DS Seminar 2018
Felipe Riquelme

- Katok conjecture on the realizable entropies
- Geodesic flows without compacity
- Otal-Peigne topological entropy identity
- Einsdler-Kadyrov-Pohl inequality
- Thermodynamics Formalism in negative curvature, Paulin-Pollicott-Schapira

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Abstract In the talk Felipe Riquelme proved Katok conjectures on the realizable entropies for geodesic flows without compacity hypotheses using purely geometric tools. The proof uses Otal-Peigne topological entropy identity being the main machinary the theorem in ["Amount of failure of upper-semicontinuity of entropy in noncompact rank one situations, and Hausdorff dimensions", Kadyrov and Pohl] that gives an upper bound for the limsup of $h_{\mu_j}(f)$ in terms of nu, when μ_j is a sequence of f-invariant prob. meas. that conv. to the meas. ν in the weak*-top.

Motivation

Let (X, f) top. dyn. syst. The general goal is to describe the set $E(f)$ of realizable entropies with respect to μ-ergodic f-inv. measures. Katok proved in 1984 that if X is a surface and f is $C^{1+\alpha}$, then the set $E(f)$ contains the interval $[0, h_{top}(f)]$. Katok conjectured this is still valid for any smooth systems.

On the other hand, there are uniquely ergodic minimal homeomorphisms on compact manifold hanving $h_{top} > 0$. [Construction of curious minimal uniquely ergodic homeomorphisms on manifolds: the Denjoy-Rees technique, François Béguin, Sylvain Crovisier, Frédéric Le Roux,arXiv:math/0605438].

And geodesic flows on compact surfaces of negative curvature satisfies Katok conjecture [Ergodic universality of some topological dynamical systems, Anthony Quas, Terry Soo, arXiv:1208.3501].

The particular goal was to describe the set $E(f)$ of realizable entropies with respect to μ-ergodic f-inv. measures for geodesic flows g_t on $X = TM/\Gamma$ for M simply connected Riemmanian manifolds of curvature $-b^2 \leq k \leq -a^2 < 0$ and Γ a subgroup of $Isom^+(M)$ that is discrete and non-elementary.

The canonical example: $M = \mathbb{H}$, $\Gamma = PSL(2, \mathbb{Z})$, $Isom^+(M) = PSL(2, \mathbb{R})$, $X = TM/\Gamma, g_t : X \to X$.

Main theorem [Velozo, Riquelme]: (X, g_t) verifies Katok conjecture.

Proof sketch in 3 parts for the punctured torus S:

1. Preliminaries
2. Thermodynamics Formalism
3. Zero-Temperature States

Preliminaries

Thm [Otal-Peigne 2004] $h_{top}(g)$ =exponential growth rate of the group Γ ($EGRG(\Gamma)$), where

$$EGRG(\Gamma) := \limsup_{r\to\infty} \frac{1}{r} \log \#\{\gamma \in \Gamma : d(i, \gamma i) \leq r\}.$$

In the finite volume case $EGRG = dim(S) - 1$, in our case, $EGRG = 1$. In our case the cusp of S is isometrically equivalent to

$$\{z \in \mathbb{H} : Im(z) >= t\}/ < z \mapsto z + b >$$

for some $t > 0, b \in \mathbb{R} \setminus 0$.

Claim 1. $EGRG(< z \mapsto z + b >) = 1/2$.

Thermodynamic formalism

Let $F : TS \to \mathbb{R}$ continuous and bounded. The pressure is defined by

$$P(F) := sup\{h_\mu(g) + \int F d\mu : \mu \in \mathcal{M}_F\}.$$

Properties of the map $t \mapsto P(tF)$.

1. It is convex and continuous.
2. If F is Holder F admits at most one equilibrium measure [F. Paulin, M. Pollicott and B. Schapira, Equilibrium states in negative curvature. Astérisque No. 373 (2015), viii+281 pp. ISBN: 978-2-85629-818-3].
3. If $F > 0$ and decreases to zero through the cusp, then for any $t > 0$ we have that tF admits an eq. measure μ_t.
4. The map $t \mapsto P(tF)$ is C^1 on $(0, \infty)$, moreover

$$\frac{dP(tF)}{dt}|_{t=t_0} = \int F d\mu_{t_0}.$$

Claim 2. The map $t \mapsto h_{\mu_t}(g)$ is C^0.

