau_{n_\epsilon+2}(z)\mu_{\mathrm{Leb}}(S_{n_\epsilon+2}) > t\lambda_s^2\right\} \le e^{-t\lambda_s^2} + \delta.$$

We have that for any $0 < \epsilon' < \epsilon$

$$\mu_{\mathrm{Leb}}\left\{z \in \mathbb{T}^2 : \mu_{\mathrm{Leb}}(\tilde{S}_{\epsilon'})\tau_{\tilde{S}_{\epsilon'}}(z) > t\right\} \le e^{-t\lambda_s^2} + \delta.$$

The lower bound is similar. $\square$

Example of shrinking to infinitely many parallel strips

Definition 2.6.6 (Cylinders and shrinking strips)**.** *For each $n \in \mathbb{N}$, we define the set of cylinders*

$$\mathcal{A}_n = \left\{[11\dots12]_{-n+1}^{1}, [211\dots12]_{-n+1}^{1}, [2211\dots12]_{-n+1}^{1}, \dots [22\dots212]_{-n+1}^{1}\right\}$$

and the set of strips

$$\mathcal{S}_n := \{\cap_{i=-n+1}^{0} T^{-i}\mathcal{R}_{x_i} : [x_{-n+1}, x_{-n+2}, \dots, x_{-1}, x_0]_{-n+1}^{1} \in \mathcal{A}_n\},$$
$$\hat{S}_n := \cup_{\mathcal{R}\in\mathcal{S}_n}\mathcal{R} \subset \mathbb{T}^2.$$

We can consider the entry time time to the set of strips $\hat{S}_n$. It corresponds to the function defined by $\tau_n : \mathbb{T}^2 \to \mathbb{N}$, $\mathbb{T}^2 \ni z \mapsto \inf\{k \in \mathbb{N} : T^k(z) \in \hat{S}_n\} \in \mathbb{N}$. A direct consequence of Proposition 2.6.3 is the following:

Proposition 2.6.7. *For any number $t > 0$,*

$$\lim_{n\to\infty} \mu_{Leb}\left\{z \in \mathbb{T}^2 : \mu_{Leb}(\hat{S}_n)\tau_n(z) > t\right\} = e^{-t}.$$

2.6.2 A related result

On the setting of hyperbolic toral automorphisms and Gibbs measures, we show a stronger result than the Borel-Cantelli Lemma. Consider a hyperbolic toral automorphisms T given by a matrix M with eigenvalues $\lambda = \lambda_u > 1 > \lambda_s > 0$. For $n \in \mathbb{N}$ let $\mathcal{R}$ be a finite Markov partition with $\mathcal{R}^n := \bigvee_{i=-n}^{n} T^i\mathcal{R}$. Let $l_n \subset \mathcal{R}^n$ and $\hat{l}_n := \cup_{\mathcal{W}\in l_n}\mathcal{W} \subset \mathbb{T}^2$. Define

$$\mathcal{V}_n := T^{-n}\hat{l}_n,$$
$$F_n(z) := \sum_{k=1}^{n} \mathbb{1}_{\mathcal{V}_k}(z) \text{ for } z \in \mathbb{T}^2 \text{ and}$$
$$E_n := \sum_{k=1}^{n} \mu(\hat{l}_k).$$

Our result is the following:

Proposition 2.6.8. *For $n \in \mathbb{N}$,*

$$F_n(z) = E_n + \mathcal{O}(E_n^{1/2}\log^{3/2+\epsilon} E_n),\ \epsilon > 0$$

μ-a.e. for any Gibbs measure μ with a Hölder potential ϕ that is ψ-mixing with

$$\Psi(n) := \psi(n)\lambda^n \exp\left\{\left\|S^\sigma_{n-[\sqrt{n}]}\phi\right\|\right\}$$

summable.

Proof. Suppose that μ is a Gibbs measure and ψ-mixing with

$$\Psi(n) := \psi(n)\lambda^n \exp\left\{\left\|S^\sigma_{n-[\sqrt{n}]}\phi\right\|\right\}$$

summable. Remember that for $n \in \mathbb{N}$, for every $\mathcal{W} \in \mathcal{R}^n$ we have

$$c_1 \exp\left\{-2n\log\lambda + \inf_{x\in\mathcal{X}} |S^\sigma_{2n}\phi(x)|\right\} \le \mu(\mathcal{W}) \le c_2 \exp\left\{-2n\log\lambda + \|S^\sigma_{2n}\phi\|\right\}.$$

Suppose now that $m, n \in \mathbb{N}, n > m$ are fixed, then

$$\begin{aligned}
&\mu(T^{-n}\hat{l}_n \cap T^{-m}\hat{l}_m) - \mu(T^{-n}\hat{l}_n)\mu(T^{-m}\hat{l}_m)\\
&\quad = \sum_{\substack{\mathcal{Q}\in T^{-n}l_n\\ \mathcal{W}\in T^{-m}l_m}} \mu(\mathcal{Q}\cap\mathcal{W}) - \mu(\mathcal{Q})\mu(\mathcal{W})\\
&\quad \le \sum_{\substack{\mathcal{Q}\in T^{-n}l_n, \mathcal{W}\in T^{-m}l_m\\ \mathcal{Q}\cap\mathcal{W}\neq\emptyset}} \mu(\mathcal{Q}\cap\mathcal{W}) - \mu(\mathcal{Q})\mu(\mathcal{W})\\
&\quad \le \lambda^{2n-2m}\mu(l_n)|l_m|\\
&\qquad\quad \cdot \sup\left\{\mu(\mathcal{Q}\cap\mathcal{W}) - \mu(\mathcal{Q})\mu(\mathcal{W}) : \mathcal{Q}\in T^{-n}l_n, \mathcal{W}\in T^{-m}l_m, \mathcal{Q}\cap\mathcal{W}\neq\emptyset\right\}\\
&\quad \le \lambda^{2n-2m}\mu(l_n)|l_m|\\
&\qquad\quad \cdot \sup \inf_{k\in\{0,\dots,2n-2m-1\}} \left\{\psi(k)\mu(\mathcal{W})\mu(\mathcal{Q}') : \mathcal{W}\in T^{-m}l_m, \mathcal{Q}'\in\mathcal{R}^{2n-2m-k}\right\}\\
&\quad \le \lambda^{2n-2m}\mu(l_n)|l_m|\\
&\qquad\quad \cdot \inf_{k\in\{0,\dots,2n-2m-1\}} c_2\psi(k)\exp\left\{(-2n+2m+k)\log\lambda + \left\|S^\sigma_{2n-2m-k}\phi\right\|\right\}\\
&\qquad\qquad \cdot \sup\left\{\mu(\mathcal{W}) : \mathcal{W}\in T^{-m}l_m\right\}\\
&\quad \le \frac{c_2^2}{c_1}\mu(l_n)\mu(l_m) \inf_{k\in\{0,\dots,2n-2m-1\}} \psi(k)\lambda^k \exp\left\{\left\|S^\sigma_{2n-2m-k}\phi\right\|\right\}\\
&\quad \le \frac{c_2^2}{c_1}\mu(l_n)\mu(l_m)\psi([\sqrt{2n-2m}])\lambda^{[\sqrt{2n-2m}]}\exp\left\{\left\|S^\sigma_{2n-2m-[\sqrt{2n-2m}]}\phi\right\|\right\}\\
&\quad = \frac{c_2^2}{c_1}\mu(l_n)\mu(l_m)\Psi([\sqrt{2n-2m}]).
\end{aligned}$$

The inequality above proves the condition (2) of Theorem 3 in [70], from which our theorem follows. $\square$

Chapter 3

Escape rates for smooth flows

3.1 Introduction

Suppose that we have an ergodic and finite measure preserving dynamical system. If we consider a subset of the phase space, we know that the orbit of almost every point enters it. A natural object to study in this case is the measure of the points that have not entered this subset up to a time $n \in \mathbb{N}$. It is natural to think in some classical examples of uniformly hyperbolic smooth dynamical systems that this measure will decrease exponentially as n increases. The escape rate through a subset of the phase space corresponds to the asymptotic rate between n and the logarithm of the measure of the points that have not entered our subset up to time n. Once one has understood the escape rate of a set, one may wonder how the escape rates of sets whose measure converge to zero and the measure of the sets itself are asymptotically related? The answer is that for some uniformly hyperbolic smooth dynamical systems and some particular probability measures (Gibbs measures for example) one can explicitly describe this asymptotic behaviour ([39]). The question that motivates this chapter is: Can we say something similar for smooth flows? A general answer is out of the scope of this thesis, however, we will present a setting in which it is possible to obtain results for smooth semi-flows analogous to that for Gibbs measures in discrete dynamical systems. This is the content of the chapter.

Suppose Λ is a set of possible states (a compact metric space) that evolves in time according to the transformations $\Phi^t : \Lambda \to \Lambda$, $t \geq 0$. If we know the state of the system at time zero, say $x \in \Lambda$, then at time t it is $\Phi^t(x)$. To be consistent we need that $\Phi^{t+s}(x) = \Phi^s(\Phi^t(x))$ for any $s, t \geq 0$. This defines a flow $\{\Phi^t\}$. We assume that we have an invariant ergodic probability measure μ on Λ, so that in particular $(\Lambda, \mathcal{B}_\Lambda, \mu, \Phi^t)$ is a measure preserving dynamical system. For an open set $\mathcal{H} \subset \Lambda$ and $t \geq 0$, we define

$$K(\mu, t, \mathcal{H}, \Lambda) := \log \mu\{x \in \Lambda : \Phi^s x \notin \mathcal{H}, s \in [0, t]\}$$

and the escape rate through $\mathcal{H}$ by

$$R(\mu, \mathcal{H}, \Lambda) := -\limsup_{t \to \infty} \frac{1}{t} K(\mu, t, \mathcal{H}, \Lambda).$$

We are interested in the asymptotic behaviour of the escape rate $R(\mu, \mathcal{H}, \Lambda)$ as the measure of $\mathcal{H}$ decreases to zero. For discrete dynamical systems, this has been studied in [39] and in the references therein. In the discrete case we have a measure preserving dynamical system $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \mu, T)$ and define for an open set $\mathcal{H} \subset \mathcal{X}$ and a positive integer $k \in \mathbb{N}$

$$K(\mu, k, \mathcal{H}, \mathcal{X}) := \log \mu\{x \in \mathcal{X} : T^i x \notin \mathcal{H}, i \in \{0, \ldots, k-1\}\}.$$

The escape rate through $\mathcal{H}$ is defined by

$$R(\mu, \mathcal{H}, \mathcal{X}) := -\limsup_{k \to \infty} \frac{1}{k} K(\mu, k, \mathcal{H}, \mathcal{X}).$$

In [39] is considered a discrete dynamical system $(\mathcal{X}, T)$, where the space of invariant probability measures is denoted by $\mathcal{M}_T$, and fix an appropriate Banach space $\mathcal{E}_\mathcal{X} \subset \mathcal{X}^*$. For the pressure function $P : \mathcal{E}_\mathcal{X} \to \mathbb{R}$, that corresponds to $P(\cdot) = \sup_{\mu \in \mathcal{M}_T} Q(\cdot, \mu)$ for the function $Q : \mathcal{E}_\mathcal{X} \times \mathcal{M}_T \to \mathbb{R}$, $Q(\varphi, \mu) = h_\mu(T) + \int_\mathcal{X} \varphi d\mu$, where $h_\mu(T)$ is the measure theoretic entropy, in [39] there is a well defined functional $\mathscr{M} : \mathcal{E}_\mathcal{X} \to \mathcal{M}_T$ such that for $\varphi \in \mathcal{E}_\mathcal{X}$, $\mathscr{M}(\varphi) = \mu_\varphi$ corresponds to the unique solution of $P(\varphi) = Q(\varphi, \mu_\varphi)$. The measure μ_φ is called the equilibrium state (or Gibbs measure). Their result is that for $(\mathcal{X}, T)$ a non-invertible subshift of finite type or a conformal repeller, for some shrinking sequences $\{\mathcal{I}_n\}, \mathcal{I}_n \subset \mathcal{X}$ with $\cap_{n \in \mathbb{N}} \mathcal{I}_n = \{z\}$,

$$\lim_{n \to \infty} \frac{R(\mu_\varphi, \mathcal{I}_n, \mathcal{X})}{\mu_\varphi(\mathcal{I}_n)} = \gamma_\varphi(z) \text{ on } \varphi \in \mathcal{E}_\mathcal{X}, \tag{3.1}$$

where $\gamma_\varphi : \mathcal{X} \to [0, 1]$ is defined for $\varphi \in \mathcal{E}_\mathcal{X}, x \in \mathcal{X}$ by

$$\gamma_\varphi(x) := \begin{cases} 1 & \text{if } x \text{ is not periodic,} \\ 1 - e^{S_p^T \varphi(x) - pP(\varphi)} & \text{if } x \text{ has prime period } p. \end{cases}$$

A special semi-flow (Λ, Φ^t) over a discrete dynamical system $(\mathcal{X}, T)$, corresponds to a semi-flow in which every point in Λ moves with unit speed along along the non-contracting and non-expanding direction until it reaches the boundary of Λ and it jumps according T. That is, for a continuous function $f : \mathcal{X} \to \mathbb{R}^{>0}$, we consider the continuous action $\Phi^t = \Phi^t_f$ on

$$\Lambda = \Lambda_f := \{(x,t) : x \in \mathcal{X}, 0 \le t < f(x)\}$$

onto itself defined by

$$\Phi^t_f(x,s) := \left(T^m x, s + t - S^T_m f(x)\right) \text{ for } S^T_m f(x) \le s + t < S^T_{m+1} f(x),$$

where $m \ge 0$. The main result of this chapter establishes an analog of (3.1) for a special semi-flow (Λ, Φ^t) over a discrete dynamical system $(\mathcal{X}, T)$, where $(\mathcal{X}, T)$ is a subshift of finite type or a conformal repeller. We call by $\mathcal{M}_{\Phi^t}$ the space of invariant probability measures on $\mathcal{B}_\Lambda$ and fix an appropriate Banach space $\mathcal{E}_\Lambda \subset \Lambda^*$. Again we consider the pressure function $P : \mathcal{E}_\Lambda \to \mathbb{R}$ that corresponds to $P(\cdot) = \sup_{\mu \in \mathcal{M}_{\Phi^t}} Q(\cdot, \mu)$ for the function $Q : \mathcal{E}_\Lambda \times \mathcal{M}_{\Phi^t} \to \mathbb{R}$, $Q(\varphi, \mu) = h_\mu(\Phi^1) + \int_\Lambda \varphi d\mu$, where $h_\mu(\Phi^1)$ is the measure theoretic entropy (see [16]). We consider the well defined functional $\mathscr{M} : \mathcal{E}_\Lambda \to \mathcal{M}_{\Phi^t}$ such that for $\varphi \in \mathcal{E}_\Lambda$, $\mathscr{M}(\varphi) = \mu_\varphi$ corresponds to the unique solution of $P(\varphi) = Q(\varphi, \mu_\varphi)$. The measure μ_φ is again called equilibrium state, and one can prove that $\mu_\varphi = \nu^f$ with

$$d\nu^f := \frac{d\nu \times d\mu_{\text{Leb}}}{\int_{\mathcal{X}} f d\nu},$$

where $\nu = \nu_\phi$ is an equilibrium state associated to $\phi \in \mathcal{E}_{\mathcal{X}}$ and μ_{Leb} is the Lebesgue measure on $\mathcal{B}_{\mathbb{R}}$. We prove that under certain smoothness condition for the roof function and the assumption that $f : \mathcal{X} \to \mathbb{R}^{>1}$, for any shrinking sequence $\{\mathcal{I}_n\}, \mathcal{I}_n \subset \mathcal{X}$ such that (3.1) is satisfied with $\cap_{n \in \mathbb{N}} \mathcal{I}_n = \{z\}$, we have that

$$\lim_{\delta \searrow 0} \lim_{n \to \infty} \frac{R(\mu_\varphi, T^{r_n} \mathcal{I}_n \times [0, \delta], \mathcal{X})}{\mu_\varphi(\mathcal{I}_n \times [0,1])} = \tilde{\gamma}_\varphi(z) \text{ on } \mathcal{E}_\Lambda, \tag{3.2}$$

where $\{r_n\} \subset \mathbb{Z}$ is a monotonic decreasing sequence and $\tilde{\gamma}_\varphi : \mathcal{X} \to [0,1]$ is defined for $\varphi \in \mathcal{E}_\Lambda, x \in \mathcal{X}$ by

$$\tilde{\gamma}(x) := \begin{cases} 1 & \text{if } (x,0) \text{ does not belong to any periodic orbit,} \\ 1 - e^{\int_\tau \varphi(x,t)dt} & \text{if } (x,0) \in \tau \text{ and } \tau \text{ is a periodic orbit.} \end{cases}$$

We prove (3.2) in several steps. We start by considering the case that $(\mathcal{X}, T)$ is a non-invertible subshift of finite type and after the case that $(\mathcal{X}, T)$ is an invertible subshift of finite type. The non-invertible case follows from a discretisation of the flow and an application of (3.1). From these results combined with [39], we obtain (3.2) in the case that $(\mathcal{X}, T)$ is a conformal repeller. A natural question at this point is what can we say about escape rates for Axiom A flows? In the case that Λ is a basic set for an Axiom A flows Φ we are only able to prove that on an appropriate Banach space $\mathcal{E}_\Lambda \subset \Lambda^*$, for $\phi \in \mathcal{E}_\Lambda$ we have that as $\delta > 0$ decreases to zero

$$\frac{R(\mu_\phi, \mathcal{H}'_{\delta,n}, \Lambda)}{\mu_\phi(\mathcal{H}_n)} \text{ accumulates on } \mathcal{S}_\phi, \tag{3.3}$$

where $\mathcal{H}'_{\delta,n}, \mathcal{H}_n$ are some specific shrinking sets, $\mathcal{H}'_{\delta,n}$ converges to some $z \in \Lambda$ as $n \to \infty, \delta \to 0$ and $\mathcal{S}_\phi \subset [0,1]$ is a discrete set, i.e. every every point in $x \in \mathcal{S}_\phi$ has a neighbourhood $\mathcal{U} \subset [0,1]$ such that $\mathcal{U} \cap \mathcal{S}_\phi = \{x\}$. The set $\mathcal{S}_\phi$ depends on z and on ϕ. By "specific shrinking sets" we mean that our result only works for some projection of cylinders sets that lack of geometric interpretation. We formalise this in Corollary 3.4.14.

3.2 Results

We introduce a necessary condition from [39].

Definition 3.2.1. *We say that a family of open sets* $\{\mathcal{U}_n\}, \mathcal{U}_n \subset \mathcal{X}^+$ *satisfies the* nested condition *if it satisfies that:*

1. *each* $\mathcal{U}_n$ *consists of a finite union of cylinder sets, with each cylinder having length* n*;*
2. $\mathcal{U}_{n+1} \subset \mathcal{U}_n$ *for every* $n \in \mathbb{N}$ *and* $\cap_{n\in\mathbb{N}} \mathcal{U}_n = \{z\}$*;*
3. *there exist constants* $c > 0$ *and* $0 < \rho < 1$ *such that* $\mu(\mathcal{U}_n) \leq c\rho^n$ *for all* $n \in \mathbb{N}$*;*
4. *there is a sequence* $\{l_n\} \subset \mathbb{N}$ *and a constant* $\kappa > 0$ *such that* $\kappa < l_n/n \leq 1$ *and* $\mathcal{U}_n \subset [z]_{l_n}$ *for all* $n \in \mathbb{N}$*; and*
5. *if* $\sigma^p(z) = z$ *has prime period* p*, then* $\sigma^{-p}(\mathcal{U}_n) \cap [z]_p \subset \mathcal{U}_n$ *for large enough* n*,*

where $z \in \mathcal{X}^+$ *is a global variable.*

Our first theorem is the following:

Theorem 3.2.2. *Let (Λ_f, Φ_f^t) be a special flow over $(\mathcal{X}, \sigma)$. Suppose that $f : \mathcal{X} \to \mathbb{R}^{>1}$ is θ^2-Lipschitz and that $\{\mathcal{I}_n\}, \mathcal{I}_n \subset \mathcal{X}^+$ satisfies the nested condition with $\cap_{n\in\mathbb{N}}\mathcal{I}_n = \{z\}$ for $z \in \mathcal{X}^+$. Then*

$$\lim_{n\to\infty} \frac{R(\mu_\varphi^f, \sigma^{[n/2]}\mathcal{I}_n \times [0,1], \Lambda_f)}{\mu_\varphi^f(\mathcal{I}_n \times [0,1])} = \tilde{\gamma}_\varphi(z) \text{ for all Hölder functions } \varphi : \mathcal{X} \to \mathbb{R}.$$

In our proof an important role will be played by a similar behaviour of the escape rates for a non-invertible subshift of finite type.

Theorem 3.2.3 (Theorem 5.1 in [39])**.** *Suppose that $\{\mathcal{I}_n\}, \mathcal{I}_n \subset \mathcal{X}^+$ satisfies the nested condition with $\cap_{n\in\mathbb{N}}\mathcal{I}_n = \{z\}$ for $z \in \mathcal{X}^+$, then*

$$\lim_{n\to\infty} \frac{R(\mu_\varphi, \mathcal{I}_n, \mathcal{X}^+)}{\mu_\varphi(\mathcal{I}_n)} = \gamma_\varphi(z) \text{ for all Hölder functions } \varphi : \mathcal{X}^+ \to \mathbb{R}.$$

Let $\mathcal{J}$ be a conformal repeller. Given $z \in \mathcal{J}$, we definte $B(z, \epsilon)$ to be the ball centred at z of radius $\epsilon > 0$. Our second theorem is the following:

Theorem 3.2.4. *Let (Λ_F, Φ_F^t) be a special flow over $(\mathcal{J}, f)$. Suppose that $F : \mathcal{J} \to \mathbb{R}^{>1}$ is Hölder and $z \in \mathcal{J}$, then*

$$\lim_{n\to\infty} \frac{R(\mu_\varphi^F, B(z,\epsilon) \times [0,1], \Lambda_f)}{\mu_\varphi^F(B(z,\epsilon) \times [0,1])} = \tilde{\gamma}_\varphi(z) \text{ for all Hölder functions } \varphi : \mathcal{J} \to \mathbb{R}.$$

3.3 Proofs

We have separated the proofs of our results into three subsections. In §3.3.1 we state and prove Proposition 3.3.2, this a one-sided version of Theorem 3.2.2. In §3.3.2 we prove Theorem 3.2.2 by using the constructions of the previous subsection. Finally, in §3.3.3 we prove Theorem 3.2.4 by using the work of the first and second subsection.

3.3.1 Auxiliary constructions

We require certain smoothness for the roof function, indeed, we will make clear that it is enough to consider any θ-Lipschitz function. Let start with an easy observation.

Remark 3.3.1. *Given a θ-Lipschitz function $f : \mathcal{X}^+ \to \mathbb{R}^{>0}$, there exists $\eta : \mathbb{N} \to \mathbb{R}^{>0}$ converging to 0 such that*

$$\max\left\{ \sup_{x\in[y]_m} f(x) - \inf_{x\in[y]_m} f(x) : y \in \mathcal{X}^+ \right\} < \eta(m) \tag{3.4}$$

for all $m \in \mathbb{N}$. Moreover, $\eta(m) = |f|_\theta \theta^m$ for $m \in \mathbb{N}$.

Recall that given a special flow (Λ, Φ^t) over a measure preserving dynamical system $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \mu, \sigma)$, where $(\mathcal{X}, \sigma)$ is a (invertible or non-invertible) subshift of finite type, we defined a probability measure μ^f on $\mathcal{B}_\Lambda$ by

$$d\mu^f := \frac{d\mu \times d\nu_{Leb}}{\int_\mathcal{X} f d\mu}.$$

Our proof relies on the next proposition:

Proposition 3.3.2. *Let (Λ_f, Φ^t_f) be a special flow over $(\mathcal{X}^+, \mathcal{B}_{\mathcal{X}^+}, \mu_\varphi, \sigma)$, where $(\mathcal{X}^+, \sigma)$ is a non-invertible topologically mixing subshift of finite type and μ_φ is the equilibrium state associated to Hölder potential $\varphi : \mathcal{X}^+ \to \mathbb{R}$. Suppose that $f : \mathcal{X}^+ \to \mathbb{R}^{>0}$ is θ-Lipschitz and that $\{\mathcal{U}_n\}, \mathcal{U}_n \subset \mathcal{X}^+$ satisfies the nested condition with $\cap_{n\in\mathbb{N}} \mathcal{U}_n = \{z\}$ for $z \in \mathcal{X}^+$. Then*

$$\lim_{\delta \searrow 0} \lim_{n\to\infty} \frac{R(\mu^f_\varphi, \mathcal{U}_n \times [0, \delta], \Lambda_f)}{\mu_\varphi(\mathcal{U}_n)} = \frac{\gamma_\varphi(z)}{\int f d\mu_\varphi}.$$

Proof of Proposition 3.3.2. We can find $\epsilon > 0$ such that $f > \epsilon$. Once fixed ϵ, we can choose $\delta \in (0, \epsilon/3)$ and $m \in \mathbb{N}$ such that $2\delta + \eta(m) < 0.5 \int f d\mu$. We require this last condition in inequality (3.8). We define an approximation of f from above by

$$\overline{f}_{m,\delta}(x) := \left(\left[\sup_{y \in [x]_m} f(y)/\delta \right] + 2 \right) \delta,$$

and an approximation of f from below by

$$\underline{f}_{m,\delta}(x) := \left(\left[\inf_{y \in [x]_m} f(y)/\delta \right] - 2 \right) \delta.$$

To make the notation shorter we denote $\overline{f} = \overline{f}_{m,\delta}$ and $\underline{f} = \underline{f}_{m,\delta}$. We consider the special flows $(\Lambda_{\overline{f}}, \Phi^t_{\overline{f}})$ and $(\Lambda_{\underline{f}}, \Phi^t_{\underline{f}})$. We can discretise them by considering $(\Lambda_{\overline{f}}, \Phi^{k\delta}_{\overline{f}})$ and $(\Lambda_{\underline{f}}, \Phi^{k\delta}_{\underline{f}})$, where $k \geq 0$. We can associate a subshift of finite type to each discrete flow by doing the following. Define

$$\mathcal{X}_{\overline{f}} := \{(y_i)_{i=0}^\infty : y_i = \Phi^{k\delta}_{\overline{f}}([x]_m), x \in \mathcal{X}^+, k \in \mathbb{N}, A_{\overline{f}}(y_i, y_{i+1}) = 1\},$$

where

$$A_{\overline{f}} \left(\Phi^{k\delta}_{\overline{f}}([x]_m), \Phi^{(k'+1)\delta}_{\overline{f}}([x']_m) \right) = \begin{cases} 1 & \text{if } C_1 \text{ or } C_2, \\ 0 & \text{if not,} \end{cases}$$

and

$$C_1 \Leftrightarrow k = k' \;\&\; x \in [x']_m,$$

$$C_2 \Leftrightarrow \begin{cases} (k+1)\delta = \overline{f}(x) \;\& \\ (k'+1)\delta = \overline{f}(x') \;\& \\ x_{i+1} = x'_i \text{ for all } i \in \{0, \dots, m-2\}, \end{cases}$$

with the shift $\sigma_{\overline{f}} : \mathcal{X}_{\overline{f}} \to \mathcal{X}_{\overline{f}}$, $\sigma_{\overline{f}} := \Phi^{\delta}_{\overline{f}}$. We denote $\Phi^{k\delta}_{\overline{f}}([x]_m)$ by $([x]_m, k \bmod \overline{f}([x]_m)/\delta)$.

