Common fixed points of set-valued mappings in hyperconvex metric spaces

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May 14, 2018

Definition

A metric space (M, d) is said to be hyperconvex if for any family of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ in M such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, for any $\alpha, \beta \in \Gamma$, then $\bigcap_{\alpha\in\Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$.

Basic definitions and properties

Definition

A metric space (M, d) is said to be hyperconvex if for any family of closed balls $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma}$ in M such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, for any $\alpha, \beta \in \Gamma$, then $\bigcap_{\alpha\in\Gamma} B(x_\alpha, r_\alpha) \neq \emptyset$.

This definition can be seen as a binary intersection property of balls combined with a metric convexity.

In linear spaces, hyperconvexity reduces to the binary intersection property of the closed balls.

Examples of hyperconvex spaces are the real line with the usual metric and $l^\infty(I)$ for any set I.

Admissible and sub-admissible sets

Definition

Let (M, d) be a metric space. Let A be a bounded nonempty subset of M. Set

$$\operatorname{co}(A) = \cap \{B : B \text{ is a closed ball such that } A \subset B\}.$$

If A is an intersection of closed balls, we will say A is an admissible subset of M. A is called sub-admissible if for each finite subset D of A, $\operatorname{co}(D) \subset A$.

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It is worth also noting that if M is hyperconvex, then each admissible set in M is also hyperconvex.

Fundamental Properties

Theorem

Let (M, d) be a metric space.

1. *There exists an isometric embedding $i : M \to l^\infty(M)$.*
2. *If M is hyperconvex then it is complete.*
3. *M is hyperconvex iff for any metric space N which contains isometrically M, there exists a nonexpansive retraction $r : N \to M$, i.e. r is nonexpansive and $r(x) = x$ for any $x \in M$.*

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Throughout this work, we will identify M with $i(M)$. If we assume that M is hyperconvex (and since hyperconvexity is preserved by isometry), then, by statement (3), there exists a nonexpansive retraction $r : l^\infty(M) \to M$.

Set-valued mappings

Definition

For a set-valued mapping $Q : X \to 2^Y$, we associate two mappings $Q^- : Y \to 2^X$, the inverse of Q, and $Q^* : Y \to 2^X$, the dual of Q, defined by $Q^-(y) = \{x \in X : y \in Q(x)\}$, and respectively $Q^*(y) = \{x \in X : y \notin Q(x)\} = X \setminus Q^-(y)$. The values of Q^- and Q^* are called the fibers, respectively the cofibers of Q.

Set-valued mappings

Definition

For topological spaces X and Y, a set-valued mapping $Q : X \to 2^Y$ is said to be

(i) upper semicontinuous if for every open subset G of Y, the set $\{x \in X : Q(x) \subseteq G\}$ is open;

(ii) lower semicontinuous if for every open subset G of Y, the set $\{x \in X : Q(x) \cap G \neq \emptyset\}$ is open;

(iii) closed if its graph (that is, the set $\mathrm{Gr}\, Q = \{(x, y) \in X \times Y : y \in Q(x)\}$) is a closed subset of $X \times Y$;

(iv) compact if $Q(X)$ is contained in a compact subset of Y.

The following is a hyperconvex version of the Himmelberg's fixed point theorem

Theorem

Let H be a hyperconvex metric space. If $Q : H \to 2^H$ is a compact and upper semicontinuous set-valued mapping with nonempty admissible values, then there exists $x_0 \in H$ such that $x_0 \in Q(x_0)$.[a]

[a] X. Wu, B. Thompson, G.X. Yuan, *Fixed point theorems of upper semicontinuous multivalued mappings with applications in hyperconvex metric spaces*, J. Math. Anal. Appl. 276 (2002), 80–89.

Hyperconvex Fan-Browder's Theorem

The following is a hyperconvex version of the Fan-Browder's fixed point theorem

Theorem

Let H be a compact hyperconvex metric space. If $Q : H \to 2^H$ is a set-valued mapping with nonempty sub-admissible values and open fibers, then there exists $x_0 \in H$ such that $x_0 \in Q(x_0)$.[a]

[a] G.X.Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Monographs and Textbooks in Pure and Applied Mathematics, 218. Marcel Dekker, Inc., New York, 1999.

Many researchers have devoted their efforts to generalize the KKM principle to different new classes of set-valued mappings. Let us recall one of them in the context of hyperconvex metric spaces.

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Definition

Let Y be a nonempty set and H a hyperconvex metric space. A set-valued mapping $P : Y \to 2^H$ is said to be a generalized (metric) *KKM* mapping if for every finite subset $\{y_1, \ldots, y_n\}$ of Y, there exists $\{h_1, \ldots, h_n\} \subset H$ such that for each nonempty index set $I \subseteq \{1, \ldots, n\}$, we have $\mathrm{co}(\{h_i : i \in I\}) \subset \bigcup_{i \in I} P(y_i)$.[a]

[a] W. A. Kirk, B. Sims, G. X. Yuan, *The Knaster-Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications*, Nonlinear Anal. 39 (2000), Ser. A: Theory Methods, 611–627.

