The Radius of the Electron

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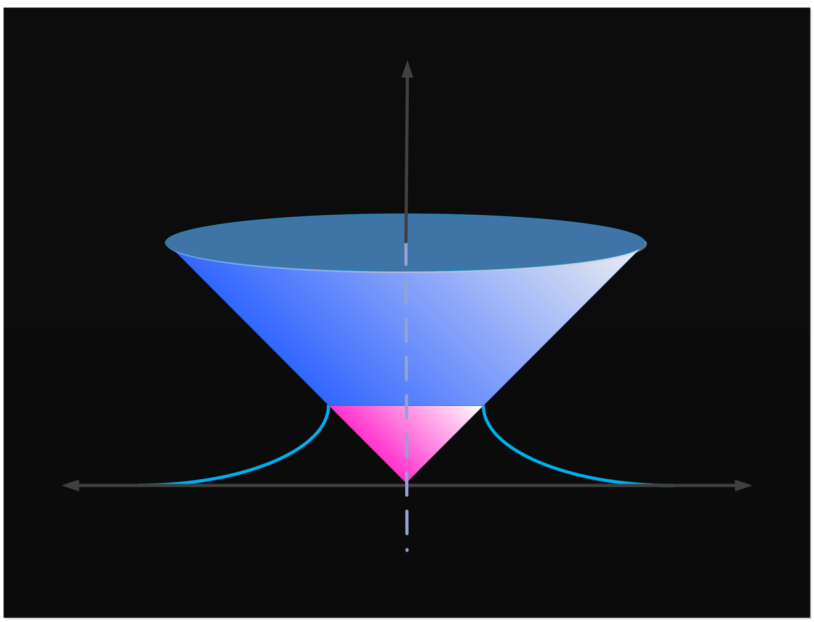

"It is therefore impossible to get all the mass to be electromagnetic in the way we had hoped. It is not a legal theory if we have nothing but electrodynamics. Something else has to be added."

R.P. Feynman, *The Feynman Lectures*, chapter 28

Abstract

A Lagrangian formulation for the vacuum gauge electron is developed by drawing on analogies with classical theories of acoustic fields in continuous media. The theory has several notable underlying features including a second rank vacuum stress tensor representing the deformed state of the vacuum in the neighborhood of the charge. The vacuum tensor is characterized by a zero determinant and operates relative to a flat metric so that vacuum deformations cannot be directly associated with spacetime curvature.

The theory of the vacuum gauge electron is based solely on the requirement to enforce causality in the electromagnetic field. Like the fields of the particle itself, the associated Lagrangian is a causal Lagrangian having a naturally built in symmetry for the propagation of vacuum waves. Among other things, the motion of the fields define a particle radius and provide stability relative to an arbitrary frame of reference.

Accelerated motions of the electron are easily accommodated by the vacuum Lagrangian with the addition of an acceleration strain tensor. The total Lagrangian then generates a Noether current identical to the symmetric stress tensor derived in the conventional electromagnetic theory.

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1.1 Moving Velocity Fields

According to Quantum Electrodynamics (QED), the interaction between two charged particles is mediated by the exchange of virtual photons—quanta of the electromagnetic field having a transverse polarization ϵ^{ν}_{T}. Such photons are associated with the acceleration fields of the charges involved. Apparently one can arrive at the r^{-2} Coulomb force of a large (classical) object interacting with a test charge by summing over a large number of interactions involving virtual photons. This is important information for anyone attempting to design a classical theory of the electron since there does not seem to be any reason to assign a Coulomb field to the particle.

On the other hand, an important property of QED is the ability create and destroy electrons and positrons by acting on eigenstates with creation and annihilation operators

$$b^{s\dagger}_{\boldsymbol{p}} \quad b^{s}_{\boldsymbol{p}} \quad d^{s\dagger}_{\boldsymbol{p}} \quad d^{s}_{\boldsymbol{p}} \tag{1.1}$$

This suggests that a valid classical theory of the electron might be possible if its fields were constrained by causality. Electric and magnetic components would then be written

$$\mathbf{E} = \mathbf{E}_{M} \cdot \vartheta \tag{1.2a}$$
$$\mathbf{B} = \mathbf{B}_{M} \cdot \vartheta \tag{1.2b}$$

The subscript M indicates the well known Maxwell fields while the causality sphere, labeled by ϑ, is a *one* propagating in all directions at light speed and consuming the *zero* which lies in front of it. This function takes an argument which can be shown to be proportional to the retarded time

$$\vartheta = \vartheta(ct_r/\gamma) \tag{1.3}$$

A function such as this placed alongside the fields of a charged particle is a powerful device which can create (or destroy) the particle similar to the operators in (1.1).

While the application of causality to a classical electron might seem like a radical idea, it is worthwhile to discuss why the theory of QED might not be affected by it. First, causality implies that energy and momentum will be transported into the vacuum by the velocity fields as a new form of radiation. This violates energy conservation but also requires the existence of a new particle to transmit the field. It is assumed to be massless, but unlike the photon, must have a polarization vector ϵ^{ν}_{L} along the direction of the fields. A relation among polarization vectors of the two particles is the covariant contraction

$$\epsilon^{\nu}_{L}\epsilon^{T}_{\nu} = 0 \tag{1.4}$$

This relation is mirrored in the fields themselves in the form of the classical expression

$$\mathbf{E}_v \cdot \mathbf{E}_a - \mathbf{B}_v \cdot \mathbf{B}_a = 0 \tag{1.5}$$

In short, the photons of QED can know nothing about quanta of a moving Coulomb field. The problem is very similar to the production of wave motion in a continuous medium. Both transverse and longitudinal waves can exist simultaneously while not interfering with each other.

This takes care of the mediators but says nothing about the leptons involved in the interaction. For this it is important to touch on the fact that there is currently no acceptable theory of the classical electron which can be properly quantized. However, this is the central challenge which can be solved using vacuum gauge electrodynamics. Specifically, we will prove that an electron endowed with moving velocity fields yields an equivalence with Dirac electron theory, and that the fields themselves obey the energy momentum relation

$$\mathcal{E} = \frac{1}{2}\mathcal{P}c \tag{1.6}$$

In a universe built around the causality principle, nothing less than this is to be expected.

It is not difficult to include moving velocity fields into the general scheme of theoretical physics since their presence amounts to only a slight modification of the 1905 postulates. The postulate of relativity remains the same but the constancy of light postulate must be changed to read:

> –Postulate 2: ***The speed of the electromagnetic field is independent of the motion of the source.***

This is mainly just a generalization of the Einstein postulate to include all possible electromagnetic fields, but is also a unifying statement which renders the descriptive title, '*static field*' as an unsubstantiated figment of the imagination. On a more philosophical level though moving velocity fields are also an acknowledgement that every charged particle in the universe had a beginning—a necessary consequence of Big Bang cosmology[1].

1.2 Causality Applied to a Charged Particle

Without actually inserting momentum into the velocity fields, it is important to show how classical electron theory can be constructed so that both the Maxwell-Lorentz equations and the symmetric stress tensor are constrained by causality.

Potential Formulation of the Maxwell-Lorentz Equations: First, define the flat spacetime Minkowski metric

$$\mathrm{g}^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{1.7}$$

[1]In the words of George Malley, "Everything (in the universe) is on its way to somewhere".

The causal field strength tensor for a charged particle created at the origin at time $ct_r = 0$ is

$$F^{\mu\nu} = F^{\mu\nu}_{M} \cdot \vartheta \tag{1.8}$$

Taking the divergence brings about a modification to the conventional notion of charge and current density associated with the particle since

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^{*\nu}_e \tag{1.9}$$

where

$$J^{*\nu}_e \equiv J^\nu_e \cdot \vartheta + J^\nu_N \qquad\qquad J^\nu_N \equiv \frac{c}{4\pi} F^{\mu\nu}_M \cdot \partial_\mu \vartheta \tag{1.10}$$

and where J^ν_N may be referred to as the ***null current*** having a four-space norm of zero.

Based on causal fields, a causal potential theory can be constructed from the homogeneous Maxwell-Lorentz equations leading to

$$\Box^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^{*\nu}_e \tag{1.11}$$

However, unlike the conventional theory, the presence of the causal light sphere mandates the application of the covariant ***vacuum gauge*** condition,

$$\boxed{\partial_\nu A^\nu_v \equiv \sqrt{\mathrm{E}^2 - \mathrm{B}^2}} \tag{1.12}$$

This is an exceptional example of the use of gauge freedom leading to a de-coupled theory of velocity and acceleration potentials:

$$\Box^2 A^\nu_v - \partial^\nu \partial_\mu A^\mu_v = \frac{4\pi}{c} J^{*\nu}_e \tag{1.13a}$$

$$\Box^2 A^\nu_a - \partial^\nu \partial_\mu A^\mu_a = \frac{4\pi}{c} J^\nu_a \tag{1.13b}$$

Solutions to these two equations can be combined to form the most general vacuum gauge solution for arbitrary motions of the charge:

$$\boxed{A^\nu(\boldsymbol{r}, t) = \frac{e(1 - a^\lambda R_\lambda)}{\rho^2} R^\nu \cdot \vartheta} \tag{1.14}$$

For reference, the fields produced by A^ν_v and A^ν_a are

$$F^{\mu\nu}_v = \frac{e}{\rho^3}\left[R^\mu \beta^\nu - R^\nu \beta^\mu\right] \cdot \vartheta \tag{1.15a}$$

$$F^{\mu\nu}_a = \frac{e}{\rho^2}\left[R^\mu a^\nu - R^\nu a^\mu - \xi(R^\mu \beta^\nu - R^\nu \beta^\mu)\right] \cdot \vartheta \tag{1.15b}$$

where $\xi \equiv a^\lambda R_\lambda/\rho$.