Given a set $\mathcal{N} \subset \mathbb{N}_0$ and $\mathcal{W} \in \xi_m$ define

$$d\mathbb{N}|_{[0,\overline{f}(\mathcal{W})/\delta)}(\mathcal{N}) = |\{n \in \mathcal{N} : n < \overline{f}(\mathcal{W})/\delta\}|,$$

the measure

$$\tilde{\mu}^{\overline{f}} := \frac{1}{\int \overline{f} d\mu} \sum_{\mathcal{W} \in \xi_m} \mu|_{\mathcal{W}} \times \delta d\mathbb{N}|_{[0,\overline{f}(\mathcal{W})/\delta)} \tag{3.5}$$

is an invariant probability measure for the subshift of finite type $(\mathcal{X}_{\overline{f}}, \sigma_{\overline{f}})$ and corresponds to the equilibrium state of a Hölder potential $\phi = \phi_{\overline{f}} : \mathcal{X}_{\overline{f}} \to \mathbb{R}$ (Lemma 3.3.3). Again, we can do the same by replacing $\overline{f}$ with $\underline{f}$. Notice that for a given roof function g, the measure with tilde $\tilde{\mu}^g$ is a discrete version of the measure μ^g.

We define for $f^* = \overline{f}$ or $\underline{f}$

$$\widetilde{\mathcal{U}_n \times \{0\}} = \widetilde{\mathcal{U}_n \times \{0\}}^{f^*} := \{[y_0, \dots y_{r-1}]_r : [x_0, \dots, x_{n-1}]_n \in \mathcal{U}_n\}$$

where

$$r = r(x_0, \dots, x_{n-2}) := \sum_{i=0}^{n-2} f^*(x_i) + 1,$$

$y_{r-1} = y_{r-1}(x_{n-1}) := (x_{n-1}, 0)$, and for $i = 0, \dots, r-2$

$$y_i = y_i(x_i) := (x_i, 0)(x_i, 1) \cdots (x_i, f^*(x_i) - 1).$$

When we are on the space $\mathcal{X}_{f^*}$ we will assume that $\widetilde{\mathcal{U}_n \times \{0\}} = \widetilde{\mathcal{U}_n \times \{0\}}^{f^*}$.

Applying Theorem 3.2.3 to the subshift of finite type that we have constructed we obtain

$$\lim_{n\to\infty} \frac{R(\tilde{\mu}^{\overline{f}}, \widetilde{\mathcal{U}_n \times \{0\}}, \mathcal{X}_{\overline{f}})}{\tilde{\mu}^{\overline{f}}(\widetilde{\mathcal{U}_n \times \{0\}})} = \gamma_\phi(z),$$

and the same can be done for $\underline{f}$.

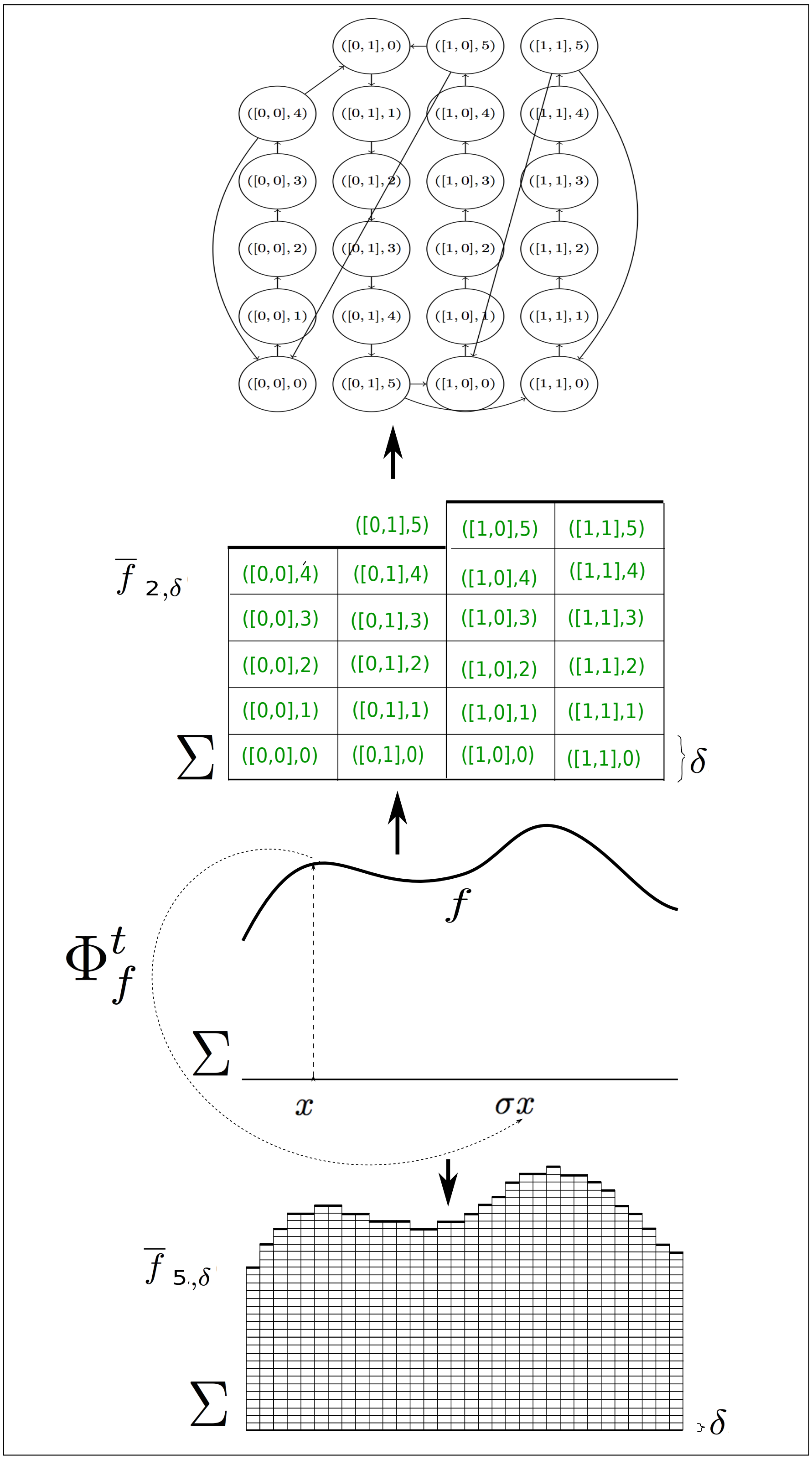


Figure 3.1: Example of our discretisation of the flow for $\mathcal{X}^+ = \{0,1\}^{\mathbb{N}_0}$, where $[i,j]$ for $i,j \in \{0,1\}$ are the cylinders of length 2.

Suppose that $n, k \in \mathbb{N}$. By definition we have for $\underline{f} = \underline{f}_{m,\delta}$ that

$$K(\tilde{\mu}^{\underline{f}}, k, \widetilde{\mathcal{U}_n \times \{0\}}, \mathcal{X}_{\underline{f}}) = K(\mu^{\underline{f}}, \delta k, \mathcal{U}_n \times [0,\delta], \Lambda_{\underline{f}}),$$

for $\overline{f} = \overline{f}_{m,\delta}$ that

$$K(\tilde{\mu}^{\overline{f}}, k, \widetilde{\mathcal{U}_n \times \{0\}}, \mathcal{X}_{\overline{f}}) = K(\mu^{\overline{f}}, \delta k, \mathcal{U}_n \times [0,\delta], \Lambda_{\overline{f}}),$$

and finally, that the map $t \mapsto K(\mu^f, t, \mathcal{U}_n \times [0,\delta], \Lambda_f)$ is decreasing in $t > 0$. From this we have that independently of $n \in \mathbb{N}$ and for any $t > \delta$

$$K(\tilde{\mu}^{\underline{f}}, \lceil t/\delta \rceil, \widetilde{\mathcal{U}_n \times \{0\}}, \mathcal{X}_{\underline{f}}) \leq K(\mu^{\underline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\underline{f}}) \tag{3.6}$$

and

$$K(\mu^{\overline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\overline{f}}) \leq K(\tilde{\mu}^{\overline{f}}, \lfloor t/\delta \rfloor, \widetilde{\mathcal{U}_n \times \{0\}}, \mathcal{X}_{\overline{f}}). \tag{3.7}$$

We will need the following inequality

$$\begin{aligned} K(\mu^{\underline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\underline{f}}) + \log\frac{1}{2} &\leq K(\mu^f, t, \mathcal{U}_n \times [0,\delta], \Lambda_f) \\ &\leq K(\mu^{\overline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\overline{f}}) + \log 2. \end{aligned} \tag{3.8}$$

In order to prove it, let consider the inclusions

$$\begin{aligned} \mathcal{A} &:= \left\{(x,s') \in \Lambda_{\underline{f}} : \Phi^s_{\underline{f}}(x,s') \notin \mathcal{U}_n \times [0,\delta], 0 \leq s \leq t\right\} \\ &\subseteq \left\{(x,s') \in \Lambda_f : s' < \underline{f}(x), \Phi^s_f(x,s') \notin \mathcal{U}_n \times [0,\delta], 0 \leq s \leq t\right\} \\ &\subseteq \left\{(x,s') \in \Lambda_f : \Phi^s_f(x,s') \notin \mathcal{U}_n \times [0,\delta], 0 \leq s \leq t\right\} =: \mathcal{V}. \end{aligned}$$

Then $\mu^{\underline{f}}(\mathcal{A}) \int \underline{f} d\mu \leq \mu^f(\mathcal{V}) \int f d\mu$ and $\log(\mu^{\underline{f}}(\mathcal{A})) + \log(\int \underline{f} d\mu) \leq \log(\mu^f(\mathcal{V})) + \log(\int f d\mu)$. Thus by definition $K(\mu^{\underline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\underline{f}}) + \log(\int \underline{f} d\mu) \leq K(\mu^f, t, \mathcal{U}_n \times [0,\delta], \Lambda_f) + \log(\int f d\mu)$. We chose m and δ so that $2\delta + \eta(m) < 0.5 \int f d\mu$, then $\underline{f} = \underline{f}_{m,\delta}$ satisfies

$$\int \underline{f} d\mu \geq \int f d\mu - \eta(m) - 2\delta \geq \frac{\int f d\mu}{2}$$

and $\overline{f} = \overline{f}_{m,\delta}$ satisfies

$$\int \overline{f} d\mu \leq \int f d\mu + \eta(m) + 2\delta \leq 2 \int f d\mu.$$

From this is clear that

$$K(\mu^{\underline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\underline{f}}) + \log\frac{1}{2} \leq K(\mu^{\underline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\underline{f}}) + \log\frac{\int \underline{f} d\mu}{\int f d\mu}$$
$$\leq K(\mu^{f}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{f}).$$

The second inequality in (3.8) is completely analogous, in this case we obtain

$$K(\mu^{f}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{f}) \leq K(\mu^{\underline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\underline{f}}) + \log\frac{\int \overline{f} d\mu}{\int \underline{f} d\mu}$$
$$\leq K(\mu^{\underline{f}}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{\underline{f}}) + \log 2.$$

In the next inequality we will use this identity:

$$\tilde{\mu}^{\overline{f}}(\widetilde{\mathcal{U}_n \times \{0\}}) = \frac{\delta}{\int \overline{f} d\mu}\mu(\mathcal{U}_n).$$

For all $t \in \mathbb{R}^{>\delta}$,

$$\begin{aligned}
&\frac{1}{\mu(\mathcal{U}_n)}\frac{1}{t}K(\mu^{f}, t, \mathcal{U}_n \times [0,\delta], \Lambda_{f}) \\
&\leq \frac{1}{\mu(\mathcal{U}_n)}\frac{1}{t}K(\tilde{\mu}^{\overline{f}}, \lfloor t/\delta \rfloor, \widetilde{\mathcal{U}_n \times \{0\}}, \mathcal{X}_{\overline{f}}) + \frac{\log 2}{\mu(\mathcal{U}_n)t} \\
&\leq \frac{\lfloor t/\delta \rfloor}{\lceil t/\delta \rceil}\frac{1}{\tilde{\mu}^{\overline{f}}(\widetilde{\mathcal{U}_n \times \{0\}})\int \overline{f} d\mu}\frac{1}{\lfloor t/\delta \rfloor}K(\tilde{\mu}^{\overline{f}}, \lfloor t/\delta \rfloor, \widetilde{\mathcal{U}_n \times \{0\}}, \mathcal{X}_{\overline{f}}) + \frac{\log 2}{\mu(\mathcal{U}_n)t}.
\end{aligned}$$

In the last inequality, taking $\limsup_{t\to\infty}$ on both sides, then letting n tend to infinity, and finally multiplying by -1, allows us to write

$$\lim_{n\to\infty}\frac{R(\mu^{f}, \mathcal{U}_n \times [0,\delta], \Lambda_{f})}{\mu(\mathcal{U}_n)} \geq \gamma_{\varphi}(z)\frac{1}{\int \overline{f} d\mu}. \tag{3.9}$$

Similarly we obtain

$$\lim_{n\to\infty}\frac{R(\mu^{f}, \mathcal{U}_n \times [0,\delta], \Lambda_{f})}{\mu(\mathcal{U}_n)} \leq \gamma_{\varphi}(z)\frac{1}{\int \underline{f} d\mu}. \tag{3.10}$$

Taking $f^* = \overline{f}$ or $\underline{f}$, by definition we have

$$\int |f - f^*| d\mu \leq 2\delta + \eta(m). \tag{3.11}$$

This combined with inequalities (3.9),(3.10) and the fact that m can be taken arbitrarily large concludes that

$$\lim_{\delta\searrow 0}\lim_{n\to\infty}\frac{R(\mu_\varphi^f,\mathcal{U}_n\times[0,\delta],\Lambda_f)}{\mu_\varphi(\mathcal{U}_n)}=\frac{\gamma_\varphi(z)}{\int f d\mu_\varphi}, \tag{3.12}$$

and this finishes the proof. $\square$

The following claim used in the proof of Proposition 3.3.2 is well known, however we include a demonstration for completeness.

Claim 3.3.3. *The probability measure*

$$\tilde{\mu}^{\overline{f}} := \frac{1}{\int \overline{f} d\mu}\sum_{\mathcal{W}\in\xi_m}\mu|_{\mathcal{W}}\times\delta d\mathbb{N}|_{[0,\overline{f}(\mathcal{W})/\delta)}$$

is an invariant probability measure for the subshift of finite type $(\mathcal{X}_{\overline{f}},\sigma_{\overline{f}})$ *and corresponds to the equilibrium state of a Hölder potential* $\phi=\phi_{\overline{f}}:\mathcal{X}_{\overline{f}}\to\mathbb{R}$.

Proof. We introduce some notation. For $n\in\mathbb{N}$ we define the set of allowed words of length n, $\mathcal{X}_n := \{x_{[0,n)} := x_0\dots x_{n-1} : x\in\mathcal{X}\}$. In what follows we take m and δ fixed in the definition of $f^*=\overline{f}$ or $\underline{f}$. We define the function $\tilde{\pi}=\tilde{\pi}_{m,\delta}:\mathcal{X}_{f^*}\to\sigma$ so that the image of

$$\overline{x}=(x_0,l_0),(x_1,l_1),\dots,(x_n,l_n),\dots$$

is given by $\tilde{\pi}(\overline{x})=x_{i_0}x_{i_1}x_{i_2}\dots$ where $i_0=0$ and for $n\in\mathbb{N}$, $i_n=\min\{k>i_{n-1}: l_k=0\}$. We extend the definition of $\tilde{\pi}$ to the case of finite sequences and given

$$w=\overline{x}_{[0,k)}=(x_0,l_0),\dots,(x_{k-1},l_{k-1})$$

for some $k\in\mathbb{N}$ where $\overline{x}\in\mathcal{X}_{f^*}$, we define $\#w := |\{n\in\{1,\dots,k-1\}:l_n=0\}|+1$.

By definition, given $\overline{x}\in\mathcal{X}_{f^*}, i,j,k\geq 0$ with $i<j$ and $w=\overline{x}_{[i,j)}$ we have that

$$\tilde{\mu}^{f^*}([w]_k^{k+j-i})=\frac{\delta}{\int f^* d\mu}\mu([\tilde{\pi}(w)]_{\#w+1}).$$

This can be seen as an alternative way to write the same measure defined in (3.5).

For $f^*=\overline{f}$ or $\underline{f}$ we need to prove that the measure

$$\tilde{\mu}^{f^*}=\frac{1}{\int f^* d\mu}\sum_{w\in\xi_m}\mu|_w\times\delta d\mathbb{N}|_{[0,f^*(w)/\delta)}$$

is an invariant probability measure for the subshift of finite type $(\mathcal{X}_{f^*},\sigma_{f^*})$ and corresponds to the equilibrium state of a Hölder potential.

From a corollary of the Kolmogorov consistency theorem on a subshift of finite type $\mathcal{X} \subset \{1, \dots, a\}^{\mathbb{N}_0}$, where $a \geq 2$, the set of σ-invariant probability measures is identified one-to-one with the set of maps $\mu : \mathcal{B}_{\mathcal{X}} \to \mathbb{R}^{>0} \cup \{0, \infty\}$ such that

$$\sum_{s=1}^{a} \mu([s]_1) = 1 \tag{3.13}$$

and for all $x \in \mathcal{X}$, for all integers $i, j, k \geq 0$ with $i < j$ we have for $w = x_{[i,j)}$

$$\mu([w]_k^{k+j-i}) = \sum_{s=1}^{a} \mu([w, s]_k^{k+j-i+1}) \tag{3.14}$$

and

$$\mu([w]_{k+1}^{k+j-i+1}) = \sum_{s=1}^{a} \mu([s, w]_k^{k+j-i+1}). \tag{3.15}$$

In what follows let consider m and δ fixed in the definition of $f^* = \overline{f}$ or $\underline{f}$.

Then, for the first part of the proof we need to check that $\tilde{\mu}^{f^*}$ satisfies (3.13), (3.14) and (3.15). We start by proving that $\tilde{\mu}^{f^*}$ satisfies (3.13), indeed

$$\sum_{[\overline{y}]_1 : \overline{y} \in \mathcal{X}_{f^*}} \tilde{\mu}^{f^*}([\overline{y}]_1) = \sum_{C \in \xi_m} \sum_{i=0}^{f^*(C)/\delta - 1} \frac{\delta \mu(C)}{\int f^* d\mu} = \frac{1}{\int f^* d\mu} \sum_{C \in \xi_m} \mu(C) f^*(C) = 1.$$

In order to prove (3.14) and (3.15), let us suppose that $\overline{x} \in \mathcal{X}_{f^*}$ and $i, j, k \geq 0$ with $i < j$. Denote $w = \overline{x}_{[i,j)}$ where $\overline{x}_{[i,j)} = (x_0, l_0), \dots, (x_{j-1}, l_{j-1})$, and for shorter notation define also $v(x) = \frac{f(x)}{\delta}$ for $x \in \mathcal{X}_m$. We have that:

$$\begin{aligned}
&\sum_{\substack{(x,l): \\ x \in \mathcal{X}_m, l \in [0, v(x))}} \tilde{\mu}^{f^*}\left([w, (x, l)]_k^{j-i+k+1}\right) \\
&= \mathbb{1}_{\{1+l_{j-1}\}}(\alpha(x_{j-1})) \cdot \sum_{C \in \xi_m} \frac{\delta}{\int f^* d\mu} \mu([\tilde{\pi}(w), C]_k^{k+\#w+2}) \\
&+ \left(1 - \mathbb{1}_{\{1+l_{j-1}\}}(\alpha(x_{j-1}))\right) \cdot \frac{\delta}{\int f^* d\mu} \mu([\tilde{\pi}(w)]_{\#w+1}) \\
&= \frac{\delta}{\int f^* d\mu} \mu([\tilde{\pi}(w)]_{\#w+1}) \\
&= \tilde{\mu}^{f^*}([w]_k^{k+j-i})
\end{aligned}$$

from which (3.14) follows; and

$$
\begin{aligned}
&\sum_{\substack{(x,l):\\ x\in\mathcal{X}_m, l\in[0,v(x))}} \tilde{\mu}^{f^*}\left([(x,l),w]_k^{j-i+k+1}\right)\\
&= \sum_{x\in\mathcal{X}_m} \frac{\delta}{\int f^* d\mu}\mu([x,\tilde{\pi}(w)]_k^{k+\#w+2})\\
&= \frac{\delta}{\int f^* d\mu}\mu([\tilde{\pi}(w)]_k^{k+\#w+1})\\
&= \tilde{\mu}^{f^*}([w]_k^{j-i+k})
\end{aligned}
$$

hence (3.15).

To prove that $\tilde{\mu}^{f^*}$ is the equilibrium state of a Hölder potential we will find explicitly a Hölder potential $\tilde{\varphi} = \tilde{\varphi}_{m,\delta}$ associated to it. Suppose that μ is the equilibrium state of an α-Hölder potential φ, then the candidate is $\tilde{\varphi}(\overline{x}) = \tilde{\varphi}(\tilde{\pi}(\overline{x}))$.

We observe that $d(\overline{x},\overline{y}) \le \theta^{k\|f\|/\delta}$ implies $d(\tilde{\pi}(\overline{x}),\tilde{\pi}(\overline{y})) \le \theta^{mk}$ and

$$d(\tilde{\pi}(\overline{x}),\tilde{\pi}(\overline{y}))^{\|f\|/\delta} \le d(\overline{x},\overline{y})^m.$$

Therefore

$$\sup_{\overline{x}\neq\overline{y}} \frac{d\left(\tilde{\varphi}(\overline{x}),\tilde{\varphi}(\overline{y})\right)}{d(\overline{x},\overline{y})^{\alpha\delta m/\|f\|}} \le \sup_{\overline{x}\neq\overline{y}} \frac{d\left(\varphi(\tilde{\pi}(\overline{x})),\varphi(\tilde{\pi}(\overline{y}))\right)}{d(\tilde{\pi}(\overline{x}),\tilde{\pi}(\overline{y}))^{\alpha}} < \infty$$

because we assumed that φ is α-Hölder. This proves that $\tilde{\varphi}$ is $\frac{\alpha\delta m}{\|f\|}$-Hölder.

To prove that $\tilde{\mu}^{f^*}$ is an equilibrium state (or Gibbs measure) we need to check that it satisfies Definition 1.2.12. Suppose $m\delta/\|f\| = 1$, and for notational convenience call $s = \lfloor m\delta k/\|f\|\rfloor$. We have the following bounds:

$$
\begin{aligned}
&\sup_{\overline{x}\in\mathcal{X}_{f^*}} \frac{\tilde{\mu}^{f^*}\left([\overline{x}_{[0,k)}]_k\right)}{\exp\{-Pk + S_k^{\sigma}\tilde{\varphi}(\overline{x})\}}\\
&\le \frac{\delta}{\int f^* d\mu} \sup_{x\in\mathcal{X}} \frac{\mu([x_{[0,s+1)}]_{s+1})}{\exp\{-Ps/[m\delta/\|f\|] + S_s^{\sigma}\varphi(x)/[m\delta/\|f\|]\}}\\
&= \frac{\delta}{\int f^* d\mu} \sup_{x\in\mathcal{X}} \frac{\mu([x_{[0,s+1)}]_{s+1})}{\exp\{-Ps + S_s^{\sigma}\varphi(x)\}}\\
&\le \frac{\delta c_2}{\int f^* d\mu},
\end{aligned}
$$

and

$$\sup_{\overline{x}\in\mathcal{X}_{f^*}} \frac{\tilde{\mu}^{f^*}\left([\overline{x}_{[0,k)}]_k\right)}{\exp\{-Pk+S_k^\sigma\tilde{\phi}(\overline{x})\}} \geq \frac{\delta}{\int f^* d\mu} \sup_{x\in\mathcal{X}} \frac{\mu\left([x_{[0,k)}]_k\right)}{\exp\{-Pk+S_k^\sigma\varphi(x)\}}$$
$$\geq \frac{\delta c_1}{\int f^* d\mu}.$$

This concludes the demonstration. □

3.3.2 Proof of Theorem 3.2.2

In the hypothesis of Theorem 3.2.2 we have a special flow with roof function $f : \mathcal{X} \to \mathbb{R}^{>1}$, as we will use Proposition 3.3.2 to prove it, we will require to induce a roof function $f^+ : \mathcal{X}^+ \to \mathbb{R}^{>0}$. An useful lemma for this is the following:

Lemma 3.3.4 (Proposition 1.2 in [69])**.** *For a continuous ψ there exists ϕ such that $\psi \sim \phi$ and $\phi(x) = \phi(y)$ if and only if $x_i = y_i$ for all $i \geq 0$.*

Given a θ^2-Lipschitz function $f : \mathcal{X} \to \mathbb{R}^{>1}$, let $\tilde{f} : \mathcal{X} \to \mathbb{R}^{>0}$ such that $f \sim \tilde{f}$ and $\tilde{f}(x) = \tilde{f}(y)$ if and only if $x_i = y_i$ for all $i \geq 0$. We define the function $f^+ : \mathcal{X}^+ \to \mathbb{R}^{>0}$ by $f^+\left((x_n)_{n=0}^\infty\right) = \tilde{f}\left((x_n)_{n=0}^\infty\right)$ for some election of $(x_n)_{n=-\infty}^{-1}$ such that $(x_n)_{n=-\infty}^\infty \in \mathcal{X}$. The function $f^+ : \mathcal{X}^+ \to \mathbb{R}^{>0}$ is θ-Lipschitz.