The proof uses that the map $t \mapsto \mu_t$ is C^0 for the weak*-top. and the identity $h_{\mu_t}(g) = P(tF) - t \int F d\mu_t$, where $t \mapsto \int F d\mu_t$ is continuos if F decreases to zero through the cusp.

The Einsdler-Kadyrov-Pohl inequality says that if ν is an acc. point of μ_j then

$$\limsup_{j\to\infty} h_{\mu_j}(g) \leq \|\nu\| h_\nu(g) + (1 - \|\nu\|)\frac{1}{2}.$$

In particular, if $\|\nu\| = 1$ gives upper semi-continuity and if $\|\nu\| = 1$ gives that $\limsup_{j\to\infty} h_{\mu_j}(g) \leq \frac{1}{2}$.

Prop.

$$\limsup_{t\to t_0} P(tF) \leq \|t\nu\|(h_\nu(g) + t_0 \int F d\nu) + (1 - \|\nu\|)\frac{1}{2}.$$

In Riquelme-Velozo-Iommi, Velozo-Riquelme and Velozo it is exploited an interesting relationship between the entropy at infinity and the critical exponent of the parabolic points. A result that shares similitudes with the known relationship between the topological entropy and the EGRG.

Zero-Temperature-States

Def. A ZTS is an acc. point of μ_t as $t \to \infty$.

Prop. A ZTS satisfies:

1. $\|\mu_\infty\| = 1$.
2. $\int F d\mu_\infty = \sup_\mu \int F d\mu$.
3. $\lim_{t\to\infty} h_{\mu_t}(g) = h_{\mu_\infty}(g)$.

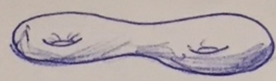
Proof Sketch. By Einsdler-Kadyrov-Pohl inequality $\limsup_{t\to\infty} h_{\mu_t}(g) \leq h_{\mu_\infty}(g)$. For the other inequality,

$$h_{\mu_\infty}(g)+t\int F d\mu_\infty \leq h_{\mu_t}(g)+t\int F d\mu_t \leq h_{\mu_t}(g)+t\int F d\mu_\infty.$$

Proof main theorem

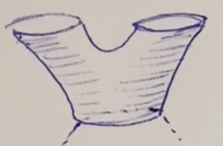
Define $F(v) = \frac{1}{1+d(v,\mathcal{O})}$, where $\mathcal{O}$ is a periodic orbit for the flow. It satisfies: maximal on $\mathcal{O}$, decreases to zero through the cusp, positive and Holder. Then $\lim_{t\to\infty} h_{\mu_t}(g) = 0$.

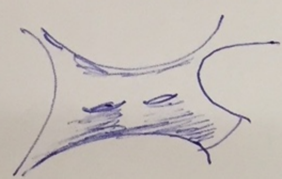
Examples of "good" manifolds for the thm.



compact manifold

punctured torus

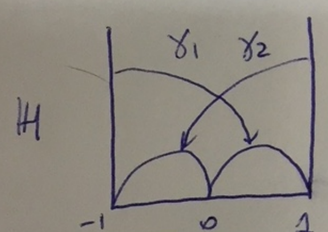




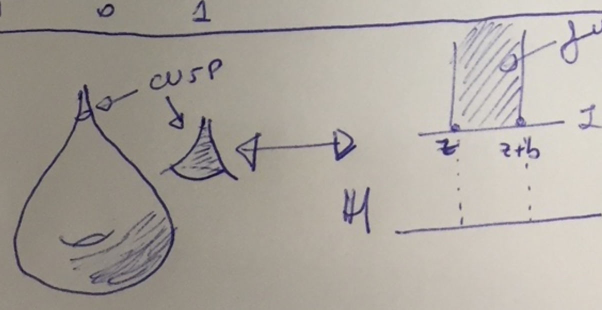
geometrically finite

$\mathbb{Z}$- covering of a compact surface

Example for Γ



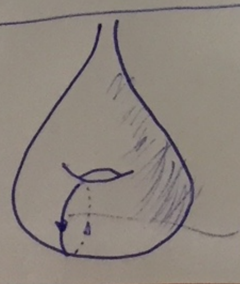
$\Gamma = \langle \gamma_1, \gamma_2 \rangle$



CUSP IS ISOMETRICALLY EQUIVALENT TO

$\{z \in \mathbb{H} : Im(z) \geq t\} / \langle z \mapsto z+ \ldots$

Proof of main thm



θ: periodic orbit for the flow