The structure of (1.14) also allows the potentials to be further divided as

$$A^\nu = A_e^\nu + A_\ell^\nu + A_a^\nu \tag{1.16}$$

where

$$A_e^\nu = \frac{e}{\rho}\beta^\nu \cdot \vartheta \qquad A_\ell^\nu = \frac{e}{\rho}\mathcal{U}^\nu \cdot \vartheta \qquad A_a^\nu = -\frac{e}{\rho^2}(a^\lambda R_\lambda)R^\nu \cdot \vartheta \tag{1.17}$$

Each of these potentials can be associated with its own vector current density, and each current density points in the same direction as its associated potentials. A covariant form for $J_e^{*\nu}$ follows as an integral over proper time[2] while remaining current densities are

$$J_\ell^\nu = \frac{ec}{2\pi\rho^3}\mathcal{U}^\nu \cdot \vartheta \qquad J_a^\nu = -\frac{ec}{2\pi\rho^4}\chi R^\nu \cdot \vartheta \tag{1.18}$$

Source wave equations for the potentials A_e^ν and A_ℓ^ν can also be written as

$$\Box^2 A_e^\nu = \frac{4\pi}{c}J_e^\nu \cdot \vartheta \qquad \Box^2 A_\ell^\nu = \frac{4\pi}{c}J_\ell^\nu \cdot \vartheta \tag{1.19}$$

Electromagnetic Lagrangian and Stress Tensor: The separation of the velocity and acceleration fields generated by the vacuum gauge condition allows for the possibility of constructing a classical Lagrangian for a charged particle as a function of independent field quantities. In consideration of notational simplicity it is useful to make the temporary replacements

$$A_v^\nu \to \mathcal{V}^\nu \qquad \partial^\mu A_v^\nu \to \mathcal{V}^{\mu\nu} \tag{1.20a}$$

$$A_a^\nu \to \mathcal{A}^\nu \qquad \partial^\mu A_a^\nu \to \mathcal{A}^{\mu\nu} \tag{1.20b}$$

so that Hamilton's principle can be written

$$\delta\mathcal{S} = \delta\int \mathcal{L}_{vg}[\mathcal{V}^{\mu\nu}, \mathcal{A}^{\mu\nu}, \mathcal{V}^\nu, \mathcal{A}^\nu, x^\nu]d^4x \tag{1.21}$$

The functional form of $\mathcal{L}_{vg}$ can be derived beginning with the classical electromagnetic Lagrangian

$$\mathcal{L}_{em} = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\mu A_\nu)(\partial^\mu A^\nu - \partial^\mu A^\nu) - \frac{1}{c}J_e^{*\nu}A_\nu \tag{1.22}$$

and making the replacements

$$A^\nu \to \mathcal{V}^\nu + \mathcal{A}^\nu \qquad J_e^{*\nu} \to J_e^{*\nu} + J_a^\nu \tag{1.23}$$

[2] A covariant formula for the conventional J_e^ν is available at the end of chapter 12 in Jackson, *Classical Electrodynammics*, Second Edition.

The vacuum gauge result is then

$$\mathcal{L}_{_{vg}} = -\frac{1}{8\pi}[\mathcal{V}^{\mu\nu}\mathcal{V}_{\mu\nu} - \mathcal{V}^{\nu\mu}\mathcal{V}_{\mu\nu} + 2\mathcal{A}^{\mu\nu}\mathcal{V}_{\mu\nu} - 2\mathcal{A}^{\nu\mu}\mathcal{V}_{\mu\nu} + \mathcal{A}^{\mu\nu}\mathcal{A}_{\mu\nu} - \mathcal{A}^{\nu\mu}\mathcal{A}_{\mu\nu}]$$
$$- \frac{1}{c}J_{e}^{*\nu}\,\mathcal{V}_{\nu} - \frac{1}{c}J_{e}^{*\nu}\,\mathcal{A}_{\nu} - \frac{1}{c}J_{a}^{\nu}\,\mathcal{V}_{\nu} - \frac{1}{c}J_{a}^{\nu}\,\mathcal{A}_{\nu} \tag{1.24}$$

Obviously, this Lagrangian differs from the conventional electromagnetic Lagrangian based on the requirement to include the acceleration current density, but this is not an issue since the additional terms containing J_a^ν have a functional value of zero. In fact, the functional value of the free field Lagrangian is identical to the conventional theory since terms containing the field quantity $\mathcal{A}^{\mu\nu}$ also add to zero. Euler-Lagrange equations for the independent field quantities now follow from Hamilton's principle leading to equations of motion given by

$$\partial_\mu \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{V}_{\mu\nu}} - \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{V}_{\nu}} = 0 \tag{1.25a}$$

$$\partial_\mu \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{A}_{\mu\nu}} - \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{A}_{\nu}} = 0 \tag{1.25b}$$

Not surprisingly, each set of Lagrange equations produces identical equations of motion

$$\partial_\mu F_{v}^{\mu\nu} = \frac{4\pi}{c}J_{e}^{*\nu} \qquad\qquad \partial_\mu F_{a}^{\mu\nu} = \frac{4\pi}{c}J_{a}^{\nu} \tag{1.26}$$

Treating velocity and acceleration fields as independent quantities is of no consequence to classical electron theory.

Beginning with $\mathcal{L}_{_{vg}}$ the canonical stress tensor is

$$T_{_{vg}}^{\mu\nu} = \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{V}_{\mu\lambda}}\,\mathcal{V}^{\nu}{}_{\lambda} + \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{A}_{\mu\lambda}}\,\mathcal{A}^{\nu}{}_{\lambda} - \mathrm{g}^{\mu\nu}\mathcal{L}_{_{vg}} \tag{1.27}$$

It is convenient to write this equation as

$$T_{_{vg}}^{\mu\nu} = \Theta_{_{vg}}^{\mu\nu} + T_{_D}^{\mu\nu}$$

where

$$\Theta_{_{vg}}^{\mu\nu} = \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{V}_{\mu\lambda}}(\mathcal{V}^{\nu}{}_{\lambda} - \mathcal{V}_{\lambda}{}^{\nu}) + \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{A}_{\mu\lambda}}(\mathcal{A}^{\nu}{}_{\lambda} - \mathcal{A}_{\lambda}{}^{\nu}) - \mathrm{g}^{\mu\nu}\mathcal{L}_{_{vg}} \tag{1.28}$$

$$T_{_D}^{\mu\nu} = \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{V}_{\mu\lambda}}\,\mathcal{V}_{\lambda}{}^{\nu} + \frac{\partial \mathcal{L}_{_{vg}}}{\partial \mathcal{A}_{\mu\lambda}}\mathcal{A}_{\lambda}{}^{\nu} \tag{1.29}$$

The tensor $\Theta_{_{vg}}^{\mu\nu}$ is the most general, symmetric and traceless, gauge invariant tensor. Using the relations

$$F_{v}^{\mu\nu} = \mathcal{V}^{\mu\nu} - \mathcal{V}^{\nu\mu} \qquad\qquad F_{a}^{\mu\nu} = \mathcal{A}^{\mu\nu} - \mathcal{A}^{\nu\mu} \tag{1.30}$$

it can be written

$$\underset{vg}{\Theta}^{\mu\nu} \equiv \frac{1}{4\pi}\left[F^{\mu\lambda}F_{\lambda}{}^{\nu} + \frac{1}{4}\mathrm{g}^{\mu\nu}F_{\alpha\lambda}F^{\alpha\lambda}\right] \tag{1.31}$$

To determine $\underset{D}{T}^{\mu\nu}$ simply write

$$\begin{aligned}
\underset{D}{T}^{\mu\nu} &= -\frac{1}{4\pi}\left(\mathcal{V}^{\mu\lambda} - \mathcal{V}^{\lambda\mu} + \mathcal{A}^{\mu\lambda} - \mathcal{A}^{\lambda\mu}\right)\mathcal{V}_{\lambda}{}^{\nu} - \frac{1}{4\pi}\left(\mathcal{A}^{\mu\lambda} - \mathcal{A}^{\lambda\mu} + \mathcal{V}^{\mu\lambda} - \mathcal{V}^{\lambda\mu}\right)\mathcal{A}_{\lambda}{}^{\nu} \\
&= -\frac{1}{4\pi}(\underset{v}{F}^{\mu\lambda} + \underset{a}{F}^{\mu\lambda})\,\partial_{\lambda}\underset{v}{A}^{\nu} - \frac{1}{4\pi}(\underset{a}{F}^{\mu\lambda} + \underset{v}{F}^{\mu\lambda})\,\partial_{\lambda}\underset{a}{A}^{\nu} \\
&= -\frac{1}{4\pi}F^{\mu\lambda}\partial_{\lambda}A^{\nu}
\end{aligned} \tag{1.32}$$