Theorem

Let Y be a nonempty set and H be a hyperconvex metric space. Suppose $P : Y \to 2^H$ is a generalized KKM mapping with closed nonempty values and $P(y_0)$ is a compact set, for at least one $y_0 \in Y$. Then $\bigcap_{y \in Y} P(y) \neq \emptyset$.[a]

[a] W. A. Kirk, B. Sims, G. X. Yuan, *The Knaster-Kuratowski and Mazurkiewicz theory in hyperconvex metric spaces and some of its applications*, Nonlinear Anal. 39 (2000), Ser. A: Theory Methods, 611–627.

First Common Fixed Point Theorem

Using KKM principle, we establish our first common fixed point theorem.

Theorem

Let H be a hyperconvex metric space and Y be a nonempty set. Let $T : H \times Y \to 2^H$ be a compact set-valued mapping with nonempty admissible values. Assume that:

(i) *for each $y \in Y$, the mapping $T(\cdot, y)$ is closed;*

(ii) *for each $x \in H$, the mapping $T(x, \cdot)$ is a generalized KKM mapping.*

Then, the family of set-valued mappings $\{T(\cdot, y)\}_{y \in Y}$ has a common fixed point, that is, there exists $x_0 \in H$ such that $x_0 \in \bigcap_{y \in Y} T(x_0, y)$

Definition

Let Y be a nonempty set and H a hyperconvex metric space. Let $\mathcal{T}$ be a family of set-valued mappings with nonempty values from Y into H. We say that the family $\mathcal{T}$ is generalized equi-*KKM* if for any finite subset $\{y_1, \ldots, y_n\}$ in Y, there exists $\{h_1, \cdots, h_n\} \subset H$ such that

$$\mathrm{co}(\{h_i;\ i \in I\}) \ \subset \bigcup_{i \in I} T(y_i),$$

for any nonempty subset I of $\{1, \cdots, n\}$ and for all $T \in \mathcal{T}$.

Second Common Fixed Point Theorem

Theorem

Let H be a hyperconvex metric space and Y a nonempty set. Let $T : H \times Y \to 2^H$ be a compact set-valued mapping satisfying the following conditions:

(i) *for each $y \in Y$, the set $\{x \in H : \ x \in T(x, y)\}$ is closed;*

(ii) *the family of set-valued mappings $\{T(x, \cdot)\}_{x \in H}$ is generalized equi-KKM on Y.*

Then the family of set-valued mappings $\{T(\cdot, y)\}_{y \in Y}$ has a common fixed point.

Intersection Result

In order to obtain new common fixed point theorems we need the following intersection result.

Theorem

Let H and Y be two hyperconvex metric spaces such that H is compact. If $P : Y \to 2^H$ is a closed set-valued mapping with nonempty admissible values and sub-admissible cofibers, then $\bigcap_{y \in Y} P(y) \neq \emptyset$.

Theorem

Let H and Y be hyperconvex metric spaces. Assume that H is compact. Let $T : H \times Y \to 2^H$ be a set-valued mapping with nonempty admissible values. Assume that:

(i) *the set-valued mappings $T(x, \cdot)$ and $T(\cdot, y)$ are closed for all $x \in H$ and $y \in Y$;*

(ii) *for each $(x, h) \in H \times H$, the set $\{y \in Y : h \notin T(x, y)\}$ is sub-admissible.*

Then, the family of set-valued mappings $\{T(\cdot, y)\}_{y \in Y}$ has a common fixed point.

Fourth Fixed Point Theorem

Theorem

Let H and Y be hyperconvex metric spaces. Assume that H is compact. Let $T : H \times Y \to 2^H$ be a closed set-valued mapping with nonempty admissible values. Assume that:

(i) *for each $y \in Y$, the set $\{x \in H : x \in T(x, y)\}$ is admissible;*
(ii) *for each $x \in H$, the set $\{y \in Y : x \notin T(x, y)\}$ is sub-admissible.*

Then, the family of set-valued mappings $\{T(\cdot, y)\}_{y \in Y}$ has a common fixed point.

Theorem

Let H and Y be compact hyperconvex metric spaces and $T : H \times Y \to 2^H$ be a set-valued mapping satisfying the following conditions:

(i) *T has open graph and nonempty sub-admissible values;*

(ii) *for each $y \in Y$, the set $\{x \in H : x \in T(x, y)\}$ is sub-admissible;*

(iii) *for each $x \in H$, the set $\{y \in Y : x \notin T(x, y)\}$ is admissible.*

Then, the family of set-valued mappings $\{T(\cdot, y)\}_{y \in Y}$ has a common fixed point.

Applications

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- (i) Minimax inequality
- (ii) Variational Inequality
- (iii) Inward fixed point theorems.

Definition

Let M be a metric space. A function $f : M \to \mathbb{R}$ is said to be metric quasiconvex (resp., metric quasiconcave) if for each $\lambda \in \mathbb{R}$, the set $\{x \in M : f(x) \leq \lambda\}$ (resp., the set $\{x \in M : f(x) \geq \lambda\}$) is sub-admissible.