Lemma 3.3.5. *Let (Λ_f, Φ_f^t) be a special flow over $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \mu, \sigma)$, where $(\mathcal{X}, \sigma)$ is an invertible subshift of finite type and μ is the equilibrium state associated for a Hölder potential on $\mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathbb{R}^{>1}$ is θ^2-Lipschitz and that $\{\mathcal{I}_n\}, \mathcal{I}_n \subset \mathcal{X}^+$ satisfies the nested condition with $\cap_{n\in\mathbb{N}}\mathcal{I}_n = \{z\}$ for $z \in \mathcal{X}^+$. Then*

$$\lim_{\delta\searrow 0}\lim_{n\to\infty} \frac{R(\mu^f, \mathcal{I}_n \times [0,\delta], \Lambda_f)}{\mu(\mathcal{I}_n)} = \frac{\gamma(z)}{\int \tilde{f} d\mu}.$$

Proof. For sake of brevity denote $x_+ = \pi_+ x \in \mathcal{X}^+$ for $x \in \mathcal{X}$. For $s, s' > 0$, if $s + s' \notin \{S_m^\sigma \tilde{f}(x) : m \in \mathbb{N}\}$, then $\Phi_{\tilde{f}}^s(x, s') \notin \mathcal{I}_n \times \{0\}$ and $\Phi_{f^+}^s(x_+, s') \notin \mathcal{I}_n \times \{0\}$. On the other hand, if $s, s' > 0$ such that $s + s' \in \{S_m^\sigma \tilde{f}(x) : m \in \mathbb{N}\}$, i.e. there exists $m_0 \in \mathbb{N}$ such that $s + s' = S_{m_0}^\sigma \tilde{f}(x)$, then

$$\begin{aligned} \Phi_{\tilde{f}}^s(x, s') \notin \mathcal{I}_n \times \{0\} &\Leftrightarrow \sigma^{m_0} x \notin \mathcal{I}_n \\ &\Leftrightarrow \sigma^{m_0} x_+ \notin \mathcal{I}_n. \end{aligned}$$

This implies that for $\delta > 0$ small enough, for all $t > 0$

$$\begin{aligned} &\mu \times \mu_{\text{Leb}} \left\{(x, s') \in \Lambda_{\tilde{f}} : \Phi_{\tilde{f}}^s(x, s') \notin \mathcal{I}_n \times [0,\delta], 0 < s \leq t\right\} \\ &= \mu \times \mu_{\text{Leb}} \left\{(x, s') \in \Lambda_{\tilde{f}} : \Phi_{f^+}^s(x_+, s') \notin \mathcal{I}_n \times [0,\delta], 0 < s \leq t\right\}. \end{aligned}$$

It is a consequence of the proof of the existence of μ in [16] that there exists an equilibrium state μ_+ on $\mathcal{B}_{\mathcal{X}^+}$ associated for a Hölder potential on $\mathcal{X}^+$ such that

$$\begin{aligned}
&\mu \times \mu_{\text{Leb}} \left\{(x, s') \in \Lambda_{\tilde{f}} : \Phi^s_{f^+}(x_+, s') \notin \mathcal{I}_n \times [0, \delta], 0 < s \leq t\right\} \\
&= \mu_+ \times \mu_{\text{Leb}} \left\{(x, s') \in \Lambda_{f^+} : \Phi^s_{f^+}(x_+, s') \notin \mathcal{I}_n \times [0, \delta], 0 < s \leq t\right\}.
\end{aligned}$$

Using Proposition 3.3.2 we have that

$$\lim_{\delta \searrow 0} \lim_{n \to \infty} \frac{R(\mu_+^{f^+}, \mathcal{I}_n \times [0, \delta], \Lambda_{f^+})}{\mu_+(\mathcal{I}_n)} = \frac{\gamma(z)}{\int f^+ d\mu_+},$$

which combined with the previous identity and the fact that

$$\int \tilde{f} d\mu = \int f^+ d\mu_+$$

concludes the proof. □

The previous lemma will be used to prove the following:

Lemma 3.3.6. *Let (Λ_f, Φ^t_f) be a special flow over $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, \sigma)$, where $(\mathcal{X}, \sigma)$ is an invertible subshift of finite type and μ is the equilibrium state associated for a Hölder potential on $\mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathbb{R}^{>1}$ is θ^2-Lipschitz and that $\{\mathcal{I}_n\}, \mathcal{I}_n \subset \mathcal{X}^+$ satisfies the nested condition with $\cap_{n \in \mathbb{N}} \mathcal{I}_n = \{z\}$ for $z \in \mathcal{X}^+$. Then*

$$\lim_{\delta \searrow 0} \lim_{n \to \infty} \frac{R(\mu^{\tilde{f}}, \sigma^{[n/2]}\mathcal{I}_n \times [0, \delta], \Lambda_{\tilde{f}})}{\mu(\mathcal{I}_n)} = \frac{\gamma(z)}{\int \tilde{f} d\mu}.$$

Proof. For any $x \in \mathcal{X}$, denote $\mathcal{S}_{\tilde{f}}(x) = \{S^\sigma_m \tilde{f}(x) : m \in \mathbb{N}\}$ and $\mathcal{S}_{f^+}(x_+) = \{S^\sigma_m f^+(x) : m \in \mathbb{N}\}$, where $x_+ = \pi_+ x \in \mathcal{X}^+$. By definition we have for $n \in \mathbb{N}$ the equivalences:

$$\begin{aligned}
\Phi^s_{\tilde{f}}(x, s') \notin \sigma^{[n/2]}\mathcal{I}_n \times \{0\} &\Leftrightarrow s + s' \notin \mathcal{S}_{\tilde{f}}(x) \text{ or } \sigma^m x \notin \sigma^{[n/2]}\mathcal{I}_n \times \{0\} \\
&\Leftrightarrow s + s' \notin \mathcal{S}_{\tilde{f}}(x) \text{ or } \sigma^{m-[n/2]} x \notin \mathcal{I}_n \times \{0\},
\end{aligned}$$

for every $x \in \mathcal{X}$. If $n, m \in \mathbb{N}$ and $m > n$ we replace $S^\sigma_m \tilde{f}(x)$ by

$$S^\sigma_{m-[n/2]} \tilde{f}(x) + \sum_{k=m-[n/2]}^{m-1} \tilde{f}(\sigma^k x).$$

Choose $\epsilon > 0$ such that $\tilde{f} > \epsilon$ on $\mathcal{X}$. We have the inequalities:

$$[n/2]\epsilon \leq \sum_{k=m-[n/2]}^{m-1} \tilde{f}(\sigma^k x) \leq [n/2]\|\tilde{f}\|,$$

for every $x \in \mathcal{X}$. Using the inequality above and the monotonicity of log, we conclude that for $\delta > 0$ small enough and for all $t > 0$

$$\begin{aligned} K(\mu^{\tilde{f}}, t + [n/2]\epsilon, \mathcal{I}_n \times [0,\delta], \Lambda_{\tilde{f}}) &\leq K(\mu^{\tilde{f}}, t, \sigma^{[n/2]}\mathcal{I}_n \times [0,\delta], \Lambda_{\tilde{f}}) \\ &\leq K(\mu^{\tilde{f}}, t - [n/2]\|\tilde{f}\|, \mathcal{I}_n \times [0,\delta], \Lambda_{\tilde{f}}), \end{aligned}$$

and therefore,

$$\begin{aligned} &-\frac{t + [n/2]\epsilon}{t}\frac{1}{t + [n/2]\epsilon} K(\mu^{\tilde{f}}, t + [n/2]\epsilon, \mathcal{I}_n \times [0,\delta], \Lambda_{\tilde{f}}) \\ &\geq -\frac{1}{t} K(\mu^{\tilde{f}}, t, \sigma^{[n/2]}\mathcal{I}_n \times [0,\delta], \Lambda_{\tilde{f}}) \\ &\geq -\frac{t - [n/2]\|\tilde{f}\|}{t}\frac{1}{t - [n/2]\|\tilde{f}\|} K(\mu^{\tilde{f}}, t - [n/2]\|\tilde{f}\|, \mathcal{I}_n \times [0,\delta], \Lambda_{\tilde{f}}). \end{aligned}$$

Using Lemma 3.3.5 we conclude the result. □

As we have replaced the original roof function f by $\tilde{f}$, we require to prove that the escape rates for the special flows with f and $\tilde{f}$ are close. This is the content of the next lemma.

Lemma 3.3.7. *Suppose $f : \mathcal{X} \to \mathbb{R}^{>0}$ is continuous. Let μ be an equilibrium state for a Hölder potential on $\mathcal{X}$. Suppose $\mathcal{U} \subset \mathcal{X}$ and call*

$$\mathcal{I}^t_{\tilde{f}}(\mathcal{U}) := \Big\{(x, s') \in \Lambda_{\tilde{f}} : \Phi^s_{\tilde{f}}(x, s') \notin \mathcal{U} \times \{0\}, s \in [0, t]\Big\}$$

and

$$\mathcal{J}^t_{\tilde{f}}(\mathcal{U}) := \mathcal{I}^t_{\tilde{f}}(\mathcal{U}) \cap \{(x, s') \in \mathcal{X} \times \mathbb{R} : \tilde{f}(x) > s' > f(x)\}.$$

Then for all $t > 0$ and for every $\epsilon \in (0, \inf_{x\in\mathcal{X}}\{|f(x)|\})$, we have that

$$\mu \times \mu_{Leb}(\mathcal{I}^t_{\tilde{f}}(\mathcal{U})) - \mu \times \mu_{Leb}(\mathcal{J}^t_{\tilde{f}}(\mathcal{U})) \geq \frac{\epsilon}{\|\tilde{f}\|} \cdot \mu \times \mu_{Leb}(\mathcal{I}^t_{\tilde{f}}(\mathcal{U})).$$

Proof. Fix $t > 0$. Then there exists $\beta_t \in \mathcal{L}^1(\mathcal{X}, \mu)$ such that

$$\mu \times \mu_{\text{Leb}}(\mathcal{I}^t_{\tilde{f}}(\mathcal{U})) = \int_{\mathcal{X}} \beta_t(x) d\mu(x).$$

In particular $\beta_t(x) < \tilde{f}(x)$ for all $x \in \mathcal{X}$ and

$$\mu \times \mu_{\text{Leb}}(\mathcal{J}^t_{\tilde{f}}(\mathcal{U})) = \int_{\{x:\beta_t(x)>f(x)\}} (\beta_t(x) - f(x))\, d\mu(x).$$

Too see this, we "think" of $\beta_t(x)$ as the fibre $\{x\} \times [0, \beta_t(x))$. By definition of the space $\Lambda_{\tilde{f}}$ we have that $\beta_t(x) < g(x)$. Then

$$\begin{aligned}
&\mu \times \mu_{\text{Leb}}(\mathcal{I}^t_{\tilde{f}}(\mathcal{U})) - \mu \times \mu_{\text{Leb}}(\mathcal{J}^t_{\tilde{f}}(\mathcal{U})) \\
&= \int_{\{x:\beta_t(x)<f(x)\}} \beta_t(x) d\mu(x) + \int_{\{x:\beta_t(x)>f(x)\}} f(x) d\mu(x) \\
&\geq \int_{\{x:\beta_t(x)<f(x)\}} \beta_t(x) d\mu(x) + \int_{\{x:\beta_t(x)>f(x)\}} f(x) \frac{\beta_t(x)}{\tilde{f}(x)} d\mu(x) \\
&\geq \int_{\{x:\beta_t(x)<f(x)\}} \beta_t(x) d\mu(x) + \frac{\epsilon}{\|\tilde{f}\|} \int_{\{x:\beta_t(x)>f(x)\}} \beta_t(x) d\mu(x) \\
&\geq \frac{\epsilon}{\|\tilde{f}\|} \cdot \mu \times \mu_{\text{Leb}}(\mathcal{I}^t_{\tilde{f}}(\mathcal{U})).
\end{aligned}$$

□

In the proof of the main result we require to consider a small modification of the previous lemma, this is the content of the next remark.

Remark 3.3.8. *Suppose $f : \mathcal{X} \to \mathbb{R}^{>0}$ is continuous. Let μ be an equilibrium state for a Hölder potential on $\mathcal{X}$. Suppose $\mathcal{U} \subset \mathcal{X}$ and call*

$$\mathcal{I}^t_f(\mathcal{U}) := \left\{(x, s') \in \Lambda_f : \Phi^s_f(x, s') \notin \mathcal{U} \times \{0\}, s \in [0, t]\right\}$$

and

$$\mathcal{J}^t_f(\mathcal{U}) := \mathcal{I}^t_f(\mathcal{U}) \cap \{(x, s') \in \mathcal{X} \times \mathbb{R} : f(x) > s' > g(x)\}.$$

Then for all $t > 0$ and for every $\epsilon \in (0, \inf_{x \in \mathcal{X}}\{|f(x)|\})$, we have that

$$\mu \times \mu_{Leb}(\mathcal{I}^t_f(\mathcal{U})) - \mu \times \mu_{Leb}(\mathcal{J}^t_f(\mathcal{U})) \geq \frac{\epsilon}{\|f\|} \cdot \mu \times \mu_{Leb}(\mathcal{I}^t_f(\mathcal{U}))$$

for all $t > 0$. To prove this we can use the same proof that Lemma 3.3.7 but replacing f by $\tilde{f}$ and $\tilde{f}$ by f.

Lemma 3.3.9. *Let (Λ_f, Φ^t_f) be a special flow over $(\mathcal{X}, \sigma)$. Suppose that $f : \mathcal{X} \to \mathbb{R}^{>1}$ is θ^2-Lipschitz and that $\{\mathcal{I}_n\}, \mathcal{I}_n \subset \mathcal{X}^+$ satisfies the nested condition with $\cap_{n\in\mathbb{N}}\mathcal{I}_n =$*

$\{z\}$ for $z \in \mathcal{X}^+$. Then

$$\lim_{\delta\searrow 0}\lim_{n\to\infty}\frac{R(\mu_\varphi^f, \sigma^{[n/2]}\mathcal{I}_n \times [0,\delta], \Lambda_f)}{\mu_\varphi(\mathcal{I}_n)} = \frac{\tilde{\gamma}_\varphi(z)}{\int f d\mu_\varphi} \text{ for all Hölder functions } \varphi : \mathcal{X} \to \mathbb{R}.$$

Proof. Under the assumptions of the theorem we can find a constant $C = C_f$ such that $|S_m^\sigma f(x) - S_m^\sigma f(x)| \leq C$ for all $m \in \mathbb{N}$. Then, omitting (in particular all that follows up to the inequality (3.19) is independent of $n \in \mathbb{N}$) the dependence of $\mathcal{I}_f$ and $\mathcal{I}_{\tilde{f}}$ on $\mathcal{I} = \sigma^{[n/2]}\mathcal{I}_n$,

$$\mathcal{I}_f^{t+C} \subset \mathcal{I}_{\tilde{f}}^t \cup \mathcal{J}_f^{t+C} \tag{3.16}$$

and

$$\mathcal{I}_{\tilde{f}}^t \subset \mathcal{I}_f^{t-C} \cup \mathcal{J}_{\tilde{f}}^t, \tag{3.17}$$

where $t \in \mathbb{R}^{>C}$.

Use Lemma 3.3.7 and Remark 3.3.8 in (3.16) and (3.17) to obtain:

$$\frac{\epsilon}{\|f\|} \cdot \mu \times \mu_{\text{Leb}}(\mathcal{I}_f^{t+C}) \leq \mu \times \mu_{\text{Leb}}(\mathcal{I}_{\tilde{f}}^t) \tag{3.18}$$

and

$$\frac{\epsilon}{\|\tilde{f}\|} \cdot \mu \times \mu_{\text{Leb}}(\mathcal{I}_{\tilde{f}}^t) \leq \mu \times \mu_{\text{Leb}}(\mathcal{I}_f^{t-C}). \tag{3.19}$$

Using Lemma 3.3.6 we conclude the result for $\tilde{f}$. This together with the inequalities (3.18) and (3.19) conclude the proof. □

We prove inclusions (3.16) and (3.17), the inequalities (3.18) and (3.19) and the last step in the proof of Theorem 3.2.2.

Proof of Inclusions. To prove the inclusion (3.16) take

$$(x, s') \in \mathcal{I}_f^{t+C}$$

with $s' \leq g(x)$. By definition

$$\Phi_f^s(x, s') \notin \mathcal{U} \times \{0\} \text{ if } 0 < s \leq t + C,$$

that can be read as

$$\sigma^m(x) \notin \mathcal{U} \text{ if } S_m^\sigma f(x) = s + s' \text{ and } 0 < s \leq t + C,$$

then, because $S_k^\sigma f(x) - C \leq S_k^\sigma g(x) \leq S_k^\sigma f(x) + C$ is valid for all $x \in \mathcal{X}$ and all

$k \geq 0$ we deduce that

$$\sigma^m(x) \notin \mathcal{U} \text{ if } S_m^\sigma g(x) = s + s' \text{ and } 0 < s \leq t,$$

which meaning is

$$\Phi_g^s(x, s') \notin \mathcal{U} \times \{0\} \text{ if } 0 < s \leq t.$$

This concludes the proof of the first inclusion. The second inclusion can be obtained in a similar way by replacing the roles of f and g. $\square$

Proof of Inequalities. We will prove inequality (3.18).

From the inclusion (3.16) we deduce that

$$\mu \times \mu_{\mathrm{Leb}}(\mathcal{I}_f^{t+C}) \leq \mu \times \mu_{\mathrm{Leb}}(\mathcal{I}_g^t) + \mu \times \mu_{\mathrm{Leb}}(\mathcal{J}_f^{t+C})$$

so we can apply Lemma 3.3.7 to the left hand side of

$$\mu \times \mu_{\mathrm{Leb}}(\mathcal{I}_f^{t+C}) - \mu \times \mu_{\mathrm{Leb}}(\mathcal{J}_f^{t+C}) \leq \mu \times \mu_{\mathrm{Leb}}(\mathcal{I}_g^t)$$

to get

$$\frac{\epsilon}{\|f\|} \cdot \mu \times \mu_{\mathrm{Leb}}(\mathcal{I}_f^{t+C}) \leq \mu \times \mu_{\mathrm{Leb}}(\mathcal{I}_g^t).$$

The inequality (3.19) comes from replacing the roles of f and g, using inclusion (3.17) instead of (3.16) and Corollary 3.3.8 instead of Lemma 3.3.7. $\square$

Proof of Conclusion. We use inequality (3.18) and replace the measure $\mu \times \mu_{\mathrm{Leb}}$ by $\int f d\mu \cdot \mu^f$ in the left hand side and by $\int g d\mu \cdot \mu^g$ in the right hand side. We obtain

$$\frac{\epsilon}{\|f\|} \cdot \int f d\mu \cdot \mu^f(\mathcal{I}_f^{t+C}) \leq \int g d\mu \cdot \mu^g(\mathcal{I}_g^t).$$

This allows to conclude that

$$\limsup_{t \to \infty} \frac{t+C}{t} \frac{1}{t+C} \log \mu^f(\mathcal{I}_f^{t+C}) \leq \limsup_{t \to \infty} \frac{1}{t} \log \mu^g(\mathcal{I}_g^t).$$

Multiplying both sides by -1, dividing by $\mu(\hat{\mathcal{U}}_n)$, taking limit when n tends to infinity and applying Corollary 3.3.6 to g we obtain that

$$\lim_{n \to \infty} \frac{1}{\mu(\hat{\mathcal{U}}_n)} \cdot \left(- \limsup_{t \to \infty} \frac{1}{t+C} \log \mu^f(\mathcal{I}_f^{t+C}(\hat{\mathcal{U}}_n)) \right) \geq \frac{\gamma(z)}{\int g d\mu}.$$

Doing analogous calculations with inequality (3.19) we conclude that

$$\lim_{n\to\infty} \frac{1}{\mu(\hat{\mathcal{U}}_n)} \cdot \left(-\limsup_{t\to\infty} \frac{1}{t-C} \log \mu^f(\mathcal{I}_f^{t-C}(\hat{\mathcal{U}}_n)) \right) \leq \frac{\gamma(z)}{\int g d\mu}.$$

This two inequalities and a change of variable when taking the lim sup we find

$$\lim_{n\to\infty} \frac{1}{\mu(\hat{\mathcal{U}}_n)} \cdot \left(-\limsup_{t\to\infty} \frac{1}{t} \log \mu^f(\mathcal{I}_f^{t}(\hat{\mathcal{U}}_n)) \right) = \frac{\gamma(z)}{\int g d\mu}.$$

Finally, because of the σ invariance of μ and the definition of g we have that

$$\frac{\gamma(z)}{\int g d\mu} = \frac{\gamma(z)}{\int f d\mu}$$

hence the result. □

We conclude this subsection with the proof of our theorem.

Proof. The invariant probability measure ν is an equilibrium state of $\varphi : \Lambda_f \to \mathbb{R}^{>0}$, then if we define $\phi(x) := \int_0^{f(x)} \varphi(x,t)dt$ and μ is its equilibrium state, we have that $\nu = \mu^f$ and we can use Proposition 3.3.2 to conclude that

$$\lim_{\delta \searrow 0} \lim_{n\to\infty} \frac{R(\nu, \mathcal{U}_n \times [0,\delta], \Lambda_f)}{\mu(\mathcal{U}_n)} = \frac{\gamma(z)}{\int f d\mu}.$$

However, $\nu(\mathcal{U}_n \times [0,1]) = \frac{\mu(\mathcal{U}_n)}{\int f d\mu}$, so

$$\lim_{\delta \searrow 0} \lim_{n\to\infty} \frac{R(\nu, \mathcal{U}_n \times [0,\delta], \Lambda_f)}{\nu(\mathcal{U}_n \times [0,1])} = \gamma(z).$$

To prove that we can write $\tilde{\gamma}$ instead of γ it is enough to notice that $P(\phi) = 0$ if and only if $P(\varphi) = 0$, and that, if z is periodic, then there exists a periodic orbit τ such that $z \in \tau$ and $S_p^\sigma \phi(z) = \int_\tau \varphi dt$, if z is not periodic, then there does not exist any periodic orbit τ such that $z \in \tau$. This allows to conclude that $\gamma(z) = \tilde{\gamma}(z)$. □

3.3.3 Proof of Theorem 3.2.4

Consider a conformal repeller $(\mathcal{J}, f)$, $F : \mathcal{J} \to \mathbb{R}^{>1}$ a Hölder function and let μ_φ be the equilibrium measure associated for a Hölder potential $\varphi : \mathcal{J} \to \mathbb{R}$. The main result of this subsection is the following lemma, that continues the work done in [39], Section 6.

Lemma 3.3.10. *Given $z \in \mathcal{J}$, we have for $B(z,\epsilon)$, the ball centred at z of radius $\epsilon > 0$ that*
$$\lim_{\delta \searrow 0} \lim_{\epsilon \to 0} \frac{R(\mu_\varphi^F, B(z,\epsilon) \times [0,\delta], \Lambda_F)}{\mu_\varphi(B(z,\epsilon))} = \frac{\gamma_\varphi(z)}{\int F d\mu_\varphi}.$$
Under the extra hypothesis $F > 1$ we conclude the proof of Theorem 3.2.4.

Before going through the proof of this lemma, we state without proof a useful result.

Proposition 3.3.11 (Stated in [39], proof of Theorem 1.1)**.** *Let $\{\mathcal{V}_n\}$ be a family of sets satisfying the nested condition (Definition 3.2.1) with $\cap_{n\in\mathbb{N}} \mathcal{V}_n = \{z\}$ and $z \in \mathcal{J}$ periodic or satisfying:*

1. *for every $n \in \mathbb{N}$, $\mathcal{V}_{n+1} \subset \mathcal{V}_n$ and $\mathcal{V}_n$ is a finite union of cylinders;*
2. *$\cap_{n\in\mathbb{N}} \mathcal{V}_n$ consists of finitely many non periodic points $\{z^1, z^2, \ldots, z^l\}$;*
3. *$\exists c > 0, 0 < \rho < 1$ such that $\mu(\mathcal{V}_n) < c\rho^{k_n}$ for all $n \in \mathbb{N}$, where k_n is the maximum of the length of the cylinders in $\mathcal{V}_n$; and*
4. *$\exists \{l_n\} \subset \mathbb{N}$ and $\kappa > 0$ constant such that $\kappa < l_n/k_n$ and $\mathcal{V}_n \subset \cup_{i=1}^{l} \left[z^i\right]_{l_n}$ for all $n \in \mathbb{N}$.*

Then
$$\lim_{n\to\infty} \frac{R(\mu_\varphi, \mathcal{V}_n, \mathcal{J})}{\mu_\varphi(\mathcal{V}_n)} = \gamma_\varphi(z).$$
The following remark will be essential to prove Lemma 3.3.10.

Remark 3.3.12. *We can allow the family of sets $\{\mathcal{U}_n\}$ in Proposition 3.3.2 to satisfy 1-4 (in Proposition 3.3.11) instead of the nested condition (Definition 3.2.1). Indeed, Proposition 3.3.2 is indifferent of the nested condition or 1-4 in Proposition 3.3.11, because it only requires the conclusion of Theorem 3.2.3.*

To prove Lemma 3.3.10 we recall that there exists a semi-conjugacy $\pi : \mathcal{X}^+ \to \mathcal{J}$, where $\mathcal{X}^+$ is a non-invertible subshift of finite type of s symbols for some $s \in \mathbb{N}$, that is, π is a continuous surjection and $f \circ \pi = \pi \circ \sigma$ for σ the shift action on $\mathcal{X}^+$. Choosing $\lambda^{-\alpha} < \theta < 1$, and considering the metric space $(\mathcal{X}^+, d_\theta)$ we have that $\tilde{\varphi} := \varphi \circ \pi : \mathcal{X}^+ \to \mathbb{R}$ is d_θ-Lipschitz. Define $\tilde{F} := F \circ \pi : \mathcal{X}^+ \to \mathbb{R}^{>0}$, $\tilde{\varphi} := \varphi \circ \pi : \mathcal{X}^+ \to \mathbb{R}^{>0}$ and call by $\tilde{\mu}$ the equilibrium measure associated to $\tilde{\varphi}$.

Clearly, $\int \tilde{F} d\tilde{\mu} = \int F d\mu$ and for $\delta > 0$ small enough
$$\frac{R(\mu^F, B(z,\epsilon) \times [0,\delta], \Lambda_F)}{\mu(B(z,\epsilon))} = \frac{R(\mu^{\tilde{F}}, \pi^{-1}B(z,\epsilon) \times [0,\delta], \Lambda_{\tilde{F}})}{\tilde{\mu}(\pi^{-1}B(z,\epsilon))}.$$

Suppose that $\{\epsilon_n\} \subset \mathbb{R}^{>0}$ tends to zero as n tends to infinity. Then we would like to find families of sets $\{\overline{\mathcal{V}_n}\}, \{\underline{\mathcal{V}_n}\}$ with $\overline{\mathcal{V}_n}, \underline{\mathcal{V}_n} \subset \mathcal{X}^+$ satisfying 1-4 in Proposition 3.3.11 when $\lim_{n\to\infty} B(z,\epsilon_n) \to \{z\}$ and $z \in \mathcal{J}$ non periodic, or satisfying the nested condition when $\lim_{n\to\infty} B(z,\epsilon_n) \to \{z\}$ and $z \in \mathcal{J}$ periodic, such that

$$(1-\eta)\frac{R(\mu^{\tilde{F}}, \underline{\mathcal{V}_n} \times [0,\delta], \Lambda_{\tilde{F}})}{\tilde{\mu}(\underline{\mathcal{V}_n})} \leq \frac{R(\mu^{\tilde{F}}, \pi^{-1}B(z,\epsilon_n) \times [0,\delta], \Lambda_{\tilde{F}})}{\tilde{\mu}(\pi^{-1}B(z,\epsilon_n))} \tag{3.20}$$

and

$$\frac{R(\mu^{\tilde{F}}, \pi^{-1}B(z,\epsilon_n) \times [0,\delta], \Lambda_{\tilde{F}})}{\tilde{\mu}(\pi^{-1}B(z,\epsilon_n))} \leq (1-\eta)^{-1}\frac{R(\mu^{\tilde{F}}, \overline{\mathcal{V}_n} \times [0,\delta], \Lambda_{\tilde{F}})}{\tilde{\mu}(\overline{\mathcal{V}_n})} \tag{3.21}$$

where $1/2 > \eta > 0$ can be taken arbitrarily small.

We can find explicitly the families of sets $\{\overline{\mathcal{V}_n}\}, \overline{\mathcal{V}_n} \subset \mathcal{X}^+$ and $\{\underline{\mathcal{V}_n}\}, \underline{\mathcal{V}_n} \subset \mathcal{X}^+$. For this we will need the next lemma (that we prove in what follows) and Lemma 3.3.16 (that we state at the end without proof).

Lemma 3.3.13. *Suppose that $\{\epsilon_n\} \subset \mathbb{R}^{>0}$ tends to zero as n tends to infinity. If $B(z,\epsilon_n) \to \{z\}$ as n tends to infinity with $z \in \mathcal{J}$ non periodic; then we can find a family of sets $\{\overline{\mathcal{V}_n}\}, \overline{\mathcal{V}_n} \subset \mathcal{X}^+$ satisfying 1-4 in Proposition 3.3.11 and the inequality (3.20) for $1/2 > \eta > 0$ fixed but arbitrarily small.*

We use the following propositions from [39].