This term can now be removed[3] from (1.28) with the observation that $\partial_{\mu}\underset{D}{T}^{\mu\nu} = 0$, and leading to the source equation:

$$\partial_{\mu}\underset{vg}{\Theta}^{\mu\nu} = \frac{1}{c}F^{\lambda\nu}\underset{e}{J}^{*}_{\lambda} \tag{1.33}$$

It is important to mention here that details of the derivation of (1.33) show no contribution from the acceleration current density which is something of a spectator in the calculation. The equivalence of electromagnetic and vacuum gauge theories for a charged particle in arbitrary motion is therefore

$$\underset{vg}{\Theta}^{\mu\nu} \equiv \underset{em}{\Theta}^{\mu\nu} = \Theta^{\mu\nu} \tag{1.34}$$

For later reference, the specific form of $\Theta^{\mu\nu}$ can be written

$$\Theta^{\mu\nu} = \underset{1}{\Theta}^{\mu\nu} + \underset{2}{\Theta}^{\mu\nu} + \underset{3}{\Theta}^{\mu\nu} \tag{1.35a}$$

with individual components given by

$$\underset{1}{\Theta}^{\mu\nu} = \frac{e^2}{4\pi\rho^4}\left[\frac{1}{\rho}(R^{\mu}\beta^{\nu} + R^{\nu}\beta^{\mu}) - \frac{1}{\rho^2}R^{\mu}R^{\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}\right]\cdot\vartheta \tag{1.35b}$$

$$\underset{2}{\Theta}^{\mu\nu} = \frac{e^2}{4\pi\rho^4}\left[a^{\mu}R^{\nu} + R^{\mu}a^{\nu} - \xi(R^{\mu}\beta^{\nu} + R^{\nu}\beta^{\mu}) + \frac{2\xi}{\rho}R^{\mu}R^{\nu}\right]\cdot\vartheta \tag{1.35c}$$

$$\underset{3}{\Theta}^{\mu\nu} = -\frac{e^2}{4\pi\rho^4}(\xi^2 + a_{\lambda}a^{\lambda})R^{\mu}R^{\nu}\cdot\vartheta \tag{1.35d}$$

[3]For details on symmetrizing the canonical stress tensor, see Jackson, *Classical Electrodynamics*, section 12.10; Landau and Lifshitz, *Classical Theory of Fields*, Section 94.

2 Deformation of the Vacuum by Electromagnetic Fields

As already discussed, enforcement of causality in the electromagnetic field violates energy conservation and requires both momentum and energy in the velocity fields. The insertion of this momentum and energy is not overly challenging, but relies heavily on an interpretation of vacuum gauge velocity potentials as a deformation of the surrounding medium in the neighborhood of the charge. The vacuum gauge condition is a precise measure of this deformation which can be demonstrated by drawing on analogies with the classical theory of elastic media.

2.1 Wave Propagation in Elastic Media

In three-dimensional space, the displacement of an elastic continuum from its equilibrium configuration is represented by a vector field $\mathbf{u} = \mathbf{u}(\boldsymbol{r})$. The displacement results in stresses and strains on the surrounding medium which are adequately described by a symmetric stress tensor typically written

$$T_{ij} = -\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \right) - \lambda\, \delta_{ij} \boldsymbol{\nabla} \cdot \mathbf{u} \tag{2.1}$$

where the positive constants μ and λ associated with the medium are measured experimentally. The elastic strain tensor is defined according to

$$\epsilon_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \right) \qquad \text{and} \quad \epsilon \equiv \sum_i \epsilon_{ii} \tag{2.2}$$

and allows the stress tensor to take the form

$$T_{ij} = -2\mu\epsilon_{ij} - \lambda\, \delta_{ij}\epsilon \tag{2.3}$$

The amount of stored potential energy $\mathcal{V}$ per unit volume associated with the deformed configuration can be determined by combining the stress and strain tensors as

$$\mathcal{V} = -\frac{1}{2} \sum_{i,j} T_{ij}\epsilon_{ij} \tag{2.4}$$

Assuming no first order changes in the density ρ of the material, and noting that the velocity of a point in the medium is given by $\partial\mathbf{u}/\partial t$, an appropriate Lagrangian density inclusive of an interacting source $\boldsymbol{f}$ can be written

$$\mathcal{L} = \tfrac{1}{2}\rho\, |\dot{\mathbf{u}}|^2 - \mu \sum_{i,j} \epsilon_{ij}\epsilon_{ij} - \tfrac{1}{2}\lambda\epsilon^2 + \mathbf{u} \cdot \boldsymbol{f} \tag{2.5}$$

Lagrange's equations are

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial \mathrm{u}_i/\partial t)} + \sum_j \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial(\partial \mathrm{u}_i/\partial x_j)} = \frac{\partial \mathcal{L}}{\partial \mathrm{u}_i} \tag{2.6}$$

which can be used to derive the dynamical equation

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \mu\boldsymbol{\nabla}^2\mathbf{u} + (\lambda+\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u} + \boldsymbol{f} \tag{2.7}$$

If the deformation and the force density are written in terms of transverse and longitudinal components

$$\mathbf{u} = \mathbf{u}_t + \mathbf{u}_l \qquad\qquad \boldsymbol{f} = \boldsymbol{f}_t + \boldsymbol{f}_l \tag{2.8}$$

the dynamical equation de-couples with $\mathbf{u}_t$ and $\mathbf{u}_l$ independently satisfying identical equations

$$\rho\frac{\partial^2\mathbf{u}_{t,l}}{\partial t^2} = \mu\boldsymbol{\nabla}^2\mathbf{u}_{t,l} + (\lambda+\mu)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u}_{t,l} + \boldsymbol{f}_{t,l} \tag{2.9}$$

However, a considerable simplication results from the solenoidal character of $\mathbf{u}_t$ and the irrotational character of $\mathbf{u}_l$ implying the relations

$$\boldsymbol{\nabla}\cdot\mathbf{u}_t = 0 \qquad\qquad \boldsymbol{\nabla}\times\mathbf{u}_l = 0 \tag{2.10}$$

Source wave equations for the separate components are then

$$\Box_t^2\mathbf{u}_t = \frac{1}{c_t^2}\frac{\partial^2\mathbf{u}_t}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{u}_t = \tilde{\boldsymbol{f}}_t \tag{2.11a}$$

$$\Box_l^2\mathbf{u}_l = \frac{1}{c_l^2}\frac{\partial^2\mathbf{u}_l}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{u}_l = \tilde{\boldsymbol{f}}_l \tag{2.11b}$$

where the source terms have been re-defined accordingly, and where the transverse and longitudinal velocities of propagation are given by

$$c_t = \sqrt{\frac{\mu}{\rho}} \qquad\qquad c_l = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{2.12}$$

2.2 Vacuum as a Deformable Medium

For a charged particle, the four-space analog of the dynamical equation in (2.7) will be the Maxwell-Lorentz equations for the velocity potentials.

$$\Box^2 A_v^\nu - \partial^\nu\partial_\mu A_v^\mu = \frac{4\pi}{c}J_e^{*\nu} \tag{2.13}$$

It has already been shown that the velocity equation can be divided into two separate equations with associated four-currents J_e^ν and J_ℓ^ν. This allows the vacuum gauge source equations in (1.19) to follow the source equations in (2.11) according to

$$\Box^2 A_e^\nu = \frac{4\pi}{c} J_e^\nu \qquad \longleftrightarrow \qquad \Box_t^2 \mathbf{u}_t = \tilde{\boldsymbol{f}}_t \tag{2.14a}$$

$$\Box^2 A_\ell^\nu = \frac{4\pi}{c} J_\ell^\nu \qquad \longleftrightarrow \qquad \Box_l^2 \mathbf{u}_l = \tilde{\boldsymbol{f}}_l \tag{2.14b}$$

$$J_e^\nu (J_\ell)_\nu = 0 \qquad \longleftrightarrow \qquad \boldsymbol{f}_t \cdot \boldsymbol{f}_l = 0 \tag{2.14c}$$

The potentials themselves map to the three-space deformation according to

$$A_v^\nu = A_e^\nu + A_\ell^\nu \qquad \longleftrightarrow \qquad \mathbf{u} = \mathbf{u}_t + \mathbf{u}_l \tag{2.15a}$$

$$\partial_\nu A_e^\nu = 0 \qquad \longleftrightarrow \qquad \boldsymbol{\nabla} \cdot \mathbf{u}_t = 0 \tag{2.15b}$$

$$\partial^\mu A_\ell^\nu - \partial^\nu A_\ell^\mu = 0 \qquad \longleftrightarrow \qquad \boldsymbol{\nabla} \times \mathbf{u}_l = 0 \tag{2.15c}$$

$$A_\ell^\nu = \partial^\nu \varphi \qquad \longleftrightarrow \qquad \mathbf{u}_l = -\boldsymbol{\nabla}\varphi \tag{2.15d}$$

Now suppose an elastic disturbance $\mathbf{u}(\boldsymbol{r}, t)$ has a longitudinal component defined by the wave vector $\boldsymbol{k}$. The projection operation analogy takes the form

$$A_e^\nu = A_v^\nu + (A_v^\mu \mathcal{U}_\mu)\mathcal{U}^\nu \qquad \longleftrightarrow \qquad \mathbf{u}_t = \mathbf{u} - (\mathbf{u} \cdot \boldsymbol{k})\boldsymbol{k} \tag{2.16a}$$

$$A_\ell^\nu = -(A_v^\mu \mathcal{U}_\mu)\mathcal{U}^\nu \qquad \longleftrightarrow \qquad \mathbf{u}_l = (\mathbf{u} \cdot \boldsymbol{k})\boldsymbol{k} \tag{2.16b}$$