Quasiconvex Functions

Definition

Let M be a metric space. A function $f : M \to \mathbb{R}$ is said to be metric quasiconvex (resp., metric quasiconcave) if for each $\lambda \in \mathbb{R}$, the set $\{x \in M : f(x) \leq \lambda\}$ (resp., the set $\{x \in M : f(x) \geq \lambda\}$) is sub-admissible.

Definition

Let M be a metric space. We say that a family $\mathcal{F}$ of real functions defined on M is metric quasiconvex if for each nonempty finite subfamily $\{f_1, \ldots, f_n\}$ of $\mathcal{F}$ and any set $\{x_1, \ldots, x_n\} \subset M$,

$$\min_{1 \leq i \leq n} f_i(x) \leq \max_{1 \leq i \leq n} f_i(x_i) \text{ for all } x \in co\{x_1, \ldots, x_n\}.$$

.

Minimax inequality

Theorem

Let H be a compact hyperconvex metric space and $\mathcal{F}$ be a family of real functions defined on $H \times H$ satisfying the following conditions:

(i) *each $f \in \mathcal{F}$ is lower semicontinuous on $H \times H$;*

(ii) *each $f \in \mathcal{F}$ is metric quasiconvex in the second variable;*

(iii) *for each $x \in H$ the family of functions $\{f(x, \cdot) : f \in \mathcal{F}\}$ is metric quasiconvex.*

Then, we have

$$\inf_{x \in H} \sup_{f \in \mathcal{F}} f(x, x) \leq \sup_{f \in \mathcal{F}} \sup_{x \in H} \min_{h \in H} f(x, h).$$

Proof

We may suppose that

$$s := \sup_{f\in\mathcal{F}} \sup_{x\in H} \min_{y\in H} f(x,h) < \infty.$$

If $r \geq s$ from first theorem, we know that there exists $x_r \in H$ such that $f(x_r, x_r) \leq r$ for each $f \in \mathcal{F}$. As H is compact, we may assume that $\lim_{r\to s+} x_r =: \bar{x}$ exists. Since the functions f are lower semicontinuous, it follows that

$$f(\bar{x},\bar{x}) \leq \liminf_{r\to s+} f(x_r, x_r) \leq \liminf_{r\to s+} \; r = s,$$

for all $f \in \mathcal{F}$. Consequently, we have

$$\inf_{x\in H} \sup_{f\in\mathcal{F}} f(x,x) \leq \sup_{f\in\mathcal{F}} f(\bar{x},\bar{x}) \leq \sup_{f\in\mathcal{F}} \sup_{x\in X} \min_{h\in H} f(x,h).$$

(h, y)-Metric Quasiconcave

Definition

Let H and Y be two metric spaces. A function $f : H \times Y \to \mathbb{R}$ is said to be (h, y)-metric quasiconcave if for each finite subset $\{(h_1, y_1), \ldots, (h_n, y_n)\}$ of $H \times Y$, and each $h \in \text{co}(\{h_1, \ldots, h_n\})$ there exists $y \in \text{co}(\{y_1, \ldots, y_n\})$ such that

$$\min_{1 \le i \le n} f(h_i, y_i) \le f(h, y).$$

Theorem

Assume that H and Y are two hyperconvex metric spaces and H is compact. Let $S : H \to 2^Y$ be an upper semicontinuous multivalued mapping with nonempty compact admissible values and $\varphi : H \times Y \times H \to \mathbb{R}$ an upper semicontinuous function. Assume that:

(i) *for each $(x, z) \in H \times H$, there exist $y \in S(x)$ and $h \in H$ such that $\varphi(h, y, z) \geq 0$;*

(ii) *for each $z \in H$, $\varphi(\cdot, \cdot, z)$ is (h, y)-metric quasiconcave;*

(iii) *for each $(h, y) \in H \times Y$, the set $\{z \in H : \varphi(h, y, z) < 0\}$ is sub-admissible;*

(iv) *for each $(h, z) \in H \times H$, the set $\{y \in Y : \varphi(h, y, z) \geq 0\}$ is admissible.*

Then, there exist $x_0 \in H$ and $y_0 \in S(x_0)$ such that $\varphi(x_0, y_0, z) \geq 0$ for all $z \in H$.

We can prove the following extensions of the Schauder' fixed point theorem:

Theorem

Let C be a nonempty compact convex set in a Hilbert space H. Let $T : C \to H$ be a continuous and inward map. Then T has a fixed point.

Inward Extensions of The Schauder's Theorem

We can prove the following extensions of the Schauder' fixed point theorem:

Theorem

Let C be a nonempty compact convex set in a Hilbert space H. Let $T : C \to H$ be a continuous and inward map. Then T has a fixed point.

For normed space we have:

Theorem

Let C be a nonempty compact convex set in a normed space E. Let $T : C \to E$ be a continuous and inward map. Then T has a fixed point.

Thank you very much !