Proposition 3.3.14 (Lemma 6.4)**.** *For all $z \in \mathcal{J}$, there exist constants $c_1, s > 0$ such that $\mu(B(z,\epsilon)) \leq c_1\epsilon^s$ for every $\epsilon > 0$.*

Proposition 3.3.15 (Proposition 6.5)**.** *There exist constants $D, c_2 > 0$ such that for all $z \in \mathcal{J}, \epsilon > 0$ and $0 < \delta < 1$ we have that μ satisfies*

$$\mu\left(B(z,\epsilon) \setminus B(z,(1-\delta)\epsilon)\right) \leq c_2\delta^D\mu(B(z,\epsilon)).$$

Proof of Lemma 3.3.13. Given $\{\epsilon_n\} \subset \mathbb{R}^{>0}$ and $k > 0$, define

$$\mathcal{U}_{k,n} := \left\{\mathcal{U} \in \bigvee_{i=0}^{k-1} f^{-i}\mathcal{R} : \mathcal{U} \cap B(z,\epsilon_n) \neq \emptyset\right\}.$$

Because f is uniformly expansive, there exist $c_3 > 0$ and $0 < \rho < 1$ such that

$\operatorname{diam}(\mathcal{U}) \le c_3\rho^k$ for all $\mathcal{U} \in \mathcal{U}_{k,n}$. This implies that

$$\mathcal{U}_{k,n} \subset B(z, \epsilon + c_3\rho^k) := B(z, (1-\alpha_k)^{-1}\epsilon)$$

for $\alpha_k = \frac{c_3\rho^k}{\epsilon + c_3\rho^k}$. Choose $k_0 \gg 1$, then using Proposition 3.3.15, for $1/2 > \eta > c_2\alpha_{k_0}^D$ we have that for all $k \ge k_0$

$$(1-\eta)\mu(\mathcal{U}_{k,n}) \le \mu(B(z,\epsilon)).$$

For each $n \in \mathbb{Z}, n > n_0$ with n_0 fixed, we find $k(n) \in \mathbb{Z}, k(n) > k_0$ such that

$$\rho^{k(n)} \le \frac{\epsilon_n}{c_3((c_2\eta^{-1})^{1/D} - 1)} \le \rho^{k(n)-1}.$$

The candidate for $\overline{\mathcal{V}_n}$ is $\pi^{-1}\mathcal{U}_{k(n),n}$, that clearly satisfies 1 and 2. The way in which we chose $k(n)$ and Proposition 3.3.14 implies

$$\begin{aligned}\tilde{\mu}(\pi^{-1}\mathcal{U}_{k(n),n}) = \mu(\mathcal{U}_{k(n),n}) &\le (1-\eta)^{-1}\mu(B(z,\epsilon_n)) \\ &\le c_1\epsilon_n^s \le c_1(c_3((c_2\eta^{-1})^{1/D} - 1)^s)\rho^{s(k(n)-1)}.\end{aligned}$$

So our candidate satisfies 3. Suppose that $\pi^{-1}z = \{z^1, z_2, \ldots, z^d\}$. To prove 4, we notice that $\overline{\mathcal{V}_n} \subset \cup_{i=1}^d [z^i]_{l_n}$ where l_n is the minimum such that $c_4^{-1}\varrho^{l_n} \ge 2\epsilon_n$ and $c_4 > 0, 0 < \varrho < 1$ are such that for any $k \in \mathbb{N}, i \in \{1, \ldots, d\}$, $c_4^{-1}\varrho^k \le \operatorname{diam}(\pi[z^i]_k)$. To prove the inequality (3.20), we notice that for every $\delta > 0$ small enough the escape rate $R(\mu^F, \mathcal{U} \times [0,\delta], \Lambda_F)$ is increasing in $\mathcal{U}$, then

$$\limsup_{n\to\infty} \frac{R(\mu^F, B(z,\epsilon) \times [0,\delta], \Lambda_F)}{\mu(B(z,\epsilon))} \le (1-\eta)^{-1} \limsup_{n\to\infty} \frac{R(\mu^{\tilde{F}}, \overline{\mathcal{V}_n} \times [0,\delta], \Lambda_{\tilde{F}})}{\tilde{\mu}(\overline{\mathcal{V}_n})}.$$

Finally, we can choose $k_0 \in \mathbb{N}$ arbitrarily large and then $\eta > 0$ can be chosen arbitrarily close to 0. □

Similarly, we can prove the following lemma.

Lemma 3.3.16. *Suppose that $\{\epsilon_n\} \subset \mathbb{R}^{>0}$ tends to zero as n tends to infinity.*

(a) If $B(z,\epsilon_n) \to \{z\}$ as n tends to infinity with $z \in \mathcal{J}$ periodic; then we can find a family of sets $\{\overline{\mathcal{V}_n}\}, \overline{\mathcal{V}_n} \subset \mathcal{X}^+$ satisfying the nested condition and the inequality (3.20) for $1/2 > \eta > 0$ fixed but arbitrarily small.

(b) If $B(z,\epsilon_n) \to \{z\}$ as n tends to infinity with $z \in \mathcal{J}$ non periodic; then we can find a family of sets $\{\underline{\mathcal{V}_n}\}, \underline{\mathcal{V}_n} \subset \mathcal{X}^+$ satisfying 1-4 and the inequality (3.21) for $1/2 > \eta > 0$ fixed but arbitrarily small.

(c) If $B(z, \epsilon_n) \to \{z\}$ as n tends to infinity with $z \in \mathcal{J}$ periodic; then we can find a family of sets $\{\underline{\mathcal{V}_n}\}, \underline{\mathcal{V}_n} \subset \mathcal{X}^+$ satisfying the nested condition and the inequality (3.21) for $1/2 > \eta > 0$ fixed but arbitrarily small.

This finishes the proof of Lemma 3.3.10.

3.4 Consequences

3.4.1 More general roof functions

If we think in the full-shift on two symbols represented by the unit interval, and a suspension over it with a roof function with discontinuities. It seems that there should not be any difference if we repair in some way the discontinuities of f and apply Proposition 3.3.2 for the repaired map. We should try to repair f with functions close in $\mathcal{L}^1$ and then repeat the argument in the proof of Proposition 3.3.2. This is what we do in this subsection.

Suppose μ is a probability measure on $\mathcal{X}^+$.

Definition 3.4.1. *We say that a function $g : \mathcal{X}^+ \to \mathbb{R}$ is adapted to μ, if there exists a pair of constants $C, \epsilon > 0$ with $C > g > \epsilon$ and there exists a family of set of points $\{\mathcal{Y}_n\}$ where $\mathcal{Y}_n = \{y^1, y^2, \ldots, y^{r_n}\} \subset \mathcal{X}^+$, for $r_n > 0$ such that for all $n \in \mathbb{N}$,*

$$\max \left\{ \sup_{y \in [x]_n} g(x) - \sup_{y \in [x]_n} g(x) : x \in \Sigma^+ \setminus \mathcal{Y}_n \right\} < \eta(n),$$

where $\eta : \mathbb{N} \to \mathbb{R}^{>0}$ and $\eta(n) \to 0$ as $n \to \infty$. Additionally, we require that there exists $\eta' : \mathbb{N} \to \mathbb{R}^{>0}$ and $\eta'(n) \to 0$ as $n \to \infty$ such that

$$\sum_{y \in \mathcal{Y}_n} \mu([y]_n) < \eta'(n).$$

Corollary 3.4.2 (Corollary of Proposition 3.3.2)**.** *Suppose μ is an equilibrium state of Hölder potential. Then, the conclusion of Proposition 3.3.2 is still valid if we replace the condition of Lipschitz roof function by the condition of roof function adapted to μ.*

Proof. Consider g the adapted roof function to μ of the statement. Notice that to prove the result it is enough to find $\{\underline{g}_n\}, \{\overline{g}_n\}$ sets of adapted to μ functions such that:

1. for some fixed $\epsilon_1 > 0$, $\inf_n \inf_x \underline{g}_n(x) > \epsilon_1$,

2. for every $n \in \mathbb{N}$, $\underline{g}_n \leq g \leq \overline{g}_n$,

3. $\left\| \overline{g}_n - \underline{g}_n \right\|_{\mathcal{L}^1} \to 0$ as $n \to \infty$.

Indeed, suppose we are able to find those sets of functions, then for any k, for all $t > 0$, independently of the sets $\mathcal{U}$ that K depends on

$$K(\mu^{\underline{g}_n}, t, \mathcal{U}, \Lambda_{\underline{g}_n}) + \log \int \underline{g}_n d\mu \leq K(\mu^g, t, \mathcal{U}, \Lambda_g) \leq K(\mu^{\overline{g}_n}, t, \mathcal{U}, \Lambda_{\overline{g}_n}) + \log \int \overline{g}_n d\mu.$$

Therefore,

$$\begin{aligned}
\frac{\gamma(z)}{\int \underline{g}_n d\mu} &= - \lim_{n\to\infty} \frac{1}{\mu(\mathcal{U}_n)} \lim_{t\to\infty} \frac{1}{t} K(\mu^{\underline{g}_n}, t, \mathcal{U}, \Lambda_{\underline{g}_n}) \\
&\geq - \lim_{n\to\infty} \frac{1}{\mu(\mathcal{U}_n)} \lim_{t\to\infty} \frac{1}{t} K(\mu^g, t, \mathcal{U}, \Lambda_g) \\
&\geq - \lim_{n\to\infty} \frac{1}{\mu(\mathcal{U}_n)} \lim_{t\to\infty} \frac{1}{t} K(\mu^{\overline{g}_n}, t, \mathcal{U}, \Lambda_{\overline{g}_n}) \\
&= \frac{\gamma(z)}{\int \overline{g}_n d\mu}.
\end{aligned}$$

However, because of our assumptions,

$$\left| \frac{\gamma(z)}{\int \underline{g}_n d\mu} - \frac{\gamma(z)}{\int \underline{g}_n d\mu} \right| < \frac{\int |\overline{g}_n - \underline{g}_n| d\mu}{\epsilon_0 \epsilon_1} \to_{n\to\infty} 0,$$

and this allows to conclude the result.

For a fixed k, we "repair" g to define $\underline{g}_k$ and $\overline{g}_k$ by:

$$\underline{g}_k|_{\{[y]_k : y \in \mathcal{Y}_k\}^c} = \overline{g}_k|_{\{[y]_k : y \in \mathcal{Y}_k\}^c} = g|_{\{[y]_k : y \in \mathcal{Y}_k\}^c},$$

and for all $y \in \mathcal{Y}_k$,

$$\begin{aligned}
\overline{g}_k|_{[y]_k} &= \sup_{z \in [y]_k} g(z), \\
\underline{g}_k|_{[y]_k} &= \inf_{z \in [y]_k} g(z).
\end{aligned}$$

Clearly, by definition, $\underline{g}_k \leq g \leq \overline{g}_k$ and both $\underline{g}_k, \overline{g}_k$ satisfy the following two conditions for $f = \underline{g}_k, \overline{g}_k$:

(a) $f > \delta$ for some $\delta > 0$, and

(b) there exists $\gamma : \{1, 2, \ldots\} \to \mathbb{R}_+$ converging to 0 such that

$$\max \left\{ \sup_{x \in [y]_m} f(x) - \inf_{x \in [y]_m} f(x) : y \in \Sigma \right\} < \gamma(m)$$

for any positive integer m.

We can replace the condition of Lipchitz roof function by the condition of roof function satisfying these two conditions and Proposition 3.3.2 remain true, indeed the same proof works. Finally,

$$\begin{aligned}
&\int |\overline{g}_k - \underline{g}_k| d\mu \\
&\leq \sum_{y \in \mathcal{Y}_k} \left(\sup_{z \in [y]_k} g(z) - \inf_{z \in [y]_k} g(z) \right) \mu([y]_k) \\
&\leq (C - \epsilon) \sum_{y \in \mathcal{Y}_k} \mu([y]_k) \leq (C - \epsilon)\eta'(k) \to_{k \to \infty} 0.
\end{aligned}$$

□

Clearly, we can replace in all the statements in Chapter 3 the assumption of Lipchitz roof function by adapted to μ, and the results are still valid.

3.4.2 Borel-Cantelli lemma for special flows

Motivated by [27] and following the ideas in the previous chapter, we plan to extend in a natural way to suspension flows the Borel-Cantelli lemma. Let $T : \mathcal{X} \to \mathcal{X}$ be a transformation preserving a probability measure μ. Suppose that $\{\mathcal{A}_n\}$ is sequence of subsets of $\mathcal{X}$. Define $\mathcal{V}_n = T^{-n}\mathcal{A}_n$ and

$$\limsup_{n \to \infty} \mathcal{V}_n = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \mathcal{V}_n,$$

the set of points that belong to infinitely many $\mathcal{V}_n$'s. The Borel-Cantelli lemma says that:

1. If $\sum \mu(\mathcal{V}_n) < \infty$, then $\mu\left(\limsup_{n\to\infty} \mathcal{V}_n\right) = 0$.
2. If $\sum \mu(\mathcal{V}_n) = \infty$ and $\mathcal{V}_n$ are independent, then $\mu\left(\limsup_{n\to\infty} \mathcal{V}_n\right) = 1$.

Define for $n \in \mathbb{N}$ $F_n(x) := \sum_{k=1}^{n} \mathbb{1}_{\mathcal{V}_k}(x)$ and $E_n := \sum_{k=1}^{n} \mu(\mathcal{A}_k)$.

The strong Borell-Cantelli lemma says that if $\sum \mu(\mathcal{V}_n) = \infty$ and $\mathcal{V}_n$ are independent, then $\frac{F_n(x)}{E_n} \to 1$ μ-a.e. as $n \to \infty$.

Definition 3.4.3 (SP condition)**.** *We say that the sequence $\{\mathcal{A}_n\}$ satisfies SP if there exists a constant C such that*

$$\sum_{i,j=m}^{n} (\mu(\mathcal{V}_j \cap \mathcal{V}_i) - \mu(\mathcal{V}_j)\mu(\mathcal{V}_i)) \leq C \sum_{i=m}^{n} \mu(\mathcal{V}_i), \text{ for all } m, n \in \mathbb{N}, n \geq m.$$

When $\mathcal{V}_n$ are not independent we have the following theorem.

Theorem 3.4.4 (Theorem 1.4 in [27], Walter Philipp's Theorem)**.** *If $\{\mathcal{A}_n\}$ satisfies SP, then it satisfies the strong Borell-Cantelli lemma and for $\epsilon > 0$*

$$F_n = E_n + \mathcal{O}(E_n^{1/2} \log^{3/2+\epsilon} E_n) \ \mu\text{-a.e.}$$

Theorem 3.4.5 (Essentially Theorem 2 in [70])**.** *Let $T : [0,1] \to [0,1]$ and μ a mixing probability measure on $[0,1]$. Let $\{\mathcal{A}_n\}$ be an arbitrary sequence of intervals in $[0,1]$. Then for $\epsilon > 0$*

$$F_n = E_n + \mathcal{O}(E_n^{1/2} \log^{3/2+\epsilon} E_n) \ \mu\text{-a.e.}$$

The extension to suspension flows of this result needs a setting. Suppose that $([0,1], \mathcal{B}_{[0,1]}, \mu, T)$ is a topologically mixing and measure preserving dynamical system, and consider the suspension flow Φ_g^t by a roof function $g : [0,1] \to \mathbb{R}^{>0}$ such that $g([0,1]) \supset [0,1]$. Consider the special smooth flow

$$\left(\Phi_g^t, \mathcal{X}_g := \{(x,y) : x \in [0,1], y \in [0, g(x))\}\right)$$

has an invariant probability measure ν. Define a family of sets $\mathcal{V}_t := \mathcal{I}_t \times [0,1]$ indexed by $t > 0$, where $\mathcal{I}_t \subset [0,1]$ are intervals. An immediate consequence of Theorem 3.4.5 is the following result.

Corollary 3.4.6. *If $\int_T \mathcal{V}_t d\nu \to \infty$ as $T \to \infty$, then the set $\left\{t > 0 : \Phi_\tau^t(\overline{x}) \in \mathcal{V}_t\right\} \subset \mathbb{R}^{>0}$ is unbounded for ν-a.e. $\overline{x} \in \mathcal{X}_g$.*

3.4.3 Alternative proofs with an application of its method

The main idea in this subsection is to use a result about the sharp concentration of the ergodic average around its space average and the Birkhoff ergodic theorem.

Doing this we can find a lower bound for the escape rate avoiding the discretisation of the special flows used in the proof of Proposition 3.3.2. Nevertheless, the result obtained in this way is weaker. The theorem that we require is the following:

Theorem 3.4.7 (Corollary 3.3 in [25])**.** *Let $f : \mathcal{X} \to \mathbb{R}$ be θ-Lipschitz and μ be the equilibrium state of a Hölder potential φ. Then*

$$\mu\left\{x : \left|\frac{1}{m}S_m^\sigma f(x) - \int f d\mu\right| \geq t\right\} \leq 2e^{-Bmt^2}$$

for every $t > 0$ and for every $m \in \mathbb{N}$, where $B := (4D|f|_\theta^2)^{-1}$ and $D = D(\varphi)$ is a constant independent of f.

Let us introduce a definition to state our result.

Definition 3.4.8. *Given a family of open sets $\{\mathcal{U}_n\}, \mathcal{U}_n \subset \mathcal{X}$, define for each $n \in \mathbb{N}$,*

$$\tilde{R}(\mu, \mathcal{U}_n, \mathcal{X}) := -\limsup_{t\to\infty} \frac{1}{t} \log \mu\{x : S_{\tau_n(x)}^\sigma f(x) \geq t\},$$

where recall that $\tau_n : \mathcal{X} \to \mathbb{N}$ is defined by $\tau_n(x) := \inf\{m \in \mathbb{N} : \sigma^m(x) \in \mathcal{U}_n\}$.

As a consequence of Theorem 3.4.7 we obtain the next proposition which provides a lower bound for the escape rate. Compare this results with Proposition 3.3.2.

Proposition 3.4.9. *Suppose that $\{\mathcal{U}_n\}, \mathcal{U}_n \subset \mathcal{X}$ satisfy the nested condition. Let $\varphi : \mathcal{X} \to \mathbb{R}$ be a Hölder continuous function, let μ denote the equilibrium state of φ and let $f : \mathcal{X} \to \mathbb{R}^{>0}$ be θ-Lipschitz; then*

$$\lim_{n\to\infty} \frac{\tilde{R}(\mu, \mathcal{U}_n, \mathcal{X})}{\mu(\mathcal{U}_n)} \geq \frac{\gamma_\varphi(z)}{\int f d\mu + \|f\|}.$$

Proof. Fix $0 < \epsilon < 1/\|f\|$ and define B as in Theorem 3.4.7.

We have that

$$\frac{-B\epsilon^3}{\mu(\mathcal{U}_n)} \to -\infty \text{ as } n \text{ tends to infinity}$$

and

$$-\frac{R(\mu, \mathcal{U}_n, \mathcal{X})}{\mu(\mathcal{U}_n)} \to -\gamma_\varphi(z) \text{ as } n \text{ tends to infinity.}$$

Therefore, there exists $n_0 \in \mathbb{N}$ and $\mathcal{N} \subset \mathbb{N}$ an infinite set such that for any

$n \in \mathbb{Z}, n > n_0$ and for any $k \in \mathcal{N}$

$$0 > \frac{\log \mu \{x : \tau_n(x) \geq k\}}{\mu(\mathcal{U}_n) k \int f d\mu} > \frac{-B\epsilon^3}{\mu(\mathcal{U}_n)},$$

which implies that

$$\mu \left\{ x : \tau_n(x) \left(\int f d\mu + \epsilon \right) \geq k \right\} > e^{-B\epsilon^3 k}. \tag{3.22}$$

We write τ_n instead of $\tau_n(x)$, $S^\sigma_{\tau_n} f$ instead of $S^\sigma_{\tau_n} f(x)$ and $S^\sigma_s f$ instead of $S^\sigma_s f(x)$ when $s \geq 0$. For any $n \in \mathbb{Z}, n > n_0$ and $[\epsilon t] \in \mathcal{N}$, using inequality (3.22) and the identity

$$\begin{aligned} \mu \left\{ x : S^\sigma_{\tau_n} f \geq t \right\} = \mu & \left\{ x : S^\sigma_{\tau_n} f \geq t, \tau_n > \epsilon t, \left| \frac{1}{[\epsilon t]} S^\sigma_{[\epsilon t]} f - \int f d\mu \right| < \epsilon \right\} \\ & + \mu \left\{ x : S^\sigma_{\tau_n} f \geq t, \tau_n > \epsilon t, \left| \frac{1}{[\epsilon t]} S^\sigma_{[\epsilon t]} f - \int f d\mu \right| \geq \epsilon \right\} \end{aligned} \tag{3.23}$$

we conclude the inequality

$$\mu \left\{ x : S^\sigma_{\tau_n} f \geq t \right\} \leq \mu \left\{ x : \tau_n \left(\epsilon + \int f d\mu + \|f\| \right) \geq t \right\} + 2e^{-B[et]\epsilon^2}. \tag{3.24}$$

Using (3.22) in the inequality above we obtain for $\epsilon, t > 0$ and $n \in \mathbb{N}$:

$$\mu \left\{ x : S^\sigma_{\tau_n} f \geq t \right\} \leq \left(1 + 2e^{B\epsilon^3} \right) \mu \left\{ x : \tau_n \cdot \left(\epsilon + \int f d\mu + \|f\| \right) \geq t \right\}. \tag{3.25}$$

Applying logarithms to both sides in (3.25), dividing on both sides by $t > 0$, then taking $-\limsup_{t \to \infty}$, and finally dividing both sides by $\mu(\mathcal{U}_n)$ and letting n tend to infinity, we conclude that

$$\lim_{n \to \infty} \frac{\tilde{R}(\mu, \mathcal{U}_n, \mathcal{X})}{\mu(\mathcal{U}_n)} \geq \frac{\gamma_\varphi(z) + \epsilon \|f\|}{\epsilon + \int f d\mu + \|f\|}. \tag{3.26}$$

Because $\epsilon > 0$ is arbitrary, we conclude the result. $\square$

We now complete the proof of some identities and inequalities used in the proof of Proposition 3.4.9.

Proof of (3.23). We prove the statement:

$$\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t\right\} = \mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| < \epsilon\right\}$$
$$+ \mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| \geq \epsilon\right\}.$$

In fact

$$\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t\right\} = \mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n \leq \epsilon t\right\} + \mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t\right\},$$

but

$$\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n \leq \epsilon t\right\} \leq \mu\left\{x : \epsilon t \|f\| \geq t\right\} = 0$$

because $0 < \epsilon < 1/\|f\|$. □

Proof of (3.24). It is enough to prove the following inequality

$$\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| < \epsilon\right\}$$
$$+ \mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| \geq \epsilon\right\}$$
$$\leq \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu\right) \geq t - \sum_{k=[\epsilon t]}^{\tau_n - 1} f \circ \sigma^k(x)\right\} + 2e^{-B[et]\epsilon^2}.$$

It involves two inequalities:

(i) the first is

$$\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| < \epsilon\right\}$$
$$\leq \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu\right) \geq t - \sum_{k=[\epsilon t]}^{\tau_n - 1} f \circ \sigma^k(x)\right\}$$

that comes from

$$\begin{aligned}
&\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| < \epsilon\right\} \\
&\leq \mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f \leq \int f d\mu + \epsilon\right\} \\
&\leq \mu\left\{x : \frac{t - \sum_{k=[\epsilon t]}^{\tau_n} f \circ \sigma^k(x)}{\tau_n} \leq \frac{S^{\sigma}_{\tau_n} f - \sum_{k=[\epsilon t]}^{\tau_n} f \circ \sigma^k(x)}{\tau_n} \leq \int f d\mu + \epsilon\right\} \\
&\leq \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu\right) \geq t - \sum_{k=[\epsilon t]}^{\tau_n - 1} f \circ \sigma^k(x)\right\},
\end{aligned}$$

(ii) the second is

$$\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| \geq \epsilon\right\} \leq 2e^{-B[et]\epsilon^2}$$

that comes from

$$\begin{aligned}
&\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t, \tau_n > \epsilon t, \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| \geq \epsilon\right\} \\
&\leq \mu\left\{x : \left|\frac{1}{[\epsilon t]} S^{\sigma}_{[\epsilon t]} f - \int f d\mu\right| \geq \epsilon\right\} \leq 2e^{-B[et]\epsilon^2}.
\end{aligned}$$

We can apply Theorem 3.4.7 and this concludes the proof. □

Proof of (3.25). We have the following inequalities:

$$\begin{aligned}
&\mu\left\{x : S^{\sigma}_{\tau_n} f \geq t\right\} \\
&\leq \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu\right) \geq t - \sum_{k=[\epsilon t]}^{\tau_n - 1} f \circ \sigma^k(x)\right\} + 2e^{-B[et]\epsilon^2} \\
&\leq \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu\right) \geq t - (\tau_n - [\epsilon t]) \|f\|\right\} + 2e^{-B[et]\epsilon^2} \\
&\leq \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu + \|f\|\right) \geq t(1 + \epsilon \|f\|)\right\} + 2e^{-B[et]\epsilon^2} \\
&\leq \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu + \|f\|\right) \geq t\right\} + 2e^{B\epsilon^3} \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu\right) \geq t\right\} \\
&\leq \left(1 + 2e^{B\epsilon^3}\right) \mu\left\{x : \tau_n\left(\epsilon + \int f d\mu + \|f\|\right) \geq t\right\}.
\end{aligned}$$

□

Proof of (3.26). Inequality (3.25) implies

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \log \mu \left\{x : S^{\sigma}_{\tau_n} f \geq t\right\} \\
&\leq \limsup_{t\to\infty} \frac{1}{t} \log \mu \left\{x : \tau_n \left(\epsilon + \int f d\mu + \|f\|\right) \geq t\right\}.
\end{aligned}$$

Finally, we can write an inequality that does not depend on $\epsilon > 0$ that concludes the result:

$$\begin{aligned}
&\lim_{n\to\infty} -\frac{1}{\mu(\mathcal{U}_n)} \limsup_{t\to\infty} \frac{1}{t} \log \mu \left\{x : S^{\sigma}_{\tau_n} f \geq t\right\} \\
&\geq \lim_{n\to\infty} -\frac{1}{\mu(\mathcal{U}_n)} \limsup_{t\to\infty} \frac{1}{t} \log \mu \left\{x : \tau_n \left(\epsilon + \int f d\mu + \|f\|\right) \geq t\right\} \\
&= \frac{\gamma(z)}{\epsilon + \int f d\mu + \|f\|}.
\end{aligned}$$

□

Application to large deviations. Now we apply Theorem 3.4.7 to obtain a large deviation result for smooth semi-flows. Recall that large deviations estimate the asymptotic measure of the bad points for the Birkhoff ergodic theorem, i.e., given a continuous observable $\varphi : X \to \mathbb{R}$, estimates the function

$$\mathbb{R}^+ \ni \epsilon \mapsto \mu \left\{x \in X : \left| \frac{1}{T} \int_0^T \varphi \circ \Phi^s(x) ds - \int \varphi d\mu \right| > \epsilon \right\}$$

as T goes to infinity. Results about large deviations for discrete dynamical systems can be found in [90, 25, 55] and references there. Our proposition about large deviations for special smooth semi-flows over non invertible subshifts of finite is the following:

Proposition 3.4.10. *Let $f : \mathcal{X}^+ \to \mathbb{R}^{>1}$ be a θ-Lipschitz function, μ an equilibrium state of Hölder potential, (Λ_f, Φ^t_f) a special smooth semi-flow over $\mathcal{X}^+$ and $F : \Lambda_f \to \mathbb{R}$ a map for which there is a constant $C > 0$ such that for every $x \in \mathcal{X}^+$ and for every $m \in \mathbb{N}$*

$$\sup_{y\in[x]_m} \int_0^{\min(f(x),f(y))} |F(x,s) - F(y,s)| ds \leq C\theta^m. \tag{3.27}$$

Then there exist constants $C_1, C_2 > 0$ depending on f and F such that for all $\epsilon > 0$, *for all* $t \in \mathbb{R}^{>\max\left(\frac{\|f\|\|F\|(1+\|f\|)}{\epsilon}, 2\|f\|\right)}$,

$$
\begin{aligned}
&\mu\left\{x \in \mathcal{X}^+ : \left|\frac{1}{t}\int_0^t F \circ \Phi_f^s(x,0)ds - \int F d\mu^f\right| \geq \epsilon\right\}\\
&\leq 2t\|f\|\exp\left\{-C_1\left(\frac{t}{\|f\|}-2\right)\left(\epsilon - \frac{\|f\|\,\|F\|}{t}(1+\|f\|)\right)^2\right\}\\
&\quad + 2t\|f\|\exp\left\{-C_2\left(\frac{t}{\|f\|}-2\right)\left(\epsilon - \frac{\|f\|\,\|F\|}{t}(1+\|f\|)\right)^2 / (\|f\|\,\|F\|)^2\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\|f\| &:= \sup_{x\in\mathcal{X}^+} |f(x)|,\\
\|F\| &:= \sup_{x\in\mathcal{X}^+}\sup_{s\in[0,f(x))}\{|F(x,s)|\} \text{ and}\\
d\mu^f &:= \frac{d\mu \times d\mu_{Leb}}{\int f d\mu}.
\end{aligned}
$$

Notice that a Hölder map $F : \Lambda_f \to \mathbb{R}$ satisfies (3.27), and that under the same hypotheses we have in particular the following well known result.