Each of the six previous equations for the potentials is also mirrored by the current densities:

$$\begin{array}{ccc} J_v^\mu = J_e^\mu + J_\ell^\mu & J_e^\mu = J_v^\mu + (J_v^\nu \mathcal{U}_\nu)\mathcal{U}^\mu & \partial_\mu J_e^\mu = 0 \\ \partial^\mu J_\ell^\nu - \partial^\nu J_\ell^\mu = 0 & J_\ell^\mu = -(J_v^\nu \mathcal{U}_\nu)\mathcal{U}^\mu & J_\ell^\mu = \partial^\mu \psi \end{array} \tag{2.17}$$

In words, the transverse and longitudinal deformations of the three-space continuum get replaced by timelike and spacelike potentials in the four-space theory of a charged particle; and these potentials are determined from associated timelike and spacelike current densities. For an accelerating charge it could be argued that the analogy would fall apart because the total potential will acquire an acceleration term

$$A^\nu = A_e^\nu + A_\ell^\nu + A_a^\nu \tag{2.18}$$

This issue is resolved by asserting that the analogy only applies to the velocity portion of the total potential. This is reasonable because A_a^μ satisfies its own independent differential equation.

Another correspondence is the connection

$$\partial_\nu A^\nu_v \qquad \longleftrightarrow \qquad \boldsymbol{\nabla}\cdot\mathbf{u} \tag{2.19}$$

But since the transverse components do not contribute to the total divergence, then

$$\partial_\nu A^\nu_v = \partial_\nu A^\nu_\ell = \Box^2\varphi \qquad \longleftrightarrow \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{u}_l = -\boldsymbol{\nabla}^2\varphi \tag{2.20}$$

According to the theory of elastic media, the three-space divergence on the right measures the net distortion, or volumetric dilatation, of the medium. This suggests that the vacuum gauge condition

$$\partial_\nu A^\nu_v = \frac{e}{\rho^2} \tag{2.21}$$

is really a ***vacuum dilatation*** gauge condition—a measure of the amount by which the surrounding vacuum is deformed by the presence of the particle. The implication here is that a comprehensive description of the vacuum gauge electron will support the introduction of a radius r_e into the theory. This is a very desirable feature of any classical theory of the electron to address—among other things—the well known problem of the divergent self energy. Moreover, a particle radius in the context of an appropriately chosen gauge condition necessarily avoids the introduction of form factors and other senseless constructs based on macroscopic classical notions. The Doppler charge density associated with this radius is easily demonstrated by evaluating (2.21) at $R = r_e$:

$$\sigma_e(\theta,\phi) = \frac{1}{4\pi}\,\partial_\nu A^\nu_v\bigg|_{R=r_e} = \frac{\sigma_e}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2} \tag{2.22}$$

The angles are measured with respect to the retarded position and the charge density can easily be integrated over the radius. Lastly, the theory of vacuum dilatation can be compared with the Liénard Wiechert potentials where the application of the Lorentz gauge condition $\partial_\mu A^\mu_e = 0$ generates a dilatation of zero and is unmistakably associated with a point-particle theory.

2.3 Propagation of Vacuum Waves

To initiate the transition to a theory of a particle radius, it is first necessary to give a precise definition to the causality step function in terms of Fourier oscillations. Individual modes of the step can then be used to re-interpret vacuum gauge potentials as outgoing spherical waves in timelike and spacelike directions. Momentum in the fields can be inserted by re-writing the velocity electric and magnetic fields exclusively in terms of vacuum gauge potentials.

Fourier Modes of the Causality Sphere: While the causality sphere in equation (1.3) is a function of the retarded time, it may also be written in terms of invariant scalars through the relation

$$ct_r/\gamma = c\tau - \rho \tag{2.23}$$

To this one may include a phase τ_e so that

$$\vartheta(\tau, \rho) = \vartheta(\tau - \rho/c + \tau_e) \tag{2.24}$$

A typical definition from signal theory is the Heaviside function

$$\mathrm{H}(t - s/v + t_o) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t - s/v + t_o) \tag{2.25}$$

where s is some distance along an axis and t_o is constant. The signum function takes values of $\pm 1/2$ depending on whether its argument is positive or negative. Unfortunately, a definition such as (2.25) will not work for the vacuum gauge electron because the argument of $\mathrm{sgn}(x)$ can never be less than zero. Moreover, it can be shown that Fourier amplitudes of $\mathrm{H}(t, s)$ in the frequency domain include an unwelcome delta function, further complicating matters.

A better definition, tailored specifically for the causal theory is

$$\vartheta(\tau - \rho/c + \tau_e) \equiv \begin{cases} \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin\omega(\tau - \rho/c + \tau_e)}{\omega}\, d\omega & \tau + \tau_e \geq \rho/c \\ \\ 0 & \tau + \tau_e < \rho/c \end{cases} \tag{2.26}$$

In the causal region the function ϑ propagates travelling waves over all frequencies. These waves have an amplitude falling off like $1/\omega$ and give a value of exactly 1 when summed over all frequencies. A schematic exhibiting two of these modes is available in figure 1. Using the definition in (2.26), a four-gradient of ϑ can be calculated as

$$\partial^\nu \vartheta = \frac{2}{\pi}\frac{R^\nu}{\rho}\int_0^\infty \cos\omega(\tau - \rho/c + \tau_e)\, d\omega \tag{2.27}$$

and using $\hat{\tau} = \tau + \tau_e$ the integrand may be expanded yielding

$$\partial^\nu \vartheta = \frac{2}{\pi}\frac{R^\nu}{\rho}\left[\int_0^\infty \cos\omega\hat{\tau}\cos\omega\rho/c\, d\omega + \int_0^\infty \sin\omega\hat{\tau}\sin\omega\rho/c\, d\omega\right] \tag{2.28}$$

Including the factor of $2/\pi$, each term in the brackets is a delta function. Both would be needed if the function ϑ were extended to the acausal region where its value would become -1. However, the mathematics is not aware that ϑ has been defined to have a value of zero in this region. This means that only one delta function is required and the appropriate gradient operation is

$$\partial^\nu \vartheta = \frac{R^\nu}{\rho} \cdot \delta \tag{2.29}$$

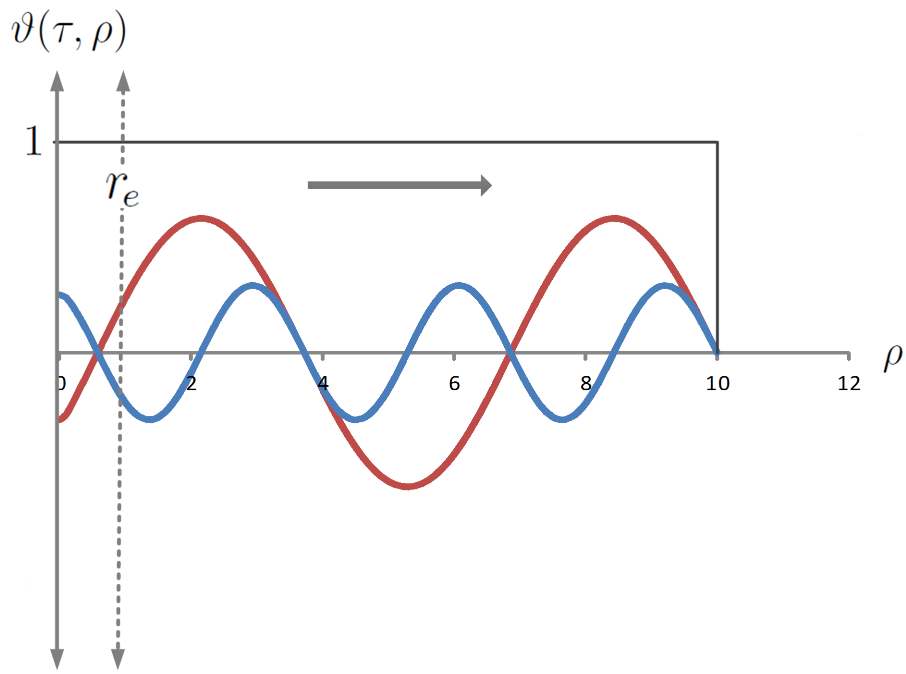

Figure 1: *Schematic showing two Fourier modes at the proper time $c\tau = 10$ transmitted by the radial step at the retarded position.*

Fourier Modes of Potentials: By themselves, Fourier modes of the step don't have much personality, but they can be given real physical meaning when applied to the vacuum gauge potentials. The vector potential is

$$A^{\nu} \cdot \vartheta = A_e^{\nu} \cdot \vartheta + A_{\ell}^{\nu} \cdot \vartheta \tag{2.30}$$