Corollary 3.4.11. *For every* $\epsilon > 0$ *we have that*

$$
\limsup_{t\to\infty}\frac{\log\mu\left\{x\in\mathcal{X}^+ : \left|\frac{1}{t}\int_0^t F\circ\Phi_f^s(x,0)ds - \int F d\mu^f\right| \geq \epsilon\right\}}{t} < 0.
$$

The proof of Proposition 3.4.10 uses standard arguments, see [65], Section 5, in particular the arguments in proofs of Theorem 5.1 and 5.3.

Proposition 3.4.10. The proof is a consequence of Theorem 3.4.7 and some inequalities.

Suppose $t > \|f\|$ and define $\tilde{F} : \mathcal{X}^+ \to \mathbb{R}, x \mapsto \int_0^{f(x)} F(x,s)ds$. Given $x \in \mathcal{X}^+$ we can write $t = S^\sigma_{n(x)}f(x) + t(x)$ for some $n(x) \in \mathbb{N}$ and $f(\sigma^n x) > t(x) \geq 0$, then $n(x) \leq t = S^\sigma_{n(x)}f(x) + t(x) \leq (n(x)+1)\|f\|$. In particular, $t \geq n(x) \geq \frac{t}{\|f\|} - 1$. Keeping this in mind we have the following inequalities:

$$
\begin{aligned}
&\mu\left\{x\in\mathcal{X}^+ : \left|\frac{1}{t}\int_0^t F\circ\Phi_f^s(x,0)ds - \int Fd\mu^f\right|\geq\epsilon\right\}\\
&=\mu\left\{x\in\mathcal{X}^+ : \left|\frac{S_{n(x)}^\sigma\tilde{F}(x)+\int_0^{t(x)}F(\sigma^{n(x)}x,s)ds}{S_{n(x)}^\sigma f(x)+t(x)}-\frac{\int\tilde{F}d\mu}{\int fd\mu}\right|\geq\epsilon\right\}\\
&\leq\mu\left\{x\in\mathcal{X}^+ : \left|\frac{S_{n(x)}^\sigma\tilde{F}(x)}{S_{n(x)}^\sigma f(x)}\frac{S_{n(x)}^\sigma f(x)}{S_{n(x)}^\sigma f(x)+t(x)}-\frac{\int\tilde{F}d\mu}{\int fd\mu}+\frac{\int_0^{t(x)}F(\sigma^{n(x)}x,s)ds}{t}\right|\geq\epsilon\right\}\\
&\leq\mu\left\{x\in\mathcal{X}^+ : \left|\frac{S_{n(x)}^\sigma\tilde{F}(x)}{S_{n(x)}^\sigma f(x)}-\frac{t(x)}{t}\frac{S_{n(x)}^\sigma\tilde{F}(x)}{S_{n(x)}^\sigma f(x)}-\frac{\int\tilde{F}d\mu}{\int fd\mu}\right|+\frac{\|f\|\,\|F\|}{t}\geq\epsilon\right\}\\
&\leq\mu\left\{x\in\mathcal{X}^+ : \left|\frac{S_{n(x)}^\sigma\tilde{F}(x)}{S_{n(x)}^\sigma f(x)}-\frac{\int\tilde{F}d\mu}{\int fd\mu}\right|+\frac{\|f\|^2\,\|F\|}{t}+\frac{\|f\|\,\|F\|}{t}\geq\epsilon\right\}=:(\star),
\end{aligned}
$$

where

$$
\epsilon_1:=\epsilon-\frac{\|f\|\,\|F\|}{t}(1+\|f\|).
$$

Furthermore,

$$
\begin{aligned}
&(\star)\\
&=\mu\left\{x\in\mathcal{X}^+ : \left|\frac{S_{n(x)}^\sigma\tilde{F}(x)}{n(x)}\frac{n(x)}{S_{n(x)}^\sigma f(x)}-\int\tilde{F}d\mu\frac{n(x)}{S_{n(x)}^\sigma f(x)}+\int\tilde{F}d\mu\frac{n(x)}{S_{n(x)}^\sigma f(x)}-\frac{\int\tilde{F}d\mu}{\int fd\mu}\right|\geq\epsilon_1\right\}\\
&\leq\mu\left\{x\in\mathcal{X}^+ : \frac{n(x)}{S_{n(x)}^\sigma f(x)}\left|\frac{S_{n(x)}^\sigma\tilde{F}(x)}{n(x)}-\int\tilde{F}d\mu\right|+\left|\int\tilde{F}d\mu\right|\left|\frac{n(x)}{S_{n(x)}^\sigma f(x)}-\frac{1}{\int fd\mu}\right|\geq\epsilon_1\right\}\\
&\leq\mu\left\{x\in\mathcal{X}^+ : \left|\frac{S_{n(x)}^\sigma\tilde{F}(x)}{n(x)}-\int\tilde{F}d\mu\right|\geq\frac{\epsilon_1}{2}\right\}\\
&\quad+\mu\left\{x\in\mathcal{X}^+ : \left|\int\tilde{F}d\mu\right|\left|\frac{S_{n(x)}^\sigma f(x)}{n(x)}-\int fd\mu\right|\geq\frac{\epsilon_1}{2}\right\}=:(\star\star),
\end{aligned}
$$

where

$$
\epsilon_2:=\frac{\epsilon_1}{2},\epsilon_3:=\frac{\epsilon_1}{2\,\|f\|\,\|F\|}\text{ and }n_1(t):=\left\lfloor\frac{T}{\|f\|}-1\right\rfloor.
$$

Finally,

$$
\begin{aligned}
(\star\star) \le & \sum_{n\in\{n_1(t),n_1(t)+1,\ldots,[t]\}} \mu\left\{x\in\mathcal{X}^+ : \left|\frac{S_n^\sigma \tilde{F}(x)}{n} - \int \tilde{F} d\mu\right| \ge \epsilon_2\right\} \\
& + \sum_{n\in\{n_1(t),n_1(t)+1,\ldots,[t]\}} \mu\left\{x\in\mathcal{X}^+ : \left|\frac{S_n^\sigma f(x)}{n} - \int f d\mu\right| \ge \epsilon_3\right\} \\
& \le 2t\left\|f\right\| \exp\left(-\frac{\tilde{C}_1}{|\tilde{F}|_\theta^2}\cdot n_1(t)\cdot\epsilon_2^2\right) + 2t\left\|f\right\|\exp\left(-\frac{\tilde{C}_2}{|f|_\theta^2}\cdot n_1(t)\cdot \epsilon_3^2\right).
\end{aligned}
$$

The map $\tilde{F}$ is θ-Lipschitz, indeed, suppose $x, y \in [z]_m$ for some $z \in \mathcal{X}^+, m \in \mathbb{N}$, and $f(x) > f(y)$ then

$$
\begin{aligned}
|\tilde{F}(x) - \tilde{F}(y)| &\le \int_{f(y)}^{f(y)+|f|_\theta\theta^m} \|F\|\, ds + \int_0^{f(y)} |F(x,s) - F(y,s)| ds \\
&\le (|f|_\theta \|F\| + C)\theta^m.
\end{aligned}
$$

Thus, we can write

$$
\begin{aligned}
&\mu\left\{x\in\mathcal{X}^+ : \left|\frac{1}{t}\int_0^t F\circ\Phi_f^s(x,0)ds - \int F d\mu^f\right| \ge \epsilon\right\} \\
&\le 2t\left\|f\right\|\exp\left(-\frac{1/(4D)}{|\tilde{F}|_\theta^2}\cdot\left\lfloor\frac{t}{\|f\|}-1\right\rfloor\cdot\left(\frac{\epsilon - \frac{\|f\|\|F\|}{t}(1+\|f\|)}{2}\right)^2\right) \\
&\quad + 2t\left\|f\right\|\exp\left(-\frac{1/(4D)}{|f|_\theta^2}\cdot\left\lfloor\frac{t}{\|f\|}-1\right\rfloor\cdot\left(\frac{\epsilon - \frac{\|f\|\|F\|}{t}(1+\|f\|)}{2\left\|f\right\|\left\|F\right\|}\right)^2\right),
\end{aligned}
$$

where D is the constant that depends on μ in [25], Theorem 3.1. □

Using Proposition 3.4.10 we can obtain a similar proposition for conformal repellers. We set some notation first. Given a conformal repeller $(\mathcal{J}, f)$ with Markov partition $\mathcal{R}$, for $w \in \mathcal{J}$ and $m \in \mathbb{N}_0$, we define $[w]_m$ to be the element $\mathcal{W} \in \bigvee_{i=0}^{m-1} f^{-i}\mathcal{R}$ such that $w \in \mathcal{W}$.

Proposition 3.4.12 (Corollary of Theorem 3.4.10)**.** *Let $(\mathcal{J}, f)$ be a conformal repeller, $F : \mathcal{J} \to \mathbb{R}^{>1}$ be a θ-Lipschitz function, μ an equilibrium state for a Hölder potential and $G : \Lambda_F \to \mathbb{R}$ a map for which there is a constant C such that for every $z \in \mathcal{J}$ and for every $m \in \mathbb{N}$*

$$
\sup_{y\in[z]_m} \int_0^{\min(F(y),F(z))} |G(z,s) - G(y,s)| ds \le C\theta^m.
$$

Then there exist constants $C_1, C_2 > 0$ depending on F and G such that for all $\epsilon > 0$, for all $t \in \mathbb{R}^{>\max\left(\frac{\|F\|\|G\|(1+\|F\|)}{\epsilon}, 2\|F\|\right)}$,

$$\begin{aligned}
&\mu\left\{z \in \mathcal{J} : \left|\frac{1}{t}\int_0^t G \circ \Phi_F^s(z,0)ds - \int G d\mu^F\right| \geq \epsilon\right\} \\
&\leq 2t\,\|F\| \exp\left\{-C_1\left(\frac{t}{\|f\|} - 2\right)\left(\epsilon - \frac{\|F\|\,\|G\|}{t}(1+\|F\|)\right)^2\right\} \\
&+ 2t\,\|F\| \exp\left\{-C_2\left(\frac{t}{\|F\|} - 2\right)\left(\epsilon - \frac{\|F\|\,\|G\|}{t}(1+\|F\|)\right)^2 / (\|F\|\,\|G\|)^2\right\},
\end{aligned}$$

where $\|F\| := \sup_{z\in\mathcal{J}}|F(z)|$, $\|G\| := \sup_{z\in\mathcal{J}}\sup_{s\in[0,F(z))}|G(z,s)|$ and $d\mu^F := \frac{d\mu\times d\mu_{Leb}}{\int F d\mu}$.

3.4.4 Axiom A diffeomorphisms

In this subsection we state and prove an escape rate result for Axiom A diffeomorphisms. We will require the results and definitions in Subsection 1.2.5. Let $f : \mathcal{M} \to \mathcal{M}$ be an Axiom A diffeomorphism where $\mathcal{M}$ is a compact $\mathcal{C}^\infty$ Riemannian manifold. Let Ω_s be a basic set, $\varphi : \Omega_s \to \mathbb{R}$ be a Hölder continuous function and $\mu = \mu_\varphi$ be the unique equilibrium state of φ. Suppose in addition that $f|_{\Omega_s}$ is topologically mixing, $\mathcal{R}$ is a Markov partition of Ω_s and $\pi : \mathcal{X} \to \Omega_s$ is the conjugation in Theorem 1.2.31. We can now state an escape rate result for Axiom A diffeomorphisms.

Corollary 3.4.13. *For any $z \in \Omega_s \setminus \cup_{j\in\mathbb{Z}} f^j(\partial^s\mathcal{R} \cup \partial^u\mathcal{R})$ such that $x = \pi^{-1}z \in \mathcal{X}$ satisfies Birkhoff ergodic theorem, we have that*

$$\frac{R(\mu, \pi([x]_{-n}^n), \Omega_s)}{\mu(\pi([x]_{-n}^n))}$$

accumulates in $\{1\} \cup \{1 - e^{S_p^f\varphi(z) - pP(\varphi)} : p \in \mathbb{N}\}$ as n tends to infinity.

Proof. Suppose that μ is the equilibrium state of φ. Let $\varphi^* = \varphi \circ \pi$ and let μ^* be the equilibrium state of φ^*. For brevity, call $x = \pi^{-1}z \in \mathcal{X}$. Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}, k > n$, then

$$\begin{aligned}
&\mu\left\{z' \in \Omega_s : f^i(z') \notin \pi\left([x]_{-n}^n\right), \forall i \in \{0, \dots, k-1\}\right\} \\
&\quad = \mu^*\left\{x' \in \mathcal{X} : \sigma^i(x') \notin [x]_{-n}^n, \forall i \in \{0, \dots, k-1\}\right\}
\end{aligned}$$

because π is 1-1 μ-a.e. We also have

$$\mu(\pi([x]_{-n}^n))) = \mu^*([x]_{-n}^n),$$

because for any μ^*-measurable set $\mathcal{S}$, $\mu^*(\mathcal{S}) = \mu(\pi^{-1}\mathcal{S})$.
Considering $\tilde{\mu}^* := \mu^*|_{\mathcal{X}^+}$, we have that

$$\begin{aligned}
&\mu^* \left\{x' \in \mathcal{X} : \sigma^i(x) \notin [x]_{-n}^n, \forall i \in \{0, \ldots, k-1\}\right\} \\
&\quad = \tilde{\mu}^* \left\{x' \in \mathcal{X}^+ : \sigma^i(x') \notin [\sigma^{-n}x]_0^{2n}, \forall i \in \{0, \ldots, k-n-1\}\right\}
\end{aligned}$$

and

$$\mu^*([x]_{-n}^n) = \tilde{\mu}^*([\sigma^{-n}x]_0^{2n}).$$

Birkhoff ergodic theorem implies that for every $p \in \mathbb{N}$ there exists a subsequence $(n_q^p)_{q\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} (\sigma^{-n_k^p}x) = x^p,$$

where $x^p \in \mathcal{X}^+$ is periodic with period p. Also, we can find a subsequence $(n_q^\infty)_{q\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} (\sigma^{-n_k^\infty}x) = x^\infty,$$

where $x^\infty \in \mathcal{X}^+$ is non periodic.

For every $p \in \mathbb{N} \cup \{\infty\}$ we can apply Theorem 3.2.3 to conclude that

$$\lim_{k\to\infty} \frac{\limsup_{m\to\infty} \frac{-1}{m} \quad \log \tilde{\mu}^* \left\{x' \in \mathcal{X}^+ : \sigma^i(x') \notin [\sigma^{-n_k^p}x]_0^{2n_k^p}, \forall i \in \{0, \ldots, m-n_k^p-1\}\right\}}{\tilde{\mu}^*\left([\sigma^{-n_k^p}x]_0^{2n_k^p}\right)}$$
$$= \gamma_\varphi(x^p),$$

which completes the proof. $\square$

3.4.5 Axiom A flows

In this subsection we present an escape rate result for Axiom A flows. In order to state it, we require the definitions and results in Subsection 1.2.6. Assume that Λ is a basic hyperbolic set for an Axiom A flow Φ, $\varphi : \Lambda \to \mathbb{R}$ is a Hölder function and μ is the unique equilibrium state of φ with support in Λ. Let ρ and $(\tilde{\Phi}_f, \tilde{\Lambda}_f, \tilde{\mu})$ be as in Proposition 1.2.35 and suppose that $\tilde{\Lambda}_f$ is a special flow over a subshift of finite type $\mathcal{X}$. In particular, we have that $\mu = \rho^*\tilde{\mu}$, i.e. $\mu(\mathcal{S}) = \tilde{\mu}(\rho^{-1}\mathcal{S})$ for all μ-measurable set $\mathcal{S} \subset \Lambda$.

Corollary 3.4.14. *Suppose that $P(\Phi|_\Lambda, \varphi) = 0$ and that the roof function is strictly*

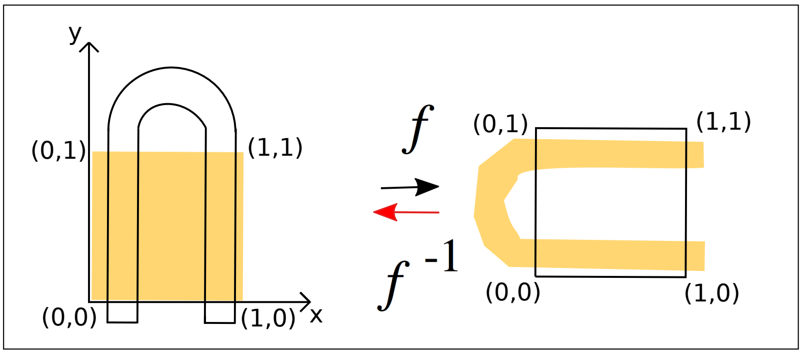


Figure 3.2: Horseshoe map.

bigger than 1. *Let* $x \in \mathcal{X}$ *be a point satisfying the Birkhoff ergodic theorem; then*

$$\lim_{\delta \searrow 0} \lim_{n \to \infty} \frac{R(\mu, \rho([x]_{-n}^{n} \times [0,\delta]), \Lambda)}{\mu(\rho([x]_{-n}^{n} \times [0,1]))}$$

accumulates in $\{1\} \cup \{1 - e^{\int_\tau \varphi(\rho x, t) dt} : \tau$ *is a periodic orbit*$\}$ *as* n *tends to infinity.*

Proof. We have that for all $n \in \mathbb{N}, t > 0$ and $\delta > 0$ small enough

$$\begin{aligned}&\mu\left\{z \in \Lambda : \Phi^s z \notin \rho\left([x]_{-n}^{n} \times [0,\delta]\right), \forall 0 < s \leq t\right\}\\ &= \tilde{\mu}\left\{(x', s') \in \tilde{\Lambda} : \tilde{\Phi}^t(x', s') \notin [x]_{-n}^{n} \times [0,\delta], \forall 0 < s \leq t\right\},\end{aligned}$$

and

$$\mu\left(\rho\left([x]_{-n}^{n} \times [0,1]\right)\right) = \tilde{\mu}\left([x]_{-n}^{n} \times [0,1]\right).$$

Using same proof of Proposition 3.3.2 and the last part of the proof of Corollary 3.4.13, we conclude the result. □

We can consider consider a concrete application of this corollary to the example of Axiom A flow in Subsection 1.2.6. Indeed, we can consider the Axiom A flow (Λ, Φ^t) that coincides with the special flow (Λ_g, Φ_g^t) over the horseshoe map $(f, \Omega), \Omega \subset [0,1]^2$ (showed in Figure 3.2) and roof function

$$g(x,y) = \begin{cases} 2 \text{ if } y < 1/2 \\ 1 \text{ if } y > 1/2. \end{cases}$$

The horseshoe map (f, Ω) is conjugated by a map $\pi : \{1,2\}^{\mathbb{Z}} \to \Omega$ with the subshift of finite type $\{1,2\}^{\mathbb{Z}}$. On the other hand, the Axiom A flow (Λ, Φ^t) is conjugated by a map $\rho : \Lambda \to \tilde{\Lambda}$ for $\tilde{\Lambda}$ a special flow over $\{1,2\}^{\mathbb{Z}}$. For some $x_0 \in \{1,2\}^{\mathbb{Z}}$ and $\delta > 0$, the holes $\rho([x_0]_{-1}^{1} \times [0,\delta]) = \pi[x_0]_{-1}^{1} \times [0,\delta]$, $\rho([x_0]_{-2}^{2} \times [0,\delta]) = \pi[x_0]_{-2}^{2} \times [0,\delta]$ and $\rho([x_0]_{-3}^{3} \times [0,\delta]) = \pi[x_0]_{-3}^{3} \times [0,\delta]$ are shown in Figure 3.3.

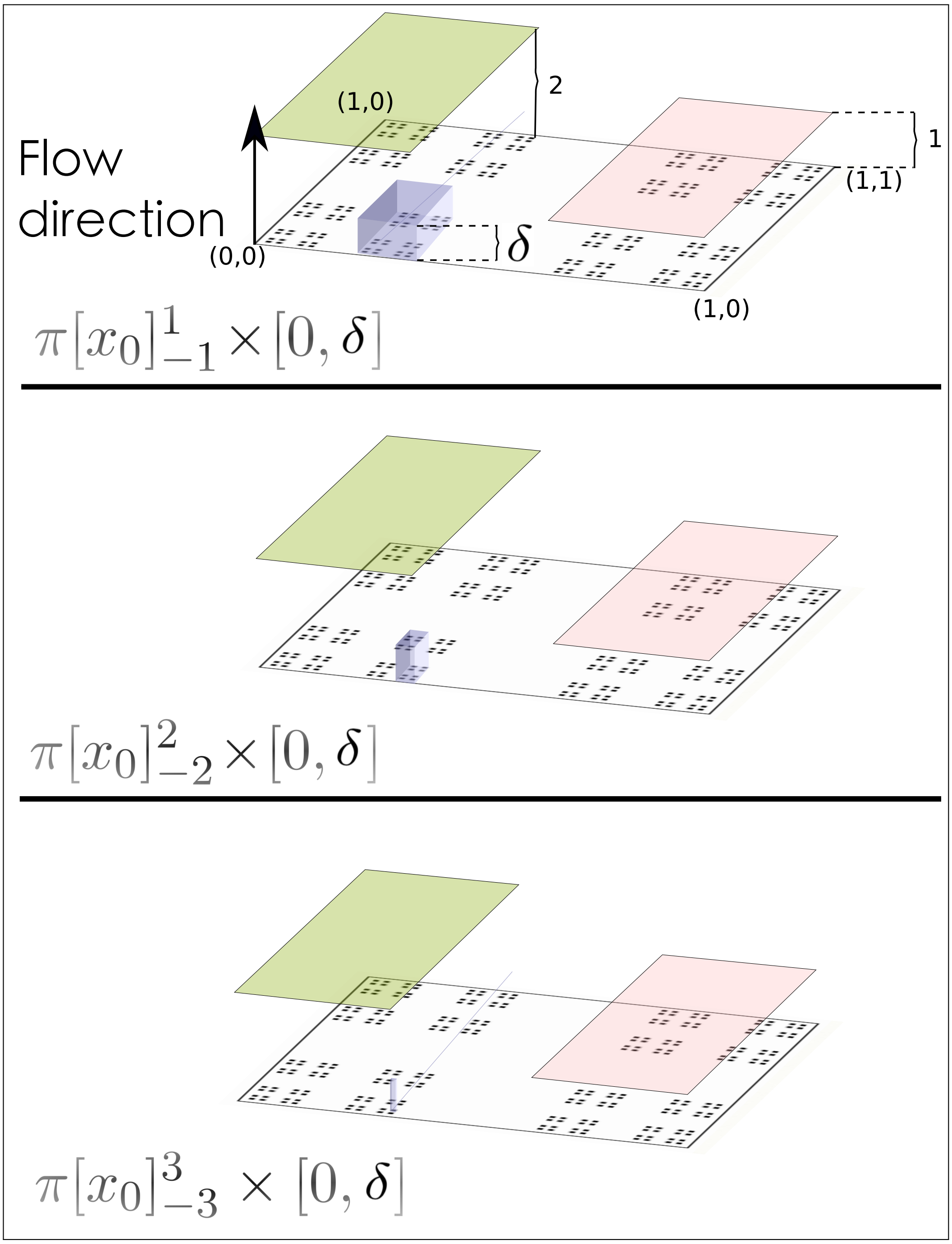


Figure 3.3: Example of a shrinking hole for an Axiom A flow.

Chapter 4

Smoothness of the stationary measures

4.1 Introduction

In this chapter we study how the stationary measure changes under perturbations of the (conformal) iterated function scheme and the weight functions that define it. Stationary measures in this setting are sometimes called self-similar measures when the weight functions are constants, and they have been studied by many authors. Self-similar measures were originally defined in [51], the same paper provides a proof of its existence and uniqueness for constant weight functions. In [83, 84] finer analytical properties of self-similar measures are studied, in [74, 60, 68] it is studied the case with overlaps, in [73] it is explored a relationship with Lyapunov exponents for random matrix products, in [67, 7] the authors study some iterated function schemes that are contracting on average. Stationary measures are believed to be strong extremal measures for irreducible systems of real analytic contractions on $\mathbb{R}^n$ [57]. New ideas in [41] considers the problem of estimation of the Wasserstein distance between stationary measures for a particular case of contracting iterated function schemes on the unit interval, the author obtains an explicit formula for the 1st Wasserstein distance and provide non-trivial upper and lower bounds for the 2nd Wasserstein distance. On the other hand, the most well studied features of iterated function schemes are their fractal properties, like the Hausdorff dimension of its limit set [38]. The Hausdorff dimension of the limit sets of iterated function schemes [38] is a vast topic of research and we omit references, as the results in this chapter only require basic knowledge of this field.

Here we focus on a complete different problem from the ones referenced above: we study the smoothness of the stationary measure and relate it with the smoothness of the perturbation of the iterated function scheme and of the weight functions that define it. Our results use basic facts of iterated function scheme and are closely related to [86]. However, our proof relies on a result of composition of operators in [35] and structural stability, whereas the proof in [86] uses Proposition 2.3 in [86] and [85]. Classical techniques from thermodynamic formalism allows us also to obtain results for the smoothness of the Hausdorff dimension of the limit set. In order to postpone technicalities for the next section, rather than stating our results here, we show two concrete examples of application.

Let us start with a definition for the convergence of measures.

Definition 4.1.1 (Convergence of measures in the weak topology)**.** *A sequence of probability measures $\{\mu_n\}$ on a metric space $\mathcal{X}$ converges in the weak topology to a probability measure μ on $\mathcal{X}$ if and only if for every bounded, continuous function $f : \mathcal{X} \to \mathbb{R}$ the sequence $\{x_n\}$ with $x_n := \int f d\mu_n$ converges to $x := \int f d\mu$.*

Our first example is an application of the results to affine maps.

Example 4.1.2. *Let $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ be the affine maps $T_1(x) = \alpha_1 x + \beta_1$ and $T_2(x) = \alpha_2 x + \beta_2$ with $0 < \alpha_1, \alpha_2 < 1$. Let us consider the weights $p_1, p_2 > 0$ with $p_1 + p_2 = 1$. The unique stationary probability measure $\mu = \mu_{\alpha_1,\alpha_2,\beta_1,\beta_2,p_1,p_2}$ in this case is given by the limit in the weak topology*

$$\mu := \lim_{n\to+\infty} \sum_{i_1,\cdots,i_n\in\{1,2\}} p_{i_1} \cdots p_{i_n} \delta_{T_{i_1}\circ\cdots\circ T_{i_n}(0)}$$

where $\delta_{T_{i_1}\circ\cdots\circ T_{i_n}(0)}$ denotes the Dirac measure supported on $T_{i_1} \cdots T_{i_n}(0)$.

If we further assume for simplicity that $\alpha_1 + \alpha_2 = 1$ and $\beta_1 = 0$, $\beta_2 = \alpha_1$ then the two images $T_1[0,1] = [0,\alpha_1]$, $T_2[0,1] = [\alpha_1, 1]$ partition the unit interval and μ will be supported on the unit interval. Finally, in this case it is simple to see that μ is then the Lebesgue measure if and only if $p_1 = \alpha_1$ and $p_2 = \alpha_2$.