Working with the complex exponential via

$$\vartheta(\tau - \rho/c + \tau_e) = \mathrm{Im}\left[\frac{2}{\pi}\int_0^{\infty} \frac{e^{i\omega(\tau-\rho/c+\tau_e)}}{\omega}\, d\omega\right] \qquad \rho \leq c\tau + \tau_e \tag{2.31}$$

timelike and spacelike potentials will then appear as

$$A_e^{\nu}(\rho, \tau) = \mathrm{Im}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} A_{e:\omega}^{\nu}(\rho) e^{i\omega\tau} d\omega\right] \tag{2.32a}$$

$$A_{\ell}^{\nu}(\rho, \tau) = \mathrm{Im}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} A_{\ell:\omega}^{\nu}(\rho) e^{i\omega\tau} d\omega\right] \tag{2.32b}$$

Here individual modes characterized by polarization vectors β^ν and $\mathcal{U}^\nu$ and phase $\tilde{\phi}_\omega = e^{i\omega\tau_e}$ are given by the outgoing spherical waves

$$A^\nu_{e:\omega}(\rho) = e \cdot \sqrt{\frac{2}{\pi}} \cdot \tilde{\phi}_\omega \frac{e^{-i\omega\rho/c}}{\omega\rho} \beta^\nu \tag{2.33a}$$

$$A^\nu_{\ell:\omega}(\rho) = e \cdot \sqrt{\frac{2}{\pi}} \cdot \tilde{\phi}_\omega \frac{e^{-i\omega\rho/c}}{\omega\rho} \mathcal{U}^\nu \tag{2.33b}$$

Adding these together, the complete four-vector mode is

$$A^\nu_\omega = \sqrt{\frac{2}{\pi}} \cdot A^\nu \cdot \frac{e^{i\omega(\tau-\rho/c+\tau_e)}}{\omega} \tag{2.34}$$

Differentiating individual components of A^ν_ω will then determine Fourier modes of the velocity fields

$$\mathbf{E}_\omega = \sqrt{\frac{2}{\pi}} \cdot \mathbf{E} \cdot \frac{e^{i\omega(\tau-\rho/c+\tau_e)}}{\omega} \tag{2.35a}$$

$$\mathbf{B}_\omega = \sqrt{\frac{2}{\pi}} \cdot \mathbf{B} \cdot \frac{e^{i\omega(t-\rho/c+\tau_e)}}{\omega} \tag{2.35b}$$

The imaginary part can now be re-integrated over all frequencies re-constructing the causal electric and magnetic field vectors $\mathbf{E} = \mathbf{E}_M \cdot \vartheta$ and $\mathbf{B} = \mathbf{B}_M \cdot \vartheta$.

Canonical Momentum in the Electromagnetic Field: To associate momentum with the electrons' velocity fields will require a re-interpretation of the fields as momentum flux fields. This is as easy as multiplying the fields by a constant. In fact, since the vacuum gauge has already produced a charge density σ_e, electric and magnetic momentum flux fields can be postulated to be

$$\boldsymbol{\pi}_{\mathrm{E}} = \sigma_e \mathbf{E} \tag{2.36a}$$

$$\boldsymbol{\pi}_{\mathrm{B}} = \sigma_e \mathbf{B} \tag{2.36b}$$

While electric flux points radially outward from the present position of the source, the direction with which magnetic flux circles the particle will depend on the frame of reference. Now record the force density $J^\nu_\pi \equiv \sigma_e J^\nu_e$ which re-defines the Maxwell-Lorentz source equations for the velocity fields:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{\mathrm{E}} = 4\pi\rho_\pi \qquad \boldsymbol{\nabla} \times \boldsymbol{\pi}_{\mathrm{B}} = \frac{4\pi}{c}\mathbf{J}_\pi + \frac{1}{c}\frac{\partial \boldsymbol{\pi}_{\mathrm{E}}}{\partial t} \tag{2.37a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\pi}_{\mathrm{E}} = -\frac{1}{c}\frac{\partial \boldsymbol{\pi}_{\mathrm{B}}}{\partial t} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_{\mathrm{B}} = 0 \tag{2.37b}$$

Covariance is established by defining the anti-symmetric electromagnetic momentum flux tensor

$$\pi^{\mu\nu} \equiv \begin{bmatrix} 0 & -\pi_{\mathrm{E}x} & -\pi_{\mathrm{E}y} & -\pi_{\mathrm{E}z} \\ \pi_{\mathrm{E}x} & 0 & -\pi_{\mathrm{B}z} & \pi_{\mathrm{B}y} \\ \pi_{\mathrm{E}y} & \pi_{\mathrm{B}z} & 0 & -\pi_{\mathrm{B}x} \\ \pi_{\mathrm{E}z} & -\pi_{\mathrm{B}y} & \pi_{\mathrm{B}x} & 0 \end{bmatrix}_v \tag{2.38}$$

This is nothing more than the field strength tensor multiplied by σ_e. The bottom line here is that with almost no effort, an entirely new interpretation of the electrons' velocity fields has been established.

Calculation of Inertial Vacuum Power: It is well known that energy associated with wave motion is determined by squaring the amplitude of a wave. Unfortunately, multiplying the Fourier component of the electric and magnetic field vectors in equation (2.35) by σ_e will not satisfy this requirement. Instead, it is necessary to use the full power of the vacuum gauge to write the momentum flux tensor as an anti-symmetric combination among timelike and spacelike components of vacuum gauge potentials. Using $a_e = 4\pi r_e^2$ this is

$$\boxed{\pi^{\mu\nu} = \frac{1}{a_e}\,[A_\ell^\mu,\, A_e^\nu]} \tag{2.39}$$

Time dependent Fourier modes of potentials can be written

$$A_{e:\omega}^\nu = \sqrt{\frac{2}{\pi}} \cdot A_e^\nu \cdot \frac{\sin\omega(\tau - \rho/c + \tau_e)}{\omega} \tag{2.40a}$$

$$A_{\ell:\omega}^\nu = \sqrt{\frac{2}{\pi}} \cdot A_\ell^\nu \cdot \frac{\sin\omega(\tau - \rho/c + \tau_e)}{\omega} \tag{2.40b}$$

Inserting these into (2.39) gives a Fourier mode of the field strength tensor in terms of easily recognizable quantities

$$\pi_\omega^{\mu\nu} = \frac{2e\sigma_e}{\pi}\,[\mathcal{U}^\mu,\, \beta^\nu] \cdot \frac{\sin^2\omega(\tau - \rho/c + \tau_e)}{\rho^2\omega^2} \tag{2.41}$$

For a calculation of radiated energy, it is necessary to determine the relation between energy flux and momentum flux in the velocity fields. The Lagrangian formulation of section 3 shows that

$$\mathbf{S}_{\mathrm{E}} = \tfrac{1}{2}\boldsymbol{\pi}_{\mathrm{E}}c \tag{2.42a}$$

$$\mathbf{S}_{\mathrm{B}} = \tfrac{1}{2}\boldsymbol{\pi}_{\mathrm{B}}c \tag{2.42b}$$

The total power radiated can be determined by integrating the flux over the surface defining the electron radius and then integrating over all wave numbers. Using the retarded distance $R = R(\rho, \theta, \phi)$ an appropriate 3-surface is given by

$$ds_\nu = \mathcal{U}_\nu R^2 d\Omega\, dc\tau \tag{2.43}$$

where angles (θ, ϕ) are measured with respect to the retarded position of the charge. If the final integration over proper time is witheld, then the flux integral is

$$\frac{d\chi^\mu}{d\tau} = \int_0^\infty \left[\oint_{\rho=r_e} \tfrac{1}{2}\, \pi_\omega^{\mu\nu} \mathcal{U}_\nu R^2 d\Omega \right] d\omega \tag{2.44a}$$

$$= \frac{ec}{\pi} \beta^\mu \int_\Omega \sigma_e(\theta, \phi) d\Omega \cdot \int_0^\infty \frac{\sin^2 \omega\tau}{\omega^2} d\omega \tag{2.44b}$$

$$= \frac{mc^2}{\tau_e} \tau \beta^\mu \tag{2.44c}$$

This result includes an integration over the Doppler charge density and can be immediately differentiated giving the four-vector power output per unit proper time

$$P^\mu_{vac} = \frac{mc^2}{\tau_e} \beta^\mu \tag{2.45}$$

The invariant radiation rate is then

$$P_{in} = \frac{dP^\mu_{vac}}{d\tau} \beta_\mu = \frac{mc^2}{\tau_e} \tag{2.46}$$

To achieve a more intriguing result suppose the operator $d/d\tau$ is applied to the integrand in equation (2.44b). This determines the four-vector power output per unit frequency:

$$\frac{dP^\mu_{vac}}{d\omega} = P_{in} \beta^\mu \left[\frac{2}{\pi} \cdot \frac{\sin 2\omega\tau}{\omega} \right] \tag{2.47}$$

According to this equation power output can be both positive and negative for a given frequency and a given time. Nevertheless, a final integration over frequencies shows that

$$P^\mu_{vac} = P_{in} \beta^\mu \cdot \vartheta(\tau) \tag{2.48}$$

indicating that no power radiated until it went thru the particle radius.

The value of P_{in} is staggering at 1.74×10^{10} Watts. The magnitude of this number is obviously related to the large value for the speed of light and it may be more reasonable to determine the total energy present in a one meter radius about the particle, which calculates to about 116 Joules. If potentials are 'meaningless constructs', then this form of energy—determined exclusively by the potentials—must be considered as not capable of being detected or harnessed, and having no mass equivalent. Its presence is rationalized only by faith that causality rules the universe.

3 Lagrangian Formulation of the Vacuum Gauge Electron

The correspondence between vacuum gauge velocity potentials and the theory of elastic continua can be used as a guide for the development of a highly specialized Lagrangian formulation for the classical electron. A strictly inflexible requirement for a successful Lagrangian must be the re-production of all the usual gauge invariant results of the conventional Maxwell-Lorentz electron theory. In addition—and based on the work of the previous section—an appropriate formulation must also provide a framework to incorporate the proliferation of vacuum waves. All of these requirements can be met beginning with the derivation of a four-space analog to the symmetric stress tensor T_{ij} given in (2.1).

3.1 Vacuum Tensor

Derivation of the Vacuum Tensor: A firm theoretical foundation for the vacuum tensor can be established by working in the Maxwell limit and defining the third rank tensor $\Psi^{\alpha\mu}{}_{\nu}$ composed of the four components of the vacuum gauge velocity potentials[4] along with two occurrences of the fourth rank, totally anti-symmetric, Levi-Civita symbol:

$$\Psi^{\alpha\mu}{}_{\nu} \equiv \tfrac{1}{2}\boldsymbol{\epsilon}^{\alpha\mu\sigma\tau}\boldsymbol{\epsilon}_{\sigma\tau\nu\kappa}A^{\kappa} \tag{3.1}$$

The field strength tensor follows from a divergence operation on the last index

$$\partial^{\nu}\Psi^{\alpha\mu}{}_{\nu} = \tfrac{1}{2}\boldsymbol{\epsilon}^{\alpha\mu\sigma\tau}\boldsymbol{\epsilon}_{\sigma\tau\nu\kappa}\,\partial^{\nu}A^{\kappa} = \tfrac{1}{2}\boldsymbol{\epsilon}^{\alpha\mu\sigma\tau}F^{\mathfrak{D}}_{\sigma\tau} = F^{\alpha\mu} \tag{3.2}$$

where $F^{\mathfrak{D}}_{\sigma\tau}$ is the dual to the field strength tensor. Contractions on the third index with either β^{ν} or $\mathcal{U}^{\nu}$ also produces the field strength tensor

$$F^{\alpha\mu} = -\frac{1}{\rho}\Psi^{\alpha\mu}{}_{\nu}\beta^{\nu} = \frac{1}{\rho}\Psi^{\alpha\mu}{}_{\nu}\mathcal{U}^{\nu} = \frac{1}{\rho}\left[A^{\alpha},\, \beta^{\mu}\right] \tag{3.3}$$

This derivation has the added benefit of showing the linear relationship between the field strength tensor and the components of the potentials themselves. Now consider a divergence operation on either of the two remaining indices. The inclusion of the metric is appropriate here to raise the last index with the result

$$\partial_{\alpha}\Psi^{\alpha\mu\nu} = \partial_{\alpha}\left[\mathrm{g}^{\nu\lambda}\Psi^{\alpha\mu}{}_{\lambda}\right] = \partial^{\nu}A^{\mu} - \mathrm{g}^{\mu\nu}\partial_{\alpha}A^{\alpha} \tag{3.4}$$

The first term on the far right is the covariant derivative of the velocity potentials—exhibiting a single symmetric term dependent on accelerations of the particle which can be isolated from the velocity terms as

$$\partial^{\nu}A^{\mu} = \eta^{\nu\mu} + J^{\mu\nu}_{\boldsymbol{a}} \tag{3.5}$$

[4] The subscript v on the velocity potentials and the velocity field strength tensor is sometimes left out. We hope no ambiguity with counterpart acceleration potentials and fields will arise.

The symbol $\eta^{\nu\mu} = \eta^{\dagger\mu\nu}$ defines the components of a ***vacuum strain tensor*** whose scalar contraction is exactly the vacuum gauge condition $\eta = g_{\mu\nu}\eta^{\mu\nu}$. Coupled with the last term on the far right of (3.4) a concise representation of the ***vacuum tensor*** can be written

$$\boxed{\varDelta^{\dagger\mu\nu} \equiv \eta^{\nu\mu} - \mathrm{g}^{\mu\nu}\eta} \tag{3.6}$$

The peculiar inversion of the indices on the right side of this equation justifies the inclusion of the superscript (†) transpose operator in the definition, however this and the un-transposed version $\varDelta^{\mu\nu}$ will both be useful for the causal theory.

An explicit form of $\varDelta^{\mu\nu}$ can be determined by separating out the gauge field from the velocity potentials. Since A^{ν}_{ℓ} is determined from the gradient of the scalar φ, then $\varDelta^{\mu\nu}_{\ell}$ will be symmetric:

$$\varDelta^{\mu\nu}_{\ell} = \partial^{\mu}\partial^{\nu}\varphi - \mathrm{g}^{\mu\nu}\Box^2\varphi \tag{3.7}$$

In terms of the null vector and the four-velocity, both tensors are

$$\varDelta^{\mu\nu}_{\ell} = \frac{e}{\rho^2}\left[\frac{2}{\rho^2}R^{\mu}R^{\nu} - \frac{2}{\rho}(R^{\mu}\beta^{\nu} + R^{\nu}\beta^{\mu}) + \beta^{\mu}\beta^{\nu}\right] \tag{3.8a}$$

$$\varDelta^{\mu\nu}_{e} = \frac{e}{\rho^2}\left[\frac{1}{\rho}R^{\mu}\beta^{\nu} - \beta^{\mu}\beta^{\nu}\right] \tag{3.8b}$$

Adding the two portions back together yields the final result

$$\varDelta^{\mu\nu} = \frac{e}{\rho^2}\left[\frac{2R^{\mu}R^{\nu}}{\rho^2} - \frac{2\beta^{\mu}R^{\nu}}{\rho} - \frac{R^{\mu}\beta^{\nu}}{\rho}\right] \tag{3.9}$$

with the observation that a term proportional to the metric is conspicuously absent from the equation. Like the velocity field strength tensor, the vacuum tensor also has a representation exclusively in terms of timelike and spacelike potentials

$$\boxed{\varDelta^{\mu\nu} = \frac{1}{e}\left[2A^{\mu}_{\ell}A^{\nu}_{\ell} - A^{\mu}_{e}A^{\nu}_{e} + A^{\mu}_{\ell}A^{\nu}_{e}\right]} \tag{3.10}$$

The velocity field strength tensor follows from the anti-symmetric combination

$$F^{\mu\nu}_{v} = \varDelta^{\mu\nu} - \varDelta^{\dagger\mu\nu} \tag{3.11}$$

and determines (2.39) when multiplying both sides by σ_e. It is also possible to replace the potentials in (3.10) with individual Fourier modes and re-derive equation (2.44). Instead, consider the scalar E_{vac} given by

$$E_{vac} = \frac{1}{2}\sigma_e \int \left[\oint_{\rho=r_e} \mathcal{U}_{\mu}\varDelta^{\mu\nu}_{\omega}\beta_{\nu}R^2 d\Omega\right] d\omega = P_{in}\tau \tag{3.12}$$

Divergence of the Vacuum Tensor: An important contraction of the vacuum tensor is the divergence operation applied to the second index

$$\partial_\nu \mathit{\Delta}^{\mu\nu} = 0 \tag{3.13}$$

This can be easily proved from the definition in equation (3.6) but it can also be proved by applying ∂_ν directly to (3.9) where the divergence of each individual term can be shown to vanish. The divergence operator applied to the first index is a different matter. It is strictly only legitimate when applied to (3.5) yielding an additional acceleration term

$$\partial_\mu \mathit{\Delta}^{\mu\nu} = \frac{4\pi}{c}\left[J_e^{*\nu} - J_a^{\nu}\right] \tag{3.14}$$

The acceleration current density J_a^ν is identical to the current density determined by the electromagnetic theory in section 1 except now it should be linked to an independent vacuum theory of acceleration strain. Keeping accelerations in the closet for now, the velocity theory is then determined by

$$\partial_\mu \mathit{\Delta}^{\mu\nu} = \frac{4\pi}{c} J_e^{*\nu} \tag{3.15}$$

As already indicated, this equation may be considered as the four-space analog of (2.7). It also represents a legitimate potential formulation for the velocity fields of the Maxwell-Lorentz electron derived from a completely different premise. Ultimately, the goal will be to show that the classical fields of the electron are indistinguishable from a theory of a deformed vacuum defined by vacuum gauge potentials. Whereas the field strength tensor is the fundamental field quantity of the electromagnetic theory, the counterpart quantity of the vacuum theory will be the vacuum strain tensor $\eta^{\mu\nu}$ with the two theories linked by the anti-symmetric combination in (3.11).