We can consider the dependence of the stationary measure on the parameters α_j, β_j and p_j $(j = 1,2)$ which form a two dimensional space. For any $\mathcal{C}^{2+\delta}$ function $w : [0,1] \to \mathbb{R}$ (with $0 < \delta \le 1$) we then have that the map

$$(0,1) \ni \alpha_1 \mapsto \int w d\mu_{\alpha_1,p_1} \in \mathbb{R},$$

is $\mathcal{C}^1$, and

$$(0,1) \ni p_1 \mapsto \int w d\mu_{\alpha_1,p_1} \in \mathbb{R},$$

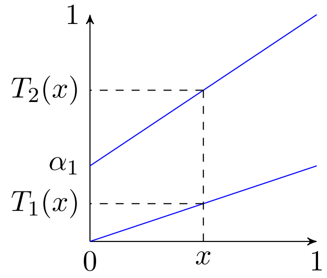


Figure 4.1: The two contractions $T_1, T_2 : [0,1] \to [0,1]$

is $\mathcal{C}^\infty$, *where we write* $\mu_{\alpha_1,p_1} = \mu_{\alpha_1,1-\alpha_1,0,\alpha_1,p_1,1-p_1}$

A more geometric example is the following:

Example 4.1.3. *For* $\lambda \in \mathcal{I}_\epsilon = (-\epsilon, \epsilon)$, *let* $\Gamma_\lambda \subset SL(2,\mathbb{C})$ *be a classical Schottky group such that* $\mathcal{I}_\epsilon \ni \lambda \mapsto \Gamma^\lambda \in SL(2,\mathbb{C})$ *is* $\mathcal{C}^m$. *Consider the conformal probability measure* μ_λ *that satisfies*

$$g^*\mu_\lambda = |dg|^{\mathscr{H}_\lambda}\mu_\lambda,$$

where $\mathscr{H}_\lambda$ *is the Hausdorff dimension of the limit set* Λ_λ *for* Γ_λ. *If* $w : \mathbb{C} \to \mathbb{R}$ *is a compactly supported* $\mathcal{C}^{s+\delta}$ *function then the map*

$$\mathcal{I}_\epsilon \ni \lambda \mapsto \int f d\mu_\lambda$$

is $\mathcal{C}^{\min(m,s-1)}$.

4.2 Results

In this section we are concerned with conformal iterated function schemes. A particularly natural special case is that of a finite family of contractions on the unit interval, since one dimensional maps are automatically conformal. For definiteness, let us consider the following setting:

Definition 4.2.1. *Assume that* $\epsilon > 0$ *small,* $\beta, \varepsilon > 0$, $k, l, m \in \mathbb{N} \setminus \{1\}$, $r \in \mathbb{N}$ *and call the interval* $(-\epsilon, \epsilon) \subset \mathbb{R}$ *by* $\mathcal{I}_\epsilon$. *Then*

1. *let* $\mathcal{T}^{(\lambda)} = \{T_i^{(\lambda)}\}_{i=1}^k$ *with* $\lambda \in \mathcal{I}_\epsilon$ *be a family of* $\mathcal{C}^{m+\beta}$ *contractions on* $[0,1]$. *Assume that we can expand for* $\lambda \in \mathcal{I}_\epsilon$,

$$T_i^{(\lambda)} = T_i + \lambda T_{i,1} + \cdots + \lambda^{m-1} T_{i,m-1} + o(\lambda^{m-1}),$$

where $T_i, T_{i,j} \in \mathcal{C}^{m+\beta}([0,1],[0,1])$, $\|dT_i\|_{\mathcal{C}^1} < 1$, $dT_i = dT_j$ for $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,m-1\}$; and

2. *let $\mathcal{G}^{(\theta)} = \{g_i^{(\theta)}\}_{i=1}^k$ with $\theta \in \mathcal{I}_\epsilon$ be a family of $\mathcal{C}^{l+\varepsilon}([0,1],\mathbb{R}^+)$ positive weight functions on $[0,1]$ satisfying the following two conditions:*

$$\sum_{i=1}^k g_i^{(\theta)} \equiv 1 \text{ and} \tag{4.1}$$

$$\sum_{i=1}^k \left\| g_i^{(\theta)} \right\|_{\mathcal{C}^0} \mathrm{Lip}\left(T_i^{(\lambda)}\right) < 1 \text{ for all } \lambda, \theta \in \mathcal{I}_\epsilon \tag{4.2}$$

where

$$g_i^{(\theta)} = g_i + \theta g_{i,1} + \cdots + \theta^r g_{i,r} + o(\theta^r) \text{ and}$$

$$g_i, g_{i,j} \in \mathcal{C}^{l+\varepsilon}([0,1],\mathbb{R}^+) \text{ for } i \in \{1,\dots,k\} \text{ and } j \in \{1,\dots,r\}.$$

In this case the stationary measure $\mu = \mu_{\lambda,\theta}$ is the unique probability measure on $[0,1]$ that satisfies

$$\int f(x)d\mu(x) = \sum_{i=1}^k \int g_i(x) f(T_i x) d\mu(x) \tag{4.3}$$

for any continuous function $f : [0,1] \to \mathbb{R}$.

The existence of such a measure is well known and discussed in Subsection 1.2.7. Observe that if the sets $T_i[0,1]$ for $i \in \{1,\dots,k\}$ are pairwise disjoint then the cumulative distribution of the stationary measure μ is a Devil staircase, i.e. the map $f : [0,1] \to [0,1]$, defined by $f(t) = \int_0^t d\mu(x)$ is singular (continuous and differentiable with derivative equal to zero μ_{Leb}-a.e in $[0,1]$, non decreasing and $f(0) < f(1)$) however it may also happen when $T_i[0,1]$ for $i \in \{1,\dots,k\}$ are not pairwise disjoint. There is an equivalent definition of stationary measure which is perhaps somewhat more intuitive and particularly useful for simulations that is given by the following rather well known lemma.

Lemma 4.2.2. *For any $x_0 \in [0,1]$ we can write μ as the weak star limit of finitely supported probability measures, indeed*

$$\mu = \lim_{n \to +\infty} \sum_{\underline{i} \in \{1,\cdots,k\}^n} g_{\underline{i}}^{(\theta)}(x_0) \delta_{T_{\underline{i}}^{(\lambda)}(x_0)}$$

where for each of the k^n strings $\underline{i} = (i_1, \cdots, i_n)$ we write (for $n \in \mathbb{N}$):

$$T_{\underline{i}}^{(\lambda)} := T_{i_1}^{(\lambda)} \circ \cdots \circ T_{i_n}^{(\lambda)} : \mathbb{R} \to \mathbb{R}; \text{ and}$$
$$g_{\underline{i}}^{(\theta)}(x_0) := g_{i_1}^{(\theta)}\left(T_{i_2}^{(\lambda)} \cdots T_{i_n}^{(\lambda)}(x_0)\right) \cdots g_{i_{n-1}}^{(\theta)}\left(T_{i_n}^{(\lambda)}(x_0)\right) \cdot g_{i_n}^{(\theta)}(x_0).$$

Our first main result is about the differentiability of the dependence of this measure.

Theorem 4.2.3. *Assume $\delta \in (0,1)$, $k, l, m, s \in \mathbb{N} \setminus \{1\}$ and $r \in \mathbb{N}$, then:*

1. *Given $\theta \in \mathcal{I}_\epsilon$, the measure $\mu_{\lambda,\theta}$ has a $\mathcal{C}^{\min(l,m,s)-1}$ dependence on $\lambda \in \mathcal{I}_\epsilon$ as an element of $\mathcal{C}^{s+\delta}([0,1], \mathbb{R})^*$.*
2. *Given $\lambda \in \mathcal{I}_\epsilon$, the measure $\mu_{\lambda,\theta}$ has a $\mathcal{C}^r$ dependence on $\theta \in \mathcal{I}_\epsilon$ as an element of $\mathcal{C}^1([0,1], \mathbb{R})^*$.*

Remark 4.2.4. *In Theorem 4.2.3, when we study the dependence of the measure $\mu = \mu_{\lambda,\theta}$ on λ, it is essential to consider the measure μ as an element of $\mathcal{C}^{s+\delta}([0,1], \mathbb{R})^*$ for $s \in \mathbb{N} \setminus \{1\}$, i.e. we identify μ with the functional $\mathscr{M} : \mathcal{C}^{s+\delta}([0,1], \mathbb{R}) \to \mathbb{R}$ defined by $\mathcal{C}^{s+\delta}([0,1], \mathbb{R}) \ni w \mapsto \int_0^1 w(\tilde{x}) d\mu(\tilde{x}) \in \mathbb{R}$.*

In the case that all of the contractions, weights and test functions are smooth enough it is possible to show a smooth dependence of the stationary measure directly.

Definition 4.2.5 (Real analytic)**.** *A function $f : \mathcal{E} \subset \mathbb{R} \to \mathbb{R}$ where $\mathcal{E}$ is an open or closed interval, is said to be real analytic on $\mathcal{E}$ if there exists an open interval $\mathcal{D} \supset \mathcal{E}$ such that for any point $y \in \mathcal{D}$ one can write*

$$f(x) = \sum_{n=0}^{\infty} a_n (x - y)^n = a_0 + a_1(x - y) + a_2(x - y)^2 + \cdots$$

where $\{a_n\} \subset \mathbb{R}$ and the series is convergent to $f(x)$ for x in a neighborhood of y

We have the following theorem for real analytic functions.

Theorem 4.2.6. *In the case that $T^{(\lambda)}$ and $\mathcal{G}^{(\theta)}$ are real analytic, with a real analytic dependence on $\lambda \in \mathcal{I}_\epsilon$ and $\theta \in \mathcal{I}_\epsilon$, we have that the stationary measure has a real analytic dependence i.e., for $w : [0,1] \to \mathbb{R}$ real analytic, $\int w d\mu_{\lambda,\theta}$ is real analytic in both $\lambda \in \mathcal{I}_\epsilon$ and $\theta \in \mathcal{I}_\epsilon$.*

Proof. We consider fixed points $x_{\underline{i}}^{(\lambda)} \in [0,1]$ for $T_{\underline{i}}^{(\lambda)} = T_{i_0}^{(\lambda)} \circ \cdots \circ T_{i_{n-1}}^{(\lambda)} : [0,1] \to [0,1]$ where $\underline{i} = (i_0, \ldots, i_{n-1}) \in \{1, \ldots, k\}^n$. We can then associate a function of several complex variables

$$\Xi(z,t,\lambda,\theta) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \frac{\prod_{j=0}^{n-1} g_{i_j}^{(\theta)}\left(x_{\sigma^j \underline{i}}^{(\lambda)}\right) e^{t\sum_{j=0}^{n-1} w\left(x_{\sigma^j \underline{i}}^{(\lambda)}\right)}}{1 - \prod_{j=0}^{n-1} \frac{dT_{i_j}^{(\lambda)}}{dx}\left(x_{\sigma^j \underline{i}}^{(\lambda)}\right)} \right),$$

where $\sigma^j \underline{i} = (i_j, \ldots, i_{j-1})$ is the cyclic permutation and $w : [0,1] \to \mathbb{R}$ real analytic.

It follows from a result in [75] that Ξ is entire as a function of $z \in \mathbb{C}$ and has a zero at $z_t = \exp(P(-\log g^{(\theta)} + tw))$. Moreover,

$$\frac{\frac{\partial \Xi(z=1,t,\lambda,\theta)}{\partial t}|_{t=0}}{\frac{\partial \Xi(z,0,\lambda,\theta)}{\partial z}|_{z=1}} = \int w d\mu_{\lambda,\theta}$$

by the implicit function theorem and the usual formula for the derivative of pressure (see [78, 76, 79]). To deduce the analyticity of $\int w d\mu_{\lambda,\theta}$ only requires the corresponding property for the function Ξ. However, it is easy to show that $\lambda \mapsto x_{\underline{i}}^{(\lambda)}$ are individually real analytic (by the implicit function theorem) and, moreover, analytic on a common domain $\mathcal{U} \supset [0,1]$. The analyticity of Ξ (and thus of $\int w d\mu_{\lambda,\theta}$) follows. □

We have the following simple corollary from Theorem 4.2.3.

Corollary 4.2.7. *Let $w : [0,1] \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Given $\theta \in \mathcal{I}_\epsilon$, the function $(-\epsilon, \epsilon) \ni \lambda \mapsto \int w d\mu_{\lambda,\theta} \in \mathbb{R}$ is $\mathcal{C}^{\min(l,m)-1}$.*

The next corollary applies under the hypothesis that the weight functions are $\mathcal{C}^\infty$. In particular, this is true in the special case of constant weight functions.

Corollary 4.2.8. *Suppose that the family $\mathcal{G}^{(\theta)} = \{g_i^{(\theta)}\}_{i=1}^k$ of weights satisfies $g_i^{(\theta)} \in \mathcal{C}^\infty([0,1], \mathbb{R}^+)$ for every $i \in \{1, \ldots, k\}$. Let $w : [0,1] \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. Given $\theta \in \mathcal{I}_\epsilon$, the function $(-\epsilon, \epsilon) \ni \lambda \mapsto \int w d\mu_{\lambda,\theta} \in \mathbb{R}$ is $\mathcal{C}^{m-1}$.*

Our second result is on the differentiability of the Hausdorff dimension of the limit set $\mathcal{K}_\lambda$ of $\mathcal{T}^{(\lambda)}$.

Theorem 4.2.9. *Let $\mathcal{T}^{(\lambda)}$ be as before for $\lambda \in \mathcal{I}_\epsilon$ with the property that the images $T_i[0,1]$, for $i = 1, \cdots, k$, are pairwise disjoint. Then the dependence $(-\epsilon, \epsilon) \ni \lambda \mapsto \dim_H(\mathcal{K}_\lambda)$ of the Hausdorff dimension of the limit set of $\mathcal{T}^{(\lambda)}$, is $\mathcal{C}^{m-2}$.*

In Section 4.3.1 we start by defining the spaces of functions in Section 4.1 and we rewrite some lemmas on composition of operators that we will use many times in this chapter (cf. in [35]). In Section 4.3.2 we prove that $\mathcal{I}_\epsilon \ni \lambda \mapsto \pi^{(\lambda)} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ is

$\mathcal{C}^{m-1}$, where $(\mathcal{X}, \sigma)$ is a subshift of finite type. In Section 4.3.3 we introduce some thermodynamic concepts that we use in section 4.3.4 to prove Theorem 4.2.3 and Theorem 4.2.9. Our work follows ideas in [76].

4.3 Proofs

This section gives a systematic review of the components of the proof of the main theorems in this chapter. It is subdivided into four subsections. The first, about composition of operators, settles some of our notation and shows results required in our proof. The second proves a useful result for smoothness of projection maps. The third reviews some basic thermodynamic formalism results that we apply to solve our problem in the last subsection.

4.3.1 First requirement: composition of functions

We will use results on composition of functions which are related to those in [35]. To introduce our setting we need to define the metric space

$$\mathcal{X} := \{x = (x_n)_{n=0}^{\infty} : x_n \in \{1, \ldots, k\}, n \in \mathbb{N}_0\} = \{1, \ldots, k\}^{\mathbb{N}_0}$$

with the metric

$$d(x, y) := \sum_{n=0}^{\infty} \frac{1 - \delta_{\{x_n\}}(y_n)}{2^n}.$$

We consider $\mathcal{X}$ with the action of the shift $\sigma : \mathcal{X} \to \mathcal{X}$, defined by $(\sigma(x))_n = x_{n+1}$ for $n \in \mathbb{N}$, where $x = (x_n)_{n=0}^{\infty} \in \mathcal{X}$. In order to apply the machinery of thermodynamic formalism we will need to consider our composition operator on the space of α-Hölder continuous functions $f : \mathcal{X} \to \mathbb{R}$.

Definition 4.3.1. *Given $0 < \alpha < 1$, let $\mathcal{C}^{\alpha}(\mathcal{X}, \mathbb{R})$ denote the Banach space of α-Hölder continuous functions $f : \mathcal{X} \to \mathbb{R}$ with norm*

$$\|f\| := \max\{\|f\|_{\alpha}, K\|f\|_{\infty}\},$$

where

$$\|f\|_{\alpha} := \sup_{x \neq y} \left\{ \frac{|f(x) - f(y)|}{d(x, y)^{\alpha}} \right\} \text{ and } \|f\|_{\infty} := \sup_{x} \{|f(x)|\}$$

and $K > 0$ is a constant.

For the first part of the proofs, we do not really need to work with the full composition operator, whose definition depends on further smoothing conditions of its domain, but with a simpler map whose definition only depends on the space $\mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$.

Definition 4.3.2. *Given a function $v:[0,1]\to\mathbb{R}$, we define the map*

$$\begin{aligned} v_*:\ \mathcal{C}^\alpha(\mathcal{X},[0,1]) &\to \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})\\ f &\mapsto v_*(f):=v\circ f. \end{aligned}$$

Most of the results in this section deal with the regularity of the map v_*. In order to state them precisely, we need to introduce the spaces of functions $\mathcal{C}^{n+\delta}([0,1],\mathbb{R})$, for $0<\delta<1$ and $n>0$, which correspond to the classic spaces of n times continuously differentiable functions with the n-th derivatives are δ-Hölder. We define these spaces rigorously.

Definition 4.3.3. *For each $i>0$, we denote the i-th derivative of $v:[0,1]\to\mathbb{R}$, when it exists, by d^iv (where $d^0v=v$).*

Given $n>0$ and $0<\delta<1$, the space $\mathcal{C}^{n+\delta}([0,1],\mathbb{R})$ is defined to be the space of functions $v:[0,1]\to\mathbb{R}$ such that v is n times differentiable and

$$\|v\|_{\mathcal{C}^0}:=\sup_{\tilde{x}\in[0,1]}|v(\tilde{x})|<\infty,$$

$$\|v\|_{\mathcal{C}^n}:=\max_{i\in\{0,\dots,n\}}\|d^iv\|_{\mathcal{C}^0}<\infty$$

and

$$\|d^nv\|_{\mathcal{C}^\delta}:=\sup_{\tilde{x}\neq\tilde{y}}\frac{|d^nv(\tilde{x})-d^nv(\tilde{y})|}{|\tilde{x}-\tilde{y}|^\delta}<\infty.$$

We endowed it with the norm

$$\|v\|_{\mathcal{C}^{n+\delta}}=\sup(\|d^nv\|_{\mathcal{C}^\delta},\|v\|_{\mathcal{C}^n}).$$

This is a Banach space and in the case $n\in\mathbb{N}$ we have that

$$\|v\|_{\mathcal{C}^{n+\delta}}=\sup(\|v\|_{\mathcal{C}^0},\|dv\|_{\mathcal{C}^{n-1+\delta}}).$$

Remark 4.3.4. *Given an integer $n>0$, any function $v\in\mathcal{C}^{n+1}([0,1],\mathbb{R})$ has i-th Lipschitz derivative for $i=0,1,\dots n$, i.e.*

$$\mathrm{Lip}(d^iv):=\sup_{\tilde{x}\neq\tilde{y}}\frac{|d^iv(\tilde{x})-d^iv(\tilde{y})|}{|\tilde{x}-\tilde{y}|}<\infty,$$

for $i\in\{0,\dots,n\}$.

This implies that $\mathcal{C}^n([0,1],\mathbb{R}) \subset \mathcal{C}^{m+\delta}([0,1],\mathbb{R})$*, for every* $0 \le \delta \le 1$ *and* $m < n$*, because Lipschitz functions are automatically* δ*-Hölder for* $0 < \delta \le 1$.

Warning! Warning! We have used the letter d to denote a metric on $\mathcal{X}$ and to denote the derivative of a function $v : [0,1] \to \mathbb{R}$, i.e. $dv = \frac{dv}{dx}$. When the notation dv may be confusing we prefer to use the notation $\frac{dv}{dx}$.

The following result is analogous to the proof of Proposition 6.2, part ii.2) in [35].

Lemma 4.3.5. *If* $v \in \mathcal{C}^{1+\delta}([0,1],\mathbb{R})$*, then the map* v_* *is* $\mathcal{C}^0$.

Proof. We can choose arbitrarily $f_1, f_2 \in \mathcal{C}^\alpha(\mathcal{X},[0,1])$ and $x, y \in \mathcal{X}$. We can then consider a path $\gamma_1 : [0,1] \to [0,1]$ joining $f_1(x)$ and $f_1(y)$ defined by $\gamma_1(t) = (1-t)f_1(x) + tf_1(y)$ and a path $\gamma_2 : [0,1] \to [0,1]$ joining $f_2(x)$ and $f_2(y)$, defined by $\gamma_2(t) = (1-t)f_2(x) + tf_2(y)$. We then have the following inequalities

$$\begin{aligned}
&|v(f_1(x)) - v(f_2(x)) - v(f_1(y)) + v(f_2(y))| \\
&\le \int_0^1 |dv(\gamma_1(t))\frac{d\gamma_1}{dt}(t) - dv(\gamma_2(t))\frac{d\gamma_2}{dt}(t)|dt \\
&\le \int_0^1 |(dv(\gamma_1(t)) - dv(\gamma_2(t)))\frac{d\gamma_1}{dt}(t)|dt + \int_0^1 |dv(\gamma_2(t))\left(\frac{d\gamma_1}{dt}(t) - \frac{d\gamma_2}{dt}(t)\right)|dt \\
&\le \|v\|_{\mathcal{C}^{1+\delta}}(|f_2(x) - f_1(x)| \\
&\quad + |f_2(y) - f_1(y)|)^\delta |f_1(x) - f_1(y)| + \|v\|_{\mathcal{C}^1}|f_1(x) - f_2(x) - f_1(y) + f_2(y)|.
\end{aligned}$$

In particular, dividing both sides of the inequality by $d(x,y)^\alpha$ and taking the supremum over the set $\{x, y : x, y \in \mathcal{X}, x \ne y\}$, we obtain

$$\begin{aligned}
\|v_*(f_1) - v_*(f_2)\|_\alpha &= \sup_{x \ne y} \frac{|(v \circ f_1 - v \circ f_2)(x) - (v \circ f_1 - v \circ f_2)(y)|}{d(x,y)^\alpha} \\
&\le 2^\delta \|v\|_{\mathcal{C}^{1+\delta}} \|f_2 - f_1\|_\infty^\delta \|f_1\|_\alpha + \|v\|_{\mathcal{C}^1}\|f_1 - f_2\|_\alpha.
\end{aligned} \tag{4.4}$$

The result follows. □

The next lemma is similar to the proof of Proposition 6.7 in [35]. In preparation, we need to introduce some definitions of differentiable operators.

Let $\mathcal{E}, \mathcal{F}$ be Banach spaces with norms $\|\cdot\|_\mathcal{E}$ and $\|\cdot\|_\mathcal{F}$, respectively. We denote the space of bounded linear functions from $\mathcal{E}$ to $\mathcal{F}$ by $L(\mathcal{E},\mathcal{F})$. Let $\mathcal{U} \subset \mathcal{E}$ be an open set. We recall that a function $f : \mathcal{U} \to \mathcal{F}$ is Fréchet differentiable at $u \in \mathcal{U}$ if we can find a bounded linear function $df(u)$ such that

$$\lim_{\epsilon\to 0}\frac{\|f(u+\epsilon h)-f(u)-\epsilon df(u)h\|_{\mathcal{F}}}{\epsilon}=0$$

for every $h\in\mathcal{E}$ and uniformly with respect to $h\in B_1(0):=\{y\in\mathcal{E}:\|y\|_{\mathcal{E}}<1\}$. We say that f is differentiable in $\mathcal{U}$ if f is differentiable at every point $u\in\mathcal{U}$. We say that f is of class $\mathcal{C}^1$ if it is differentiable and the mapping $df:\mathcal{U}\to L(\mathcal{E},\mathcal{F})$, $u\mapsto df(u)$ is continuous for the topology induced by the norm. Inductively, we define $d^n f$ to be the differential of $d^{n-1}f$ and we say that a function f is $\mathcal{C}^n$ (n times continuously differentiable) if $df:\mathcal{U}\to L(\mathcal{E},\mathcal{F})$ is $(n-1)$ times continuously differentiable.

Lemma 4.3.6. *If $v\in\mathcal{C}^{2+\delta}([0,1],\mathbb{R})$, then v_* is $\mathcal{C}^1$ and for all $f,h\in\mathcal{C}^\alpha(\mathcal{X},[0,1])$ the derivative of v_* is given by $d(v_*)(f)(h)=(dv)_*(f)\cdot h$.*

Proof. If $v\in\mathcal{C}^{2+\delta}([0,1],\mathbb{R})$, then it has a $\mathcal{C}^{2+\delta}$ extension to an open neighbourhood of $[0,1]$, i.e. $v\in\mathcal{C}^{2+\delta}((-\epsilon_1,1+\epsilon_1),\mathbb{R})$ for some $\epsilon_1>0$. This induces an extension of v_* to $\mathcal{C}^\alpha(\mathcal{X},(-\epsilon_1,1+\epsilon_1))$. Let $f\in\mathcal{C}^\alpha(\mathcal{X},[0,1])$ and $h\in\mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$.

To complete the proof we will need two simple inequalities: choose $0<\epsilon_2<1$ sufficiently small such that $\max_{t\in[0,1]}\|f+t\epsilon_2 h\|_\infty<1+\epsilon_1$, then

$$\int_0^1\|dv\circ(f+t\epsilon_2 h)-dv\circ f\|_\infty dt\le\|h\|(\|v\|_{\mathcal{C}^2}+1)\epsilon_2^\delta \tag{4.5}$$

and

$$\|dv\circ(f+t\epsilon_2 h)-dv\circ f\|_\alpha\le 2^\delta\|v\|_{\mathcal{C}^{2+\delta}}\|\epsilon_2 h\|_\infty^\delta\|f\|_\alpha+\|v\|_{\mathcal{C}^2}\|\epsilon_2 h\|_\alpha. \tag{4.6}$$

To prove (4.5), we use that for every $t\in[0,1]$ and $x\in\mathcal{X}$

$$\begin{aligned}&\frac{|dv\circ(f(x)+t\epsilon_2 h(x))-dv\circ f(x)|}{\epsilon_2}\\&=\frac{|d^2v(f(x))\cdot t\epsilon_2 h(x)+o(t\epsilon_2 h(x))|}{\epsilon_2}\\&\le|d^2v(f(x))|\cdot|h(x)|+|h(x)|\frac{o(\epsilon_2)}{\epsilon_2}\\&\le\|h\|(\|v\|_{\mathcal{C}^2}+1).\end{aligned}$$

To prove (4.6) we notice that by definition $dv\circ(f+t\epsilon_2 h)-dv\circ f=(dv)_*(f+t\epsilon_2 h)-(dv)_*f$ and use inequality (4.4) with dv instead of v, $f+t\epsilon_2 h$ instead of f_1 and f instead of f_2.

Fix $0 < \epsilon_2 < 1$ sufficiently small for equation (4.5) to hold, then

$$\begin{aligned}
&\frac{1}{\epsilon_2}\|v_*(f+\epsilon_2 h) - v_*(f) - \epsilon_2 (dv)_*(f)\cdot h\|_\alpha \\
&= \frac{1}{\epsilon_2}\|v\circ(f+\epsilon_2 h) - v\circ f - \epsilon_2(dv\circ f)\cdot h\|_\alpha \\
&= \|\int_0^1 [dv\circ(f+t\epsilon_2 h) - dv\circ f]\cdot h dt\|_\alpha \\
&\leq \|h\|_\infty \int_0^1 \|dv\circ(f+t\epsilon_2 h) - dv\circ f\|_\alpha dt \\
&\quad + \|h\|_\alpha \int_0^1 \|dv\circ(f+t\epsilon_2 h) - dv\circ f\|_\infty dt \\
&\leq \left(2^\delta \|v\|_{\mathcal{C}^{2+\delta}} \|\epsilon_2 h\|_\infty^\delta \|f\|_\alpha + \|v\|_{\mathcal{C}^2}\|\epsilon_2 h\|_\alpha\right) + \|h\|(\|v\|_{\mathcal{C}^2}+1)\epsilon_2^\delta \\
&\leq (4\|v\|_{\mathcal{C}^{2+\delta}}\max\{\|f\|_\alpha, 1\}+1)\|h\|\epsilon_2^\delta,
\end{aligned}$$

which proves the second part of the lemma. We used inequalities (4.5) and (4.6) in the penultimate inequality.