3.2 Properties of the Vacuum Lagrangian

Assume that Hamilton's principle can be applied to the Lagrangian $\mathcal{L}_{vac}$ producing a stationary value relative to potentials which differ only slightly from vacuum gauge velocity potentials.

$$\delta \mathcal{S} = \delta \int \mathcal{L}_{vac}(\partial^\mu A^\nu, A^\nu, x^\nu) d^4x = 0 \tag{3.16}$$

A specific form for $\mathcal{L}_{vac}$ could be postulated based on definitions of the vacuum tensor, the associated strain tensor, and knowledge of continuum mechanics; but a more revealing approach is to begin with the classical free term electromagnetic Lagrangian

$$\mathcal{L}_{em} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} \tag{3.17}$$

and insert equation (3.11). After re-arranging terms, and discarded a superfluous inverted null term, the form of an interacting ***vacuum Lagrangian*** is

$$\mathcal{L}_{vac} \equiv -\frac{1}{8\pi} \mathit{\Delta}^{\mu\nu} \eta_{\mu\nu} - \frac{1}{c} J_e^{*\nu} A_\nu \tag{3.18}$$

The functional value of the free term is available by insertion of the velocity potentials. Its value is negative[5] and proportional to the square of the vacuum dilatation. It can then be solved for the dilatation as

$$\boxed{|\eta| \equiv \sqrt{-8\pi\mathcal{L}_{vac}}} \tag{3.19}$$

This equation may be coined as the genuine vacuum gauge condition, defining a relativistic theory of potentials intimately linked to their own Lagrangian, and well suited for the description of a classical charged particle. The legitimacy of $\mathcal{L}_{vac}$ can be immediately verified by applying the Euler-Lagrange equations resulting in the Maxwell-Lorentz theory of (3.15):

$$\partial_\mu \varDelta^{\mu\nu} = \frac{4\pi}{c} J_e^{*\nu} \tag{3.20}$$

One might consider the vacuum Lagrangian and its equation of motion as a symmetry property of classical electron theory—resulting in a re-interpretation of both fields and potentials. The essential difference is that the potentials are physically meaningful quantities of the vacuum theory along with the fields themselves. This is in sharp contradistinction to the conventional theory which begins with the fields and determines 'unphysical' potentials from an arbitrary gauge condition.

Timelike and Spacelike Components of the Vacuum Lagrangian: To gain some additional perspective, it will be convenient to make use of equations (3.8) and separate the individual components of the vacuum Lagrangian as:

$$\mathcal{L}_{vac} = -\frac{1}{8\pi}[\ \varDelta_\ell^{\mu\nu} + \varDelta_e^{\mu\nu}\ ] \cdot [(\eta_\ell)_{\mu\nu} + (\eta_e)_{\mu\nu}] \tag{3.21}$$

$$= -\frac{1}{8\pi}\varDelta_\ell^{\mu\nu}(\eta_\ell)_{\mu\nu} - \frac{1}{8\pi}\varDelta_e^{\mu\nu}(\eta_e)_{\mu\nu} \tag{3.22}$$

Assume the first term on the right can be viewed as a spacelike dilatation stress density in contrast to the second term which represents a timelike distortional stress. Determination of functional values for each term shows that the dilatation portion is negative and takes out twice what the distortional portion puts in. Interaction terms can be added by considering the null current and the point source separately:

$$\mathcal{L}_{int} = \frac{1}{c} J_e^\nu \cdot \vartheta(A_e)_\nu + \frac{1}{c} J_N^\nu (A_\ell)_\nu \tag{3.23}$$

[5]The sign of the vacuum Lagrangian is opposite to the sign of the conventional electromagnetic free field Lagrangian. The sign difference is not simply a matter of taste either since it will be required for the inclusion of particle accelerations. Although the sign change represents a large rift with the conventional theory, it also exemplifies the antithetical nature of the vacuum gauge solution.

Both interaction terms are delta functions and independent Lagrangians follow as

$$\mathcal{L}_{distortion} = -\frac{1}{8\pi}\Delta_e^{\mu\nu}(\eta_e)_{\mu\nu} - \frac{1}{c}J_e^{\nu}\cdot\vartheta(A_e)_{\nu} \tag{3.24a}$$

$$\mathcal{L}_{dilatation} = -\frac{1}{8\pi}\Delta_\ell^{\mu\nu}(\eta_\ell)_{\mu\nu} - \frac{1}{c}J_N^{\nu}(A_\ell)_{\nu} \tag{3.24b}$$

Applying Euler-Lagrange equations produces the separate equations

$$\partial_\mu\Delta_e^{\mu\nu} = \frac{4\pi}{c}J_e^{\nu}\cdot\vartheta \qquad\qquad \partial_\mu\Delta_\ell^{\mu\nu} = \frac{4\pi}{c}J_N^{\nu} \tag{3.25}$$

These results are exactly equations (1.19).

Lagrangian for a Moving Vacuum: The tensor defined by equation (3.6) is a four-space stress tensor. The term containing the metric has three positive definite space-space diagonal components. These terms represent the admissibility property of the stress tensor allowing the vacuum continuum to self-generate. The Lagrangian in equation (3.18) can only accommodate this feature with an additional term. It must be linear in field quantities and can have no effect on the equations of motion. The simplest possibility is to write

$$\mathcal{L}_{vac} \longrightarrow -\frac{1}{8\pi}\left[\eta - 2\pi\sigma_e\right]^2 \tag{3.26}$$

Expanding the right side and applying the vacuum gauge condition allows $\mathcal{L}_{vac}$ to be written

$$\mathcal{L}_{vac} \longrightarrow -\frac{1}{8\pi}\Delta^{\mu\nu}\eta_{\mu\nu} + \frac{1}{2}\sigma_e\,\eta \tag{3.27}$$

A more meaningful approach begins with the observation that the vacuum Lagrangian posesses an internal symmetry under the variation

$$\Delta^{\mu\nu} \longrightarrow \Delta^{\mu\nu} - 4\pi\sigma_e\,\mathrm{g}^{\mu\nu} \tag{3.28}$$

which also results in (3.27). An appropriate interacting velocity Lagrangian specifically designed to accommodate the causality principle applied to the fields is then

$$\boxed{\mathcal{L}_{vac} \equiv -\frac{1}{8\pi}\Delta^{\mu\nu}\eta_{\mu\nu} + \frac{1}{2}\sigma_e\,\eta - \frac{1}{c}J_e^{*\mu}A_\mu} \tag{3.29}$$

The linear term should be properly referred to as a vacuum dilatation producing dilatation stresses which—as already indicated—function to propagate the vacuum away from the source. Its presence is reminiscent of a term added to a Lagrangian in point particle mechanics to represent forces of constraint. The analogy has some grey area but σ_e assumes the role of a Lagrange multiplier. A good understanding of the linear term is best demonstrated by first calculating the associated stress tensor.

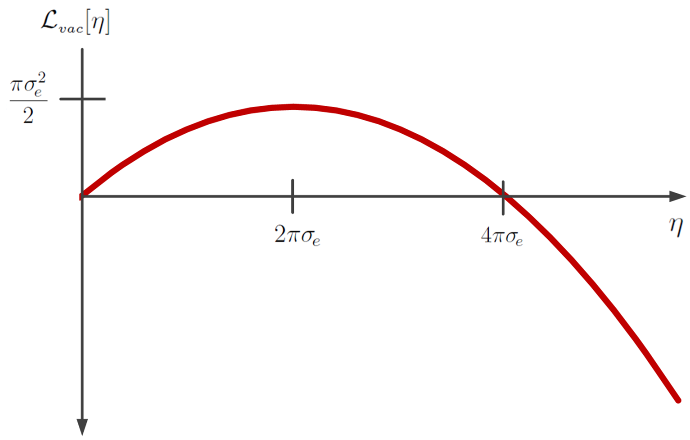

Figure 2: *Plot of $\mathcal{L}_{vac}$ versus the scalar dilatation η.*

3.3 Symmetric and Total Stress Tensors

The Noether current generated from invariance under the infinitesimal translation group $x^\mu \to x^\mu + \epsilon^\mu$ is the canonical stress tensor

$$T^{\mu\nu}_{vac} = \frac{\partial \mathcal{L}_{vac}}{\partial \eta_{\mu\lambda}} \eta^\nu{}_\lambda - \mathrm{g}^{\mu\nu} \mathcal{L}_{vac} \tag{3.30}$$

To facilitate the construction of an appropriate stress tensor based on (3.29) it is beneficial to begin by considering only terms quadratic in field quantites for which

$$T^{\mu\nu}_{vac} = \frac{1}{4\pi} \left[\tfrac{1}{2} \mathrm{g}^{\mu\nu} \Delta^{\alpha\lambda} \eta_{\alpha\lambda} - \Delta^{\mu\lambda} \eta^\nu{}_\lambda \right] \tag{3.31}$$

The velocity portion of a symmetric stress tensor can be extracted from this expression by writing

$$T^{\mu\nu}_{vac} = \Theta^{\mu\nu}_{vac} + T^{\mu\nu}_{D} \tag{3.32}$$

where the two individual tensors are given by

$$\Theta^{\mu\nu}_{vac} = \frac{1}{4\pi} \left[\tfrac{1}{2} \mathrm{g}^{\mu\nu} \eta^2 - \eta^{\mu\lambda} \eta^\nu{}_\lambda \right] \qquad\qquad T^{\mu\nu}_{D} = \frac{1}{4\pi} \eta \eta^{\nu\mu} \tag{3.33}$$

As with the conventional theory the extra term $T^{\mu\nu}_D$ can be eliminated by showing that $\partial_\mu T^{\mu\nu}_D = 0$ which leaves only the symmmetric tensor.