Now that we have the formula for the derivative of v_* :

$$d(v_*)(f)(h) = (dv)_*(f)\cdot h \tag{4.7}$$

for all $f, h \in \mathcal{C}^\alpha(\mathcal{X}, [0,1])$, we can prove that v_* is $\mathcal{C}^1$. For this, it is enough to show that $d(v_*)$ is continuous. From (4.7) we can see that $d(v_*)$ corresponds to $(dv)_*$ followed by the continuous linear map

$$\begin{aligned}
\mathscr{L} : \mathcal{C}^\alpha(\mathcal{X}, L(\mathbb{R},\mathbb{R})) &\to L(\mathcal{C}^\alpha(\mathcal{X},[0,1]), \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})), \\
\xi &\mapsto [\mathscr{L}(\xi) : h \mapsto \xi\cdot h].
\end{aligned}$$

Thus we have that $d(v_*) = \mathscr{L}\circ(dv)_*$ is continuous, since $(dv)_*$ is continuous by Lemma 4.3.5. □

The next corollary follows by induction.

Corollary 4.3.7. *If $v \in \mathcal{C}^{n+\delta}([0,1],\mathbb{R})$ for some integer $n \in \mathbb{N}$, and thus v_* is $\mathcal{C}^{n-1}$, as required.*

Proof. The case $n = 1$ is covered by Lemma 4.3.5. If the result holds for n and $v \in \mathcal{C}^{n+1+\delta}([0,1],\mathbb{R})$, then $(dv)_*$ is $\mathcal{C}^{n-1}$ by the inductive hypothesis. We can use the same argument as in the last lines of the proof of Lemma 4.3.6 to obtain that $d(v_*) = \mathscr{L}\circ(dv)_*$, where $\mathscr{L}$ is a continuous linear map, then $d(v_*)$ is $\mathcal{C}^{n-1}$. Therefore, by definition, v_* is $\mathcal{C}^n$, which concludes the proof. □

A simple argument based in the previous corollary gives the following result that we use to prove the smoothness of the stationary probability measure.

Corollary 4.3.8. *Suppose that we have a family of maps* $\{v_i \in \mathcal{C}^{n+\delta}([0,1],\mathbb{R}) : i \in \{1,\dots,k\}\}$ *for some integer* $n \in \mathbb{N}$, *and consider the map* $F : \mathcal{C}^\alpha(\mathcal{X},[0,1]) \to \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$, *defined*[1] *by* $F(\Pi)(x) := v_{x_0}(\Pi(\sigma x))$, *where* $\Pi \in \mathcal{C}^\alpha(\mathcal{X},[0,1])$ *and* $x \in \mathcal{X}$. *Then* F *is* $\mathcal{C}^{n-1}$. *Moreover, for all* $f, h \in \mathcal{C}^\alpha(\mathcal{X},[0,1])$ *the derivative of* F *is given by*

$$d(F)(f)(h)(x) = (d(v_{x_0}))_*(f(\sigma x)) \cdot h(\sigma x) \text{ for } x \in \mathcal{X}.$$

Proof. The map $l_1 : \mathcal{C}^\alpha(\mathcal{X},[0,1]) \to [\mathcal{C}^\alpha(\mathcal{X},\mathbb{R})]^k$, defined by

$$l_1(\Pi(x)) := [v_1(\Pi(x)),\dots,v_k(\Pi(x))] \in [\mathcal{C}^\alpha(\mathcal{X},\mathbb{R})]^k$$

is $\mathcal{C}^{n-1}$ by Lemma 4.3.6, and the map $l_2 : [\mathcal{C}^\alpha(\mathcal{X},\mathbb{R})]^k \to \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$, defined by

$$l_2([f_1(x),\dots,f_k(x)]) = f_{x_0}(\sigma x)$$

is linear and continuous. It follows that the map $F = l_2 \circ l_1$ is $\mathcal{C}^{n-1}$.

To prove the formula for the derivative of F we can use the chain rule and the fact that l_2 is linear to deduce that $dF = l_2 \circ dl_1$ and $dl_1 = [d(v_1)_*,\dots,d(v_k)_*]$. This together with the formula for $d(v_i)_*$ for $i \in \{1,\dots,k\}$ in Lemma 4.3.6 concludes the proof. □

To prove the smoothness of the Hausdorff dimension of the support of the stationary measure we additionally need the following results, whose proofs are analogous to the proofs in [35] combined with simple arguments similar to the used in this section.

Definition 4.3.9. *Given* $n > 0$ *and* $0 < \delta < 1$, *we define the composition operator by*

$$\begin{aligned} \mathit{Comp}: \quad \mathcal{C}^{n+\delta}([0,1],\mathbb{R}) \times \mathcal{C}^\alpha(\mathcal{X},\mathbb{R}) &\to \quad \mathcal{C}^\alpha(\mathcal{X},\mathbb{R}) \\ (v,f) &\mapsto \quad \mathit{Comp}(v,f) := v \circ f. \end{aligned}$$

Proposition 4.3.10. *Given* $n \in \mathbb{N}$ *and* $0 < \delta < 1$, *the composition operator* Comp : $\mathcal{C}^{n+\delta}([0,1],\mathbb{R}) \times \mathcal{C}^\alpha(\mathcal{X},\mathbb{R}) \to \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ *is* $\mathcal{C}^{n-1}$.

This leads to the following corollaries.

[1] The notation $v_{x_0}(\Pi(\sigma x))$ denotes $v_i(\Pi(\sigma x))$ if $x_0 = i$.

Corollary 4.3.11. *The map* $[\mathcal{C}^{n+\delta}([0,1],\mathbb{R})]^k \times \mathcal{C}^\alpha(\mathcal{X},\mathbb{R}) \ni ([v_1,\ldots,v_k], f) \mapsto v_{x_0} \circ f(x) \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ *is* $\mathcal{C}^{n-1}$.

Corollary 4.3.12. *Let* $n \in \mathbb{N}$, $0 < \delta < 1$, $\epsilon > 0$ *and suppose that we have for each* $\lambda \in \mathcal{I}_\epsilon$ *a family of maps* $\{v_i^{(\lambda)} \in \mathcal{C}^{n+\delta}([0,1],\mathbb{R}) : i \in \{1,\ldots,k\}\}$ *and a map* $f^{(\lambda)} \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$. *If the map* $\mathcal{I}_\epsilon \ni \lambda \mapsto [v_1^{(\lambda)},\ldots,v_k^{(\lambda)}] \in [\mathcal{C}^{n+\delta}([0,1],\mathbb{R})]^k$ *is* $\mathcal{C}^{n_1}$ *for some* $n_1 > 0$, *and the map* $\mathcal{I}_\epsilon \ni \lambda \mapsto f^{(\lambda)} \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ *is* $\mathcal{C}^{n_2}$ *for some* $n_2 > 0$, *then the map* $\mathcal{I}_\epsilon \ni \lambda \mapsto v_{x_0}^{(\lambda)} \circ f^{(\lambda)}(x) \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ *is* $\mathcal{C}^{\min(n_1,n_2,n-1)}$.

4.3.2 Second requirement: projection map

We will introduce a projection map $\pi^{(\lambda)} : \mathcal{X} \to [0,1]$ for $\lambda \in \mathcal{I}_\epsilon$ that will be essential to study the differentiability of the stationary measure.

Definition 4.3.13. *For each* $\lambda \in \mathcal{I}_\epsilon$ *we define the projection map* $\pi^{(\lambda)} : \mathcal{X} \to [0,1]$ *by*

$$\pi^{(\lambda)}(x) := \lim_{n\to\infty} T_{x_0}^{(\lambda)} \circ T_{x_1}^{(\lambda)} \circ \cdots \circ T_{x_n}^{(\lambda)}(0),$$

where $x = (x_i)_{i=0}^\infty$.

The following result is easily seen.

Lemma 4.3.14. *There exists* $\alpha > 0$ *such that each individual map* $\pi^{(\lambda)} : \mathcal{X} \to [0,1]$ *is* α*-Hölder continuous.*

Proof. Define $a := \max_{i\in\{1,\ldots,k\}} \sup_{\lambda\in\mathcal{I}_\epsilon}\{\|dT_i^{(\lambda)}\|_{\mathcal{C}^0}\} < 1$ and $\alpha := -\frac{\log(a)}{\log(2)}$. Suppose that $x, y \in \mathcal{X}$ and chose $n = n(x,y)$ such that $x_i = y_i$ for $i \le n$ and $x_{n+1} \ne y_{n+1}$, then

$$|\pi^{(\lambda)}(x) - \pi^{(\lambda)}(y)| \le a^n = \frac{1}{2^{\alpha n}} \le d(x,y)^\alpha.$$

This completes the proof. □

To make further use of the functional analytic approach it helps to choose a specific Banach space of Hölder continuous functions.

Remark 4.3.15. *We are now at liberty to choose values of* α *and* K *which are most convenient for us in definition of Hölder norm on* $\mathcal{X}$ *(i.e., Definition 4.3.1). Denote* $\theta_0 := \|dT_1^{(0)}\|_{\mathcal{C}^0}$ *and then fix a choice of* $\theta_0 < \theta < 1$. *We can then choose* $0 < \alpha < 1$ *sufficiently small such that* $2^\alpha\theta_0 < \frac{\theta+\theta_0}{2}$. *Finally, let us choose* $K > 0$ *sufficiently large such that*

$$\mathrm{Lip}(dT_1)\|\pi^{(0)}\|_\alpha \frac{2^\alpha}{K} < \theta - \theta_0$$

where $\mathrm{Lip}(dT_1)$ *is the Lipschitz constant of the derivative of the contraction* T_1.

We may now prove the main proposition in this section.

Proposition 4.3.16. *Provided $\alpha > 0$ is chosen sufficiently small, the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \pi^{(\lambda)} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ is $\mathcal{C}^{m-1}$.*

Proof. For each $\lambda \in (-\epsilon, \epsilon)$ we let $R^{(\lambda)} : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ be defined by

$$(R^{(\lambda)}\Pi)(x) := T_{x_0}^{(\lambda)}(\Pi(\sigma x)),$$

and we construct the map $F : \mathcal{I}_\epsilon \times \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ defined by $F(\lambda, \Pi) = \left(I - R^{(\lambda)}\right)(\Pi)$, where $\Pi \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$. As usual $D_2F(0, \pi^{(0)})$ denotes the partial derivative of F with respect to the second coordinate and evaluated in $(0, \pi^{(0)})$, i.e. for $F(0, \cdot) : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ defined by $F(0, \cdot)(\Pi) = F(0, \Pi)$, we define $D_2F(0, \pi^{(0)}) := dF(0, \cdot)(\pi^{(0)})$.

We begin with some preliminary observations.

1. First observe that $\pi^{(\lambda)}$ is a fixed point, i.e., $R^{(\lambda)}\pi^{(\lambda)} = \pi^{(\lambda)}$.

2. We next observe that the family of maps $(-\epsilon, \epsilon) \times \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \ni (\lambda, \Pi) \mapsto R^{(\lambda)}(\Pi) \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ is $\mathcal{C}^{m-1}$. Clearly it is $\mathcal{C}^{m-1}$ in λ, whilst it is $\mathcal{C}^{m-1}$ in Π by Corollary 4.3.8.

3. $D_2F(0, \pi^{(0)})$ is a linear homeomorphism of $\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ onto $\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$. Moreover, we will prove that $(I - D_2(R^{(0)}\pi^{(0)}))$ is invertible. We call

$$\mathscr{R}^{(0)} := D_2(R^{(0)}\pi^{(0)}).$$

On $\Pi \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$, $\mathscr{R}^{(0)}$ is given by

$$\mathscr{R}^{(0)}(\Pi)(x) = dT_{x_0}^{(0)}\left(\pi^{(0)}(\sigma x)\right) \cdot \Pi(\sigma x), x \in \mathcal{X},$$

and this is clear using Corollary 4.3.8. Since each T_i is a contraction it is easy to see that $\mathscr{R}^{(0)} : \mathcal{C}^0(\mathcal{X}, \mathbb{R}) \to \mathcal{C}^0(\mathcal{X}, \mathbb{R})$ satisfies $\|\mathscr{R}^{(0)}\|_\infty < 1$, i.e. $\mathscr{R}^{(0)}$ is a contraction on $\mathcal{C}^0$. Using Remark 4.3.15 we will prove that $\mathscr{R}^{(0)}$ is also a contraction on $\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$. For this, assume $\|\Pi\| \leq 1$ (and thus, in particular,

$\|\Pi\|_\alpha \leq 1$ and $\|\Pi\|_\infty \leq 1/K$). We can then use the triangle inequality to bound

$$
\begin{aligned}
&|\mathscr{R}^{(0)}(\Pi)(x) - \mathscr{R}^{(0)}(\Pi)(y)| \\
&= \left| dT^{(0)}_{x_0}\left(\pi^{(0)}(\sigma x)\right)\Pi(x) - dT^{(0)}_{y_0}\left(\pi(\sigma y)\right)\Pi(y)\right| \\
&\leq \left| dT^{(0)}_{x_0}\left(\pi^{(0)}(\sigma x)\right)\Pi(\sigma x) - dT^{(0)}_{x_0}\left(\pi^{(0)}(\sigma x)\right)\Pi(\sigma y)\right| \\
&\qquad + \left| dT^{(0)}_{x_0}\left(\pi^{(0)}(\sigma x)\right)\Pi(\sigma y) - dT^{(0)}_{y_0}\left(\pi^{(0)}(\sigma y)\right)\Pi(\sigma y)\right| \\
&\leq \|dT^{(0)}_1\|_{\mathcal{C}^0}|\Pi(\sigma x) - \Pi(\sigma y)| + \left| dT^{(0)}_{x_0}\left(\pi^{(0)}(\sigma x)\right) - dT^{(0)}_{y_0}\left(\pi^{(0)}(\sigma y)\right)\right| .\|\Pi\|_\infty \\
&\leq \|dT^{(0)}_1\|_{\mathcal{C}^0}\|\Pi\|_\alpha d(\sigma x, \sigma y)^\alpha + \mathrm{Lip}(dT^{(0)}_1)|\pi^{(0)}(\sigma x) - \pi^{(0)}(\sigma y)|\frac{1}{K} \\
&\leq \left(2^\alpha \|dT^{(0)}_1\|_{\mathcal{C}^0}\right) d(x,y)^\alpha + \left(\mathrm{Lip}(dT^{(0)}_1)\|\pi^{(0)}\|_\alpha \frac{2^\alpha}{K}\right) d(x,y)^\alpha \\
&\leq \theta d(x,y)^\alpha,
\end{aligned}
$$

where we have used Remark 4.3.15 in the last inequality. This implies that $\|\mathscr{R}^{(0)}\|_\alpha < 1$.

To end the proof we will use the implicit function theorem for Banach spaces (see for example [89]). The map F is $\mathcal{C}^{m-1}$ in a neighbourhood of $(0, \pi^{(0)})$ of $\mathcal{I}_\epsilon \times \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ and since $\max\{\|\mathscr{R}^{(0)}\|_\infty, \|\mathscr{R}^{(0)}\|_\alpha\} < 1$ we see that $D_2F(0, \pi^{(0)}) = I - \mathscr{R}^{(0)}$ is invertible. Thus the hypotheses of the implicit function theorem are satisfied and the result follows. □

Example 4.3.17. *If $T_0(x) = \lambda x$, $T_1(x) = \lambda x + t$ and $\mathcal{X} = \{0,1\}^{\mathbb{N}_0}$, then we can explicitly write the map $\pi : \mathcal{X} \to \mathbb{R}$ as an infinite series:*

$$\pi\left((x_n)_{n=0}^\infty\right) = t\sum_{n=0}^\infty \lambda^n x_n.$$

4.3.3 Third requirement: thermodynamic formalism

The basic definitions and results on thermodynamic formalism in Section 2.4 will be used in the proofs of the main theorems in the next subsection. Indeed, we can deduce by classical techniques and an argument based in composition of operators the differentiability of a Gibbs measure that we will relate with the stationary measure using the projection maps. Also, we relate the Hausdorff dimension with the zero of $t \mapsto P(-t\Phi)$ by Bowen's method for some appropriate function Φ. This will be use to deduce the differentiability of the Hausdorff dimension.

Consider the family $\mathcal{G}^{(\theta)}$ of weights $\mathcal{G}^{(\theta)} = \{g_i^{(\theta)}\}_{i=1}^k$ and $\theta \in \mathcal{I}_\epsilon := (-\epsilon, \epsilon)$. We can associate a Hölder continuous function $\psi^{(\lambda,\theta)} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ defined by

$$\psi^{(\lambda,\theta)}(x) = \log\left(g_{x_0}^{(\theta)}(\pi^{(\lambda)}(\sigma x))\right).$$

Now we proceed to the definition of the Transfer operator in this setting.

Definition 4.3.18 (Transfer operator). *We can define a transfer operator $\mathscr{L}_{\psi^{(\lambda,\theta)}} : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ by*

$$\mathscr{L}_{\psi^{(\theta,\lambda)}} w(x) = \sum_{\sigma y = x} e^{\psi^{(\lambda,\theta)}(y)} w(y) \text{ where } w \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}).$$

We see from the definition of $\mathscr{L}_{\psi^{(\lambda,\theta)}}$ and the property that $\sum_{i=1}^k g_i^{(\theta)} = 1$ that $\mathscr{L}_{\psi^{(\lambda,\theta)}} 1 = 1$, i.e., $\mathscr{L}_{\psi^{(\lambda,\theta)}}$ preserves the constant functions.

We next recall the following classical result.

Theorem 4.3.19 (Ruelle Operator Theorem). *There exists a maximal positive simple isolated eigenvalue* 1. *Moreover,*

1. *there is a positive eigenvector $w_{\psi^{(\lambda,\theta)}}$, i.e., $\mathscr{L}_{\psi^{(\lambda,\theta)}} w_{\psi^{(\lambda,\theta)}} = w_{\psi^{(\lambda,\theta)}}$;*
2. *the equilibrium state $\nu_{\psi^{(\lambda,\theta)}}$ is a fixed point for the dual operator, i.e.,*

$$\mathscr{L}^*_{\psi^{(\lambda,\theta)}} \nu_{\psi^{(\lambda,\theta)}} = \nu_{\psi^{(\lambda,\theta)}}$$

thus $\int f d\nu_{\psi^{(\lambda,\theta)}} = \int (\mathscr{L}_{\psi^{(\lambda,\theta)}} f) d\nu_{\psi^{(\lambda,\theta)}}$ for every continuous $f : \mathcal{X} \to \mathbb{R}$.

Proof. The spectral properties of the operator follow from the general results of Ruelle for transfer operators with any Hölder continuous function [16], [78]. In this particular case the fact that the maximal eigenvalue is 1 and the corresponding eigendistribution is the equilibrium state follows from the property that $\mathscr{L}_{\psi^{(\lambda,\theta)}} 1 = 1$ and [88], [61]. □

4.3.4 Proof of Theorem 4.2.3

We need to relate the Gibbs measure to the stationary measure $\mu_{\lambda,\theta}$, recall its definition in (4.3). The strategy of the proof of Theorem 4.2.3 consists of the following steps:

i. We construct a probability measure $\nu_{\lambda,\theta}$ on the Borel sets of $\mathcal{X} := \{1, \dots, k\}^{\mathbb{N}}$

such that for $w \in \mathcal{C}^{s+\delta}([0,1],\mathbb{R})$ we have

$$\int_{\mathcal{X}} w \circ \pi^{\lambda}(x) d\nu_{\lambda,\theta}(x) = \int_0^1 w(\tilde{x}) d\mu_{\lambda,\theta}(\tilde{x}), \tag{4.8}$$

where $\pi^{(\lambda)} \in \mathcal{C}^{\alpha}(\mathcal{X},[0,1])$ for $\lambda \in \mathcal{I}_\epsilon$. The probability measure $\nu_{\lambda,\theta}$ corresponds to the Gibbs measure of an explicitly constructed Hölder potential that depends on both $\mathcal{T}^{(\lambda)}$ and $\mathcal{G}^{(\theta)}$.

ii. We prove that $\mathcal{C}^{\alpha}(\mathcal{X},\mathbb{R}) \ni \Pi \mapsto w \circ \Pi \in \mathcal{C}^{\alpha}(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^{s-1}$. To achieve this, we use an argument of composition of operators (following de la Llave and Obaya) which requires $w \in \mathcal{C}^{s+\delta}([0,1],\mathbb{R})$.

iii. A similar argument is used to show that $\mathcal{I}_\epsilon \ni \lambda \mapsto \pi^{(\lambda)} \in \mathcal{C}^{\alpha}(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^{m-1}$. In order to apply the result in this case we need to use that $\mathcal{T}^{(\lambda)}$ is a family of $\mathcal{C}^{m+\beta}$ functions. We use an argument based on the implicit function theorem that requires the family $\mathcal{T}^{(\lambda)}$ to be contractions.

iv. We use a classical result about regularity of Gibbs measures to prove that $\mathcal{I}_\epsilon \ni \lambda \mapsto \nu_{\lambda,\theta} \in \mathcal{C}^{\alpha}(\mathcal{X},\mathbb{R})^*$ is $\mathcal{C}^{l-1}$.

v. As a consequence of the previous parts, we have that the map $\mathcal{I}_\epsilon \ni \lambda \mapsto (\nu_{\lambda,\theta}, w \circ \pi^{(\lambda)}) \in \mathcal{C}^{\alpha}(\mathcal{X},\mathbb{R})^* \times \mathcal{C}^{\alpha}(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^{\min(l,m,s)-1}$. On the other hand, the map $\mathcal{C}^{\alpha}(\mathcal{X},\mathbb{R})^* \times \mathcal{C}^{\alpha}(\mathcal{X},\mathbb{R}) \ni (\nu_{\lambda,\theta}, w \circ \pi^{(\lambda)}) \mapsto \nu_{\lambda,\theta}(w \circ \pi^{(\lambda)}) = \int_{\mathcal{X}} w \circ \pi^{(\lambda)}(x) d\nu_{\lambda,\theta}(x) \in \mathbb{R}$ is $\mathcal{C}^{\infty}$. This, together with equation (4.8) concludes the proof.

Now we can show the following result.

Lemma 4.3.20. *Consider the family $\mathcal{G}^{(\theta)}$ of weights $g_j^{(\theta)}$ for $j = 1, \cdots, k$ and $-\epsilon < \theta < \epsilon$. Then the stationary measure for $\mathcal{T}^{(\lambda)}$ and $\mathcal{G}^{(\theta)}$ is the image of the eigen-distribution $\nu_{\psi^{(\lambda,\theta)}}$ for $\psi^{(\lambda)}$, i.e., $(\pi^{(\lambda)})_* \nu_{\psi^{(\lambda,\theta)}} = \mu_{\lambda,\theta}$.*

Proof. By the uniqueness of the stationary measure, it is enough for us to check that

$$\int f(\tilde{x}) d\left((\pi^{(\lambda)})_* \nu_{\psi^{(\lambda,\theta)}}\right)(\tilde{x}) = \sum_{i=1}^{k} \int g_i^{(\theta)}(\tilde{x}) f(T_i \tilde{x}) d\left((\pi^{(\lambda)})_* \nu_{\psi^{(\lambda,\theta)}}\right)(\tilde{x})$$

holds for any continuous $f : [0,1] \to \mathbb{R}$ and $\tilde{x} \in [0,1]$. A straightforward manipulation yields

$$\sum_{i=1}^{k}\int g_i^{(\lambda)}(\tilde{x})f(T_i\tilde{x})d\left((\pi^{(\lambda)})_*\nu_{\psi^{(\lambda,\theta)}}\right)(\tilde{x}) = \int\left(\sum_{y\in\sigma^{-1}x} e^{\psi^{(\lambda,\theta)}(y)}f(\pi^{(\lambda)}y)\right)d\nu_{\psi^{(\lambda,\theta)}}(x)$$
$$= \int \mathscr{L}_{\psi^{(\lambda,\theta)}}(f\circ\pi^{(\lambda)})(x)d\nu_{\psi^{(\lambda,\theta)}}(x)$$
$$= \int f\circ\pi^{(\lambda)}(x)d\nu_{\psi^{(\lambda,\theta)}}(x)$$
$$= \int f(\tilde{x})d\left((\pi^{(\lambda)})_*\nu_{\psi^{(\lambda,\theta)}}\right)(\tilde{x})$$

for every continuous function $f : [0,1] \to \mathbb{R}$, where we have used that $\mathscr{L}^*_{\psi^{(\lambda,\theta)}}(\nu_{\psi^{(\lambda,\theta)}}) = \nu_{\psi^{(\lambda,\theta)}}$. □

Lemma 4.3.21. *For fixed $\theta \in \mathcal{I}_\epsilon$, the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \psi^{(\lambda,\theta)} \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^{\min(l,m)-1}$.*

Proof. Consider $\theta \in \mathcal{I}_\epsilon$ fixed. By Corollary 4.3.8 we have that $\mathcal{C}^\alpha(\mathcal{X},\mathbb{R}) \ni \Pi \mapsto g_{x_0}^{(\theta)}(\Pi(\sigma x)) \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^{l-1}$ and by Proposition 4.3.16 the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \pi^{(\lambda)} \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^{m-1}$, then the map $\mathcal{I}_\epsilon \ni \lambda \mapsto g_{x_0}^{(\theta)}(\pi^{(\lambda)}(\sigma x)) \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^{\min(m,n)-1}$. This proves that the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \psi^{(\lambda,\theta)}(x) = \log\left(g_{x_0}^{(\theta)}(\pi^{(\lambda)}(\sigma x))\right) \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^{\min(m,n)-1}$, which concludes the proof. □

Lemma 4.3.22. *For fixed $\lambda \in \mathcal{I}_\epsilon$, the map $\mathcal{I}_\epsilon \ni \theta \mapsto \psi^{(\lambda,\theta)} \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ is $\mathcal{C}^r$.*

Proof. From the hypothesis on the family $\mathcal{G}^{(\theta)}$ and the definition of $\psi^{(\lambda,\theta)}$

$$\begin{aligned}\psi^{(\lambda,\theta)}(x) &= \log\left(g_{x_0}^{(\theta)}(\pi^{(\lambda)}(\sigma x))\right)\\ &= \log\left(g_{x_0}(\pi^{(\lambda)}(\sigma x)) + \theta g_{x_0,1}(\pi^{(\lambda)}(\sigma x)) + \cdots + \theta^r g_{x_0,r}(\pi^{(\lambda)}(\sigma x)) + o(\theta^r)\right)\\ &=: t(\theta)\end{aligned}$$

where $t(\theta) = t(0) + dt(0)\theta + \frac{1}{2!}d^2t(0)\theta + \cdots + o(\theta^r)$, and where $d^it(0) \in \mathcal{C}^\alpha(\mathcal{X},\mathbb{R})$ is given by

$$d^it(0)(x) = \frac{p_i\left[g_{x_0}(\pi^{(\lambda)}(\sigma x)), g_{x_0,1}(\pi^{(\lambda)}(\sigma x)), \cdots, g_{x_0,i}(\pi^{(\lambda)}(\sigma x))\right]}{g_{x_0}(\pi^{(\lambda)}(\sigma x))^i},$$

where p_i ($i \in \{0,\ldots,r\}$) are polynomials. □

Using standard analytic perturbation theory (cf. [78]) and the previous corollary we have the following.