Applying equation (3.30) to the propagation term of the Lagrangian is rather trivial here, providing an additional stress

$$\Lambda^{\mu\nu} \equiv \frac{\sigma_e}{2} \Delta^{\dagger\mu\nu} \tag{3.34}$$

which stands on its own without need of any symmetrization. The ***Total Vacuum Stress Tensor*** then follows as

$$\boxed{\mathfrak{T}^{\mu\nu} \equiv \frac{1}{4\pi}\left[\tfrac{1}{2}\mathrm{g}^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^{\nu}{}_{\lambda}\right] + \Lambda^{\mu\nu}} \tag{3.35}$$

In the space surrounding the charge the conservation law implied by Noether's theorem is $\partial_\mu T^{\mu\nu}_{vac} = 0$. It can be verified explicitly for the vacuum theory by considering only the second term on the right in (3.31) and differentiating:

$$\partial_\mu(\varDelta^{\mu\lambda}\eta^{\nu}{}_{\lambda}) = \varDelta^{\mu\lambda}\cdot\partial_\mu\eta^{\nu}{}_{\lambda} = \varDelta^{\mu\lambda}\cdot\partial^{\nu}\eta_{\mu\lambda} = \tfrac{1}{2}\partial^{\nu}(\varDelta^{\mu\lambda}\eta_{\mu\lambda}) \tag{3.36}$$

If the point source interaction term is to be inserted into the theory it is only required to re-visit the previous calculation and include the current density from the divergence of $\varDelta^{\mu\lambda}$. Since $\partial_\mu\Lambda^{\mu\nu} = 0$, the differential law for the total vacuum stress tensor becomes

$$\partial_\mu\mathfrak{T}^{\mu\nu} = -\frac{1}{c}J^{*\lambda}_{e}\eta^{\nu}{}_{\lambda} \tag{3.37}$$

The source term here is exactly the Lorentz force density associated with a point electron interacting with its own fields. This implies an equivalence relation linking the electromagnetic and vacuum gauge theories

$$\varTheta^{\mu\nu}_{vac} \equiv \varTheta^{\mu\nu}_{1\,em} \tag{3.38}$$

This equivalence will be extended to include particle accelerations in section 5.

3.4 Stress Tensor from Canonical Energy Flux

The central disparity between the electromagnetic and vacuum gauge theories is illustrated in figure 3 showing the four extra 'diagonal' field quantities utilized by the vacuum theory. These terms serve to bolster its computational power and are critically necessary to propagate the velocity fields. In fact, the diagonal terms can be directly linked to source canonical momentum in the electric field by writing

$$\mathcal{L}_{\Lambda} = \tfrac{1}{2}\sigma_e\,\eta = \tfrac{1}{2}\boldsymbol{\pi}_{\mathrm{E}}\cdot\hat{\boldsymbol{n}} \tag{3.39}$$

Without the diagonal terms, clearly the electromagnetic theory has no ability to propagate its field. However it still seems to have at least some understanding of a moving field arising from the traditional definition of canonical momentum

$$\frac{\partial\mathcal{L}_{em}}{\partial(\partial_0\boldsymbol{A})} = \frac{1}{4\pi}\mathbf{E} \longrightarrow \boldsymbol{\pi}_{\mathrm{E}} \tag{3.40}$$

Unfortunately, if $\mathcal{L}_{em}$ is the conventional electromagnetic Lagrangian, this definition is fundamentally flawed because the vector $\mathbf{E}$ in the conventional theory includes both velocity and acceleration fields. For a causality based electromagnetic theory a more

Electromagnetic

	$\partial^0 A^1$	$\partial^0 A^2$	$\partial^0 A^3$
$\partial^1 A^0$		$\partial^1 A^2$	$\partial^1 A^3$
$\partial^2 A^0$	$\partial^2 A^1$		$\partial^2 A^3$
$\partial^3 A^0$	$\partial^3 A^1$	$\partial^3 A^2$	

Vacuum

$\partial^0 A^0$	$\partial^0 A^1$	$\partial^0 A^2$	$\partial^0 A^3$
$\partial^1 A^0$	$\partial^1 A^1$	$\partial^1 A^2$	$\partial^1 A^3$
$\partial^2 A^0$	$\partial^2 A^1$	$\partial^2 A^2$	$\partial^2 A^3$
$\partial^3 A^0$	$\partial^3 A^1$	$\partial^3 A^2$	$\partial^3 A^3$

Figure 3: *Field quantites used by electromagnetic and vacuum Lagrangians. Extra diagonal terms on the right are the vacuum gauge condition when added together.*

reasonable choice for a Lagrangian is therefore equation (1.24) which yields (3.40) for a velocity field only. There is still a problem though because momentum in the velocity fields must include the magnetic field also which is not indicated by (3.40). This suggests that canonical momentum should include all the derivatives of $\mathcal{L}_{em}$ and be written

$$\frac{\partial \mathcal{L}_{em}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4\pi} F^{\mu\nu} \longrightarrow \pi^{\mu\nu} \tag{3.41}$$

Now suppose the same reasoning is applied to the vacuum Lagrangian of equation (3.18):

$$\frac{\partial \mathcal{L}_{vac}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4\pi} \Delta^{\mu\nu} \tag{3.42}$$

This tensor contains all field quantities given on the right side of figure 3 and can still be used to produce $\pi^{\mu\nu}$ from the anti-symmetric combination in equation (3.11). On the other hand, it is more useful to transpose (3.42) and multiply through by $-2\pi\sigma_e$ re-producing $\Lambda^{\mu\nu}$. Components of this tensor can be grouped as[6]

$$\begin{aligned} &\mathrm{U}_\Lambda \equiv -\frac{1}{2}\sigma_e \boldsymbol{\nabla} \cdot \boldsymbol{A} && \frac{1}{c}\mathbf{S}_V \equiv -\frac{1}{2}\sigma_e \boldsymbol{\nabla} A \\ &\frac{1}{c}\mathbf{S}_A \equiv \frac{\sigma_e}{2}\frac{\partial \boldsymbol{A}}{\partial t} && \mathrm{T}^{ij}_\Lambda \equiv \frac{\sigma_e}{2}\left[-\frac{\partial A_i}{\partial x_j} + \delta^{ij}\partial_\alpha A^\alpha\right] \end{aligned} \tag{3.43}$$

allowing $\Lambda^{\mu\nu}$ to be written

$$\Lambda^{\mu\nu} = \begin{bmatrix} \mathrm{U}_\Lambda & \frac{1}{c}\mathbf{S}_V \\ \frac{1}{c}\mathbf{S}_A & \widehat{\mathbf{T}}_\Lambda \end{bmatrix} \tag{3.44}$$

[6] The subscript V appended to $\mathbf{S}_V$ has been borrowed from the conventional theory and only serves as a means of distinction from the momentum $\mathbf{S}_A$.

Individual elements are the canonical work-energy density U_{Λ}, the six "components" of the canonical energy flux vectors $\mathbf{S}_{V}$ and $\mathbf{S}_{A}$, and the canonical stress-strain tensor $\widehat{\mathbf{T}}_{\Lambda}$. To this tensor we may add a well known representation of the symmetric stress tensor (determined from the vacuum theory):

$$\Theta^{\mu\nu}_{vac} = \begin{bmatrix} \mathrm{U} & \frac{1}{c}\mathbf{S} \\ \\ \frac{1}{c}\mathbf{S} & -\widehat{\mathbf{T}} \end{bmatrix} \tag{3.45}$$

The elements of this tensor are the energy density U, the energy flux vector (or Poynting vector) $\mathbf{S}$, and the Maxwell stress tensor $\widehat{\mathbf{T}}$. The total stress tensor in equation (3.35) can then be written

$$\mathcal{T}^{\mu\nu} = \begin{bmatrix} \mathrm{U} & \frac{1}{c}\mathbf{S} \\ \\ \frac{1}{c}\mathbf{S} & -\widehat{\mathbf{T}} \end{bmatrix} + \begin{bmatrix} \mathrm{U}_{\Lambda} & \frac{1}{c}\mathbf{S}_{V} \\ \\ \frac{1}{c}\mathbf{S}_{A} & \widehat{\mathbf{T}}_{\Lambda} \end{bmatrix} \tag{3.46}$$

4 Stability of a Causal Electron: Vacuum Energy

The most straight forward application of the new formalism is in the rest frame where the tensor $\mathcal{T}^{\mu\nu}$ provides a simple solution for a stable particle of radius r_e. This solution is easily extended to a moving frame, but equally important is the natural emergence of a theory of inertia.

4.1 Rest Frame Solution

In the rest frame most of the energy flux components are zero and one has

$$\mathcal{T}^{\mu\nu} = \begin{bmatrix} \mathrm{U} & 0 \\ \\ 0 & -\widehat{\mathbf{T}} \end{bmatrix} + \begin{bmatrix} \mathrm{U}_{\Lambda} & \frac{1}{c}\mathbf{S}_{V} \\ \\ 0 & \widehat{\mathbf{T}}_{\Lambda} \end{bmatrix} \tag{4.1}$$

For reference, the expanded form of the symmetric stress tensor is

$$\Theta^{\mu\nu} = \frac{1}{8\pi} \begin{bmatrix} \mathrm{E}^2 & 0 & 0 & 0 \\ \\ 0 & -\mathrm{E}_x^2 + \mathrm{E}_y^2 + \mathrm{E}_z^2 & -2\mathrm{E}_x\mathrm{E}_y & -2\mathrm