Corollary 4.3.23.

1. For fixed $\theta \in \mathcal{I}_\epsilon$, the map $(-\epsilon, \epsilon) \ni \lambda \to \nu_{\psi^{(\lambda,\theta)}} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})^*$ is $\mathcal{C}^{\min(l,m)-1}$.

2. For fixed $\lambda \in \mathcal{I}_\epsilon$, the map $(-\epsilon, \epsilon) \ni \theta \to \nu_{\psi^{(\lambda,\theta)}} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})^*$ is $\mathcal{C}^r$.

In particular, this implies the following.

Corollary 4.3.24. *Given a Hölder continuous function $f \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$.*

1. For any fixed $\theta \in (-\epsilon, \epsilon)$, the map $(-\epsilon, \epsilon) \ni \lambda \mapsto \int f d\nu_{\psi^{(\lambda,\theta)}} \in \mathbb{R}$ is $\mathcal{C}^{\min(l,m)-1}$.

2. For any fixed $\lambda \in (-\epsilon, \epsilon)$, the map $(-\epsilon, \epsilon) \ni \theta \mapsto \int f d\nu_{\psi^{(\lambda,\theta)}} \in \mathbb{R}$ is $\mathcal{C}^r$.

We now turn to the proof of Theorem 4.2.3.

Proof of Theorem 4.2.3. There are two parts.

1. From Corollary 4.3.7 we deduce that for $f \in \mathcal{C}^{s+\delta}([0,1], \mathbb{R})$, the map $\mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \ni \Pi \mapsto f \circ \Pi \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ is $\mathcal{C}^{s-1}$ and we know from Proposition 4.3.16 that $\mathcal{I}_\epsilon \ni \lambda \mapsto \pi^{(\lambda)} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ is $\mathcal{C}^{m-1}$, then the map $\mathcal{I}_\epsilon \ni \lambda \to f \circ \pi^{(\lambda)} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ is $\mathcal{C}^{\min(s,m)-1}$. Using Corollary 4.3.23 we have that $(-\epsilon, \epsilon) \ni \lambda \to \nu_{\psi^{(\lambda,\theta)}} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})^*$ is $\mathcal{C}^{\min(l,m)-1}$, therefore the map $l_1 : \mathcal{I}_\epsilon \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \times \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})^*$, defined by $l_1(\lambda) = (f \circ \pi^{(\lambda)}, \nu_{\psi^{(\lambda,\theta)}})$ is $\mathcal{C}^{\min(l,m,s)-1}$. We define the map $l_2 : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \times \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})^* \to \mathbb{R}$ by $l_2(v, \nu) = \int v d\nu$ for $v \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ and $\nu \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})^*$. The map l_2 is $\mathcal{C}^\infty$.

   We consider the map $F := l_2 \circ l_1$, so $F(\lambda) = \int f \circ \pi^{(\lambda)} d\nu_{\psi^{(\lambda,\theta)}}$ is $\mathcal{C}^{\min(l,m,s)-1}$. Finally by Lemma 4.3.20, $\int f \circ \pi^{(\lambda)} d\nu_{\psi^{(\lambda,\theta)}} = \int f d\mu_{\lambda,\theta}$, which concludes the proof of part 1.

2. For $f \in \mathcal{C}^1([0,1], \mathbb{R})$, $f \circ \pi^{(\lambda)} \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ and the map $l_3 : \mathcal{I}_\epsilon \to \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})^*$ defined by $l_3(\theta) = \nu_{\psi^{(\lambda,\theta)}}$ is $\mathcal{C}^r$ by Corollary 4.3.23. We consider the map $G : \mathcal{I}_\epsilon \to \mathbb{R}$, defined by $G(\theta) = l_2(f \circ \pi^{(\lambda)}, l_3(\theta))$, where l_2 is defined in the part 1 of this proof. By Lemma 4.3.20 we have $G(\theta) = \int f d\mu_{\lambda,\theta}$ and G is $\mathcal{C}^r$ since l_3 is $\mathcal{C}^r$ and l_2 is $\mathcal{C}^\infty$. This finishes the proof.

□

4.4 Conclussions

4.4.1 Applications

A seminal paper [18], Bowen introduced a method relating the Hausdorff dimension s of an invariant set for a certain family of transformations $\mathcal{F}$ with the solution of the equation $P(s\Phi) = 0$, where P is the pressure function and Φ is an appropriate function that depends on $\mathcal{F}$. Some memorable references for applications of this approach are [76],[77],[64], [63]. The next proposition is an application of Bowen's method to compute Hausdorff dimension.

Proposition 4.4.1. *Independently of $\mathcal{G}^{(\theta)}$, there exists a unique $t = t_\lambda = dim_H(supp\ \mu_{\lambda,\theta})$ such that*

$$P\left(-t\log\left(dT_{x_0}^{(\lambda)}(\pi^{(\lambda)}(x))\right)\right) = 0.$$

We are interested in the differentiability of the map $\mathcal{I}_\epsilon \ni \lambda \mapsto t_\lambda \in \mathbb{R}$. Using Corollary 4.3.12 we can prove the main proposition we need.

Proposition 4.4.2. *The map $\mathcal{I}_\epsilon \ni \lambda \mapsto \log\left(dT_{x_0}^{(\lambda)}(\pi^{(\lambda)}(x))\right) \in \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R})$ is $\mathcal{C}^{m-2}$.*

We can now prove our second theorem.

Proof of Theorem 4.2.9. Since $P : \mathcal{C}^\alpha(\mathcal{X}, \mathbb{R}) \to \mathbb{R}$ is real analytic it follows that $\mathcal{I}_\epsilon \ni \lambda \mapsto t_\lambda \in \mathbb{R}$ is $\mathcal{C}^{m-2}$ and using Proposition 4.4.1 we conclude the proof of Theorem 4.2.9. □

We can also prove the example in the introduction of the chapter.

Proof of Example 4.1.3. Suppose that Γ_λ is generated by some Möbius transformations $\{\gamma_i^\lambda\}_{i=1}^k$ and for each $i \in \{1, \dots, k\}$ define $\mathcal{U}_i^\lambda := \{z \in \mathbb{C} : |d\gamma_i^\lambda(z)| < 1\} \subset \mathbb{C}$. For each $i, j \in \{1, \dots, k\}$ call by T_i^λ the map $\gamma_i^\lambda|_{\mathbb{C}\setminus\mathcal{U}_i} : \mathbb{C} \setminus \mathcal{U}_i \to \mathcal{U}_i$ and define the map $T_{i,j}^\lambda : \mathcal{U}_i \to \mathcal{U}_j$ such that $T_{i,j}^\lambda = T_j^\lambda|_{\mathcal{U}_i}$. We consider the shift space

$$\Sigma := \{x = (x_n)_{n=0}^\infty : x_n \in \{1, \dots, k\}, x_n \neq x_{n+1}, n \in \mathbb{N}_0\} \subset \{1, \dots, k\}^{\mathbb{N}_0}$$

and define the projection map $\pi^\lambda : \Sigma \to \Lambda_\lambda \subset \mathbb{C}$, by $x \mapsto \lim_{n\to\infty} T_{x_0}^\lambda T_{x_1}^\lambda \cdots T_{x_n}^\lambda(z_0)$ where $z_0 \in \mathbb{C}$ is fixed and $\Lambda_\lambda := \{\lim_{n\to\infty} T_{x_0}^\lambda T_{x_1}^\lambda \cdots T_{x_n}^\lambda(z_0) : x \in \Sigma\}$ is the limit set for Γ_λ. We notice that $\pi^\lambda \in \mathcal{C}^\alpha(\Sigma, \mathbb{C})$ for some small $\alpha > 0$. The conformal probability measure μ_λ satisfies that $\mu_\lambda = \pi_*^\lambda \mu^\lambda$ for $\mathcal{L}_\lambda^* \mu^\lambda = \mu^\lambda$ where

$$\mathcal{L}_\lambda w(x) = \sum_{\substack{y\in\sigma^{-1}x \\ y\in\Sigma}} |dT_{y_0,x_0}^\lambda(\pi^\lambda y)|^{\mathscr{H}_\lambda} w(T_{y_0,x_0}^\lambda(\pi^\lambda y)), w : \Sigma \to \mathbb{R}, x \in \Sigma.$$

We know from [76] that the Hausdorff dimensions of the limit set for Γ is a real analytic function on the deformation space of a Schottky group, then the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \mathscr{H}_\lambda \in \mathbb{R}$ is $\mathcal{C}^m$. On the other hand, the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \pi^\lambda \in \mathcal{C}^\alpha(\Sigma, \mathbb{C})$ is $\mathcal{C}^m$ (we can use the same proof of Proposition 4.3.16, the main difference is that now when applying Corollary 4.3.8 we obtain $\mathcal{C}^m$ and not $\mathcal{C}^{m-1}$ as the maps T_i^λ are $\mathcal{C}^\infty$ and not just $\mathcal{C}^{m+\delta}$). Then the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \mathscr{H}_\lambda \log|dT^\lambda_{y_0,x_0}(\pi^\lambda y)\pi^\lambda| \in \mathbb{R}$ is $\mathcal{C}^m$ and by perturbation theory so is the map $\mathcal{I}_\epsilon \ni \lambda \mapsto \mu^\lambda \in \mathcal{C}^\alpha(\Sigma, \mathbb{R})^*$. Finally, we have that for $w : \mathbb{C} \to \mathbb{R}$ a compactly supported $\mathcal{C}^\infty$ function $\int w \circ \pi^\lambda d\mu^\lambda = \int w d\mu_\lambda$ and therefore the map $\lambda \mapsto \int w d\mu_\lambda$ is $\mathcal{C}^m$ by an application of Corollary 4.3.7, which concludes the proof. □

4.4.2 Generalisations

A careful look at Theorem 4.2.3 and to it proof allows to obtain similar results to the ones showed in Section 4.2 under much weaker hypotheses. This is the propose of this subsection. We start by modifying Definition 4.2.1 and replacing it by:

Definition 4.4.3. *Assume that $\delta, \epsilon \in (0,1)$, $k, l, m, n, p \in \mathbb{N} \setminus \{1\}$, $q \in \mathbb{N}$ and let Λ, Θ be open intervals $\Lambda, \Theta \subset \mathbb{R}$.*

1. *Let*
$$\mathcal{T} = \mathcal{T}(\Lambda, k, l, m, \delta) := \left\{ \{T_i^{(\lambda)}\}_{i=1}^k : \lambda \in \Lambda \right\}$$
be a family of contractions such that for $\lambda \in \Lambda$ and $i \in \{1, \ldots, k\}$:
$$T_i^{(\lambda)} = \tilde{T}_i(\lambda, \cdot),$$
where
   - *(i)* $\tilde{T}_i(\lambda, \cdot) \in \mathcal{C}^{l+\delta}([0,1],[0,1])$,
   - *(ii)* $\sup_{\lambda \in \Lambda} \| \frac{\partial}{\partial x} \tilde{T}_1(\lambda, \cdot) \|_{\mathcal{C}^0} < 1$,
   - *(iii)* $\tilde{T}_1(\cdot, \cdot) \in \mathcal{C}^m(\Lambda \times [0,1],[0,1])$, *and*
   - *(iv)* $\frac{\partial}{\partial x} \tilde{T}_i(0, x) = \frac{\partial}{\partial x} \tilde{T}_j(0, x)$ *for all i, j.*

2. *On a family $\mathcal{T}$ for every $\lambda \in \Lambda$, we define the limit set $\mathcal{K}(\lambda)$ as the unique non empty closed set $\mathcal{K} \subset [0,1]$ such that*
$$\mathcal{K} = \cup_{i=1}^k T_i^{(\lambda)} \mathcal{K}.$$

3. *We define $(\mathcal{T}, \mathcal{G})$, where*
$$\mathcal{G} = \mathcal{G}(\Theta, k, n, p, \epsilon) := \left\{ \left\{ g_i^{(\theta)} \right\}_{i=1}^k : \theta \in \Theta \right\}$$
is a family of weight functions such that

(a)

$$\sum_{i=1}^{k} \left\| g_i^{(\theta)} \right\|_{\mathcal{C}^0} \operatorname{Lip}\left(T_i^{(\lambda)}\right) < 1 \text{ for all } \lambda \in \Lambda, \theta \in \Theta;$$

and

(b) for every $\theta \in \Theta, i \in \{1, \ldots, k\}$ *:*

$$g_i^{(\theta)} = \tilde{g}_i(\theta)$$

where for some $\beta \in (0, 1/2)$ *we have*

(i) $\tilde{g}_i(\theta) \in \mathcal{C}^{n+\epsilon}([0,1], \mathbb{R}^+)$,

(ii) $\tilde{g}_i(\cdot) \in \mathcal{C}^q\left(\mathcal{I}, \mathcal{C}^{n+\epsilon}([0,1], \mathbb{R}^+)\right)$.

If we do not consider the normalisation condition on the weight functions, we require a generalised definition of stationary measures. In order to deal with this we introduce the next definition.

Definition 4.4.4. *Given the families* $(\mathcal{T}, \mathcal{G})$, *define* $h_i^{(\lambda,\theta)} := \left(g_i^{(\theta)}\right)^{s^{\lambda,\theta}}$, *where* $s^{\lambda,\theta} \in [0,1]$ *is unique solution of* $P\left(s^{\lambda,\theta} \log\left(g_{x_0}^{(\theta)}(\pi^{(\lambda)}(\sigma x))\right)\right) = 0$ *and* P *is the Pressure. A generalized stationary measure* $\mu = \mu_{\lambda,\theta}$ *is the unique probability measure on* $[0,1]$ *that satisfies*

$$\int f(x) d\mu(x) = \sum_{i=1}^{k} \int h_i^{(\lambda,\theta)}(x) f(T_i^{(\lambda)}(x)) d\mu(x),$$

for any continuous function $f : [0,1] \to \mathbb{R}$.

Under the hypotheses of Definition 4.4.3, a step-by-step equal proof that the one given for Theorem 4.2.3 gives us the following result:

Theorem 4.4.5. *Let fix* $a \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\}$ *and* $\rho \in (0,1)$. *On* $(\mathcal{T}, \mathcal{G})$, *for the generalized stationary probability measure* $\mu_{\lambda,\theta}$ *with* $\lambda \in \Lambda, \theta \in \Theta$, *or in the case* $\Lambda = \Theta$, *for the generalized stationary probability measure* $\mu_{\lambda,\lambda} = \mu_\lambda$ *for* $\lambda \in \Lambda$, *we have:*

1. *For* $\theta \in \Theta$ *and* $f \in \mathcal{C}^{a+\rho}(\hat{\mathcal{K}}, \mathbb{R})$, *where* $\hat{\mathcal{K}} \supset \cup_{\lambda \in \Lambda} \mathcal{K}(\lambda)$, *the map* $F : \Lambda \to \mathbb{R}$ *defined by*

$$F(\lambda) = \int f d\mu_{\lambda,\theta}$$

belongs to $\mathcal{C}^r(\Lambda, \mathbb{R})$ *with* $r = \min\{l-1, m-1, a-1\}$.

2. *For* $\lambda \in \Lambda$ *and* $f \in \mathcal{C}^1(\hat{\mathcal{K}}, \mathbb{R})$, *the map* $F : \Theta \to \mathbb{R}$ *defined by*

$$F(\theta) = \int f d\mu_{\lambda,\theta}$$

belongs to $\mathcal{C}^q(\Theta, \mathbb{R})$.

3. *For* $\Lambda = \Theta$ *and* $f \in \mathcal{C}^{a+\rho}(\hat{\mathcal{K}}, \mathbb{R})$, *the map* $F : \Lambda \to \mathbb{R}$ *defined by*

$$F(\lambda) = \int f d\mu_\lambda$$

belongs to $\mathcal{C}^r(\Lambda, \mathbb{R})$ *with* $r = \min\{l-1, m-1, a-1, n-1, q\}$.

An easy example of application of Theorem 4.4.5 that Theorem 4.2.3 fails is the case that $x_0 \in [0,1] \setminus \cup_{\lambda \in \Lambda} \mathcal{K}(\lambda)$ and $f(x) = |x - x_0|$.

We end this subsection with two examples. In the first we can apply our theorem and it is possible to experimentally see the regularity of the map $F(\lambda)$. In the second, the hypothesis on the smoothness of the contractions is not satisfied. In this case, experimentally the map $F(\lambda)$ looks $\mathcal{C}^0$ but not $\mathcal{C}^1$, however we cannot prove it, as our method of composition of operator does not work. The first example is the following:

Example 4.4.6. *Let us consider* $\Lambda = \Theta = [1/6, 1/3]$, $x \in [0,1], n \in \mathbb{N}, \lambda \in \Lambda$,

$$\begin{aligned}
\phi(x,n) &= x^{n+1}\sin(1/x) \in \mathcal{C}^n(\mathbb{R},\mathbb{R}) \setminus \mathcal{C}^{n+1}(\mathbb{R},\mathbb{R}),\\
T_1^{(\lambda)}(x) &= \lambda x + \phi(\lambda - 0.25, 3) + 0.01,\\
T_2^{(\lambda)}(x) &= \lambda x + \frac{2}{3} + \phi(\lambda - 0.25, 3),\\
g_1^{(\lambda)}(x) &= \lambda \mathbb{1}_{[0,1/2)}(x) + (1-\lambda)\mathbb{1}_{[1/2,1]}(x),\\
g_2^{(\lambda)}(x) &= (1-\lambda)\mathbb{1}_{[0,1/2)}(x) + (\lambda)\mathbb{1}_{[1/2,1]}(x), \ \textit{and}\\
f(x) &= \begin{cases} -x & \textit{if } x \in [0,1/2) \\ x^2 & \textit{if } x \in [1/2,1]. \end{cases}
\end{aligned}$$

Then the map $F : \Lambda \to \mathbb{R}$, *defined by* $F(\lambda) = \int f(x) d\mu_\lambda(x)$, *belongs to* $C^1(\Lambda, \mathbb{R})$. *Moreover, for any interval* $\Lambda' \subset [1/6, 1/4)$ *or* $\Lambda' \subset (1/4, 1/3]$, *we have that* $F|_{\Lambda'} \in C^\infty(\Lambda', \mathbb{R})$.

The second example, where our results are not longer valid, is the following:

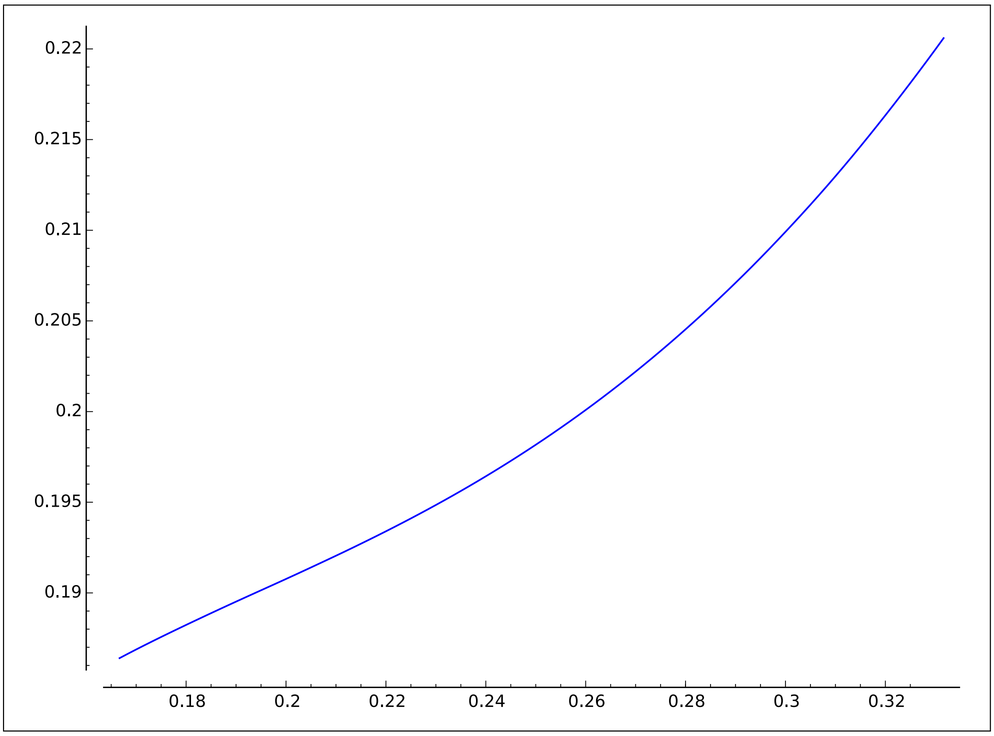


Figure 4.2: Graph of $F : \Lambda \to \mathbb{R}$ in Example 4.4.6

Example 4.4.7. *Let us consider* $\Lambda = \Theta = [1/6, 1/3]$, $x \in [0,1], n \in \mathbb{N}, \lambda \in \Lambda$,

$$\begin{aligned}
T_1^{(\lambda)}(x) &= \lambda x + \phi(\lambda - 0.25, 1) + 0.01,\\
T_2^{(\lambda)}(x) &= \lambda x + \frac{2}{3} + \phi(\lambda - 0.25, 1),\\
g_1^{(\lambda)}(x) &= \lambda \mathbb{1}_{[0,1/2)}(x) + (1-\lambda)\mathbb{1}_{[1/2,1]}(x),\\
g_2^{(\lambda)}(x) &= (1-\lambda)\mathbb{1}_{[0,1/2)}(x) + (\lambda)\mathbb{1}_{[1/2,1]}(x), \ \textit{and}\\
f(x) &= \begin{cases} -x & \textit{if } x \in [0,1/2) \\ x^2 & \textit{if } x \in [1/2,1]. \end{cases}
\end{aligned}$$

Does the map $F : \Lambda \to \mathbb{R}$, *defined by* $F(\lambda) = \int f(x) d\mu_\lambda(x)$, *belongs to* $C^0(\Lambda, \mathbb{R})$?

4.4.3 Comparison

Our results were compared to the one obtained in [86] once we finished to write the previous part. We conclude that we can apply our methods to obtain similar results that in [86], indeed we can do the following. Consider an iterated function scheme $\mathcal{T}$ as in Definition 4.4.3 such that the sets $T_i^{(\lambda)}[0,1]$ are pairwise disjoint

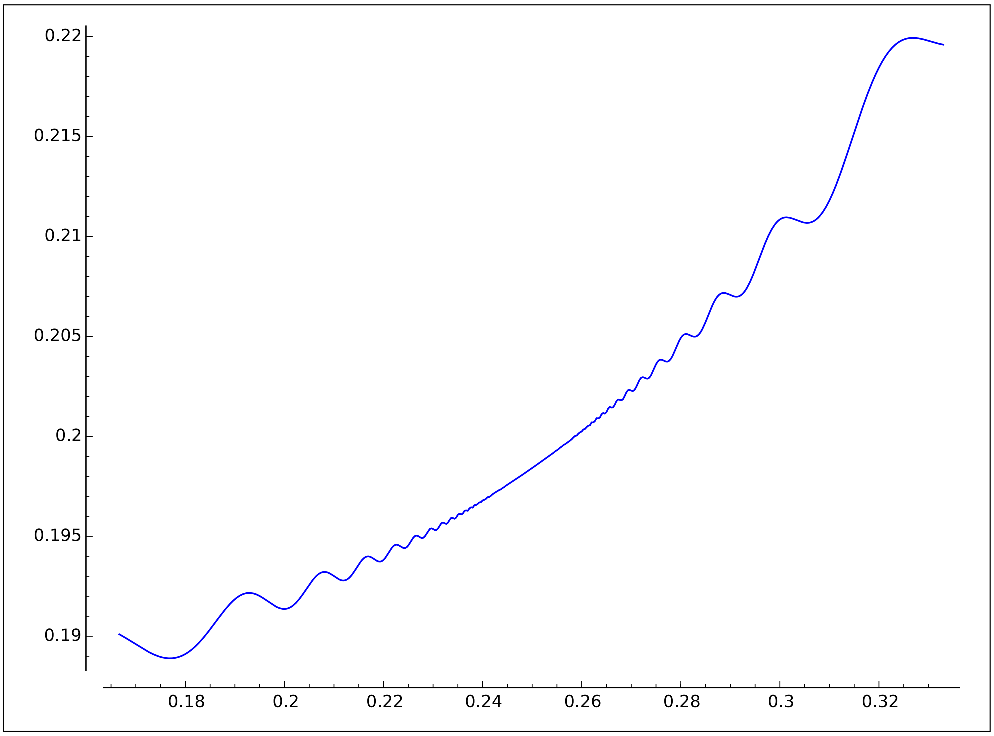


Figure 4.3: Graph of $F : \Lambda \to \mathbb{R}$ in Example 4.4.7

for $i \in \{1, \dots, k\}$ and such that $m = l$. Recall the definition of the projection map $\pi^{(\lambda)} : \mathcal{X} \to \mathbb{R}$ and the definition of the pressure P (Definition 1.2.9). It is well known that the Hausdorff dimension of the limit set $\mathcal{K}(\lambda)$, that we call by $\dim_H(\mathcal{K}(\lambda))$, corresponds to the unique $s \in [0,1]$ such that $P(s\psi^{(\lambda)}) = 0$, where $\psi^{(\lambda)} : \mathcal{X} \to \mathbb{R}$ is defined by $\psi^{(\lambda)}(x) = \log|dT^{(\lambda)}_{x_0}(\pi^{(\lambda)}(\sigma x))|$. We directly obtain from our proofs the following theorem:

Theorem 4.4.8. *1. The dependence $\mathcal{I} \ni \lambda \mapsto \dim_H(\mathcal{K}(\lambda))$ of the Hausdorff dimension of the limit set is $\mathcal{C}^{m-2}$.*

2. For $\alpha \in (0,1)$ small enough, the Gibbs measure μ_φ of $\varphi = \dim_H(\mathcal{K}(\lambda))\psi^{(\lambda)} \in C^\alpha(\mathcal{X}, \mathbb{R})$ has a $\mathcal{C}^{m-2}$ dependence on $\lambda \in \mathcal{I}$, when we consider μ_φ as an operator on $C^\alpha(\mathcal{X}, \mathbb{R})^*$.

From Theorem 1.1 and Theorem 1.2 in [86], under hypotheses similar to ours, it is possible to conclude that the regularity is C^{m-1} instead of C^{m-2} as we could prove. In their analog to the part 2. of the previous theorem, however, in [86] is necessary to consider μ_φ as an operator on $C^{\alpha'}(\mathcal{X}, \mathbb{R})^*$, where $\alpha' \in (r^\beta, 1)$ and $r \in (0,1)$ depends on the rate of contraction of $T^{(\lambda)}$. Whereas we need $\alpha \in (0,1)$ small enough so that $2^\alpha \|dT_1\|_{\mathcal{C}^0} < 1$ and $\pi^{(\lambda)} : \mathcal{X} \to \mathbb{R}$ is α-Hölder.

Glossary

μ-a.e. Subsection 1.2.1.

Birkhoff ergodic theorem Theorem 1.2.7.

Conformal repeller Definition 1.2.18.

Convergence in law Definition 2.2.10.

Diffeomorphism Definition 1.2.1.

Dynamical system Subsection 1.2.1.

Entrance time Definition 2.2.2.

Ergodic probability measure Definition 1.2.6.

Escape rate Section 3.1.

Exponential random variable Definition 2.2.9.

Gibbs measure Definition 1.2.12.

Homeomorphism Definition 1.2.1.

Invariant probability measure Definition 1.2.4.

Iterated function scheme Definition 1.2.36.

Measure preserving dynamical system Definition 1.2.5.

Measure theoretic entropy Definition 1.2.8.

Pressure Definition 1.2.9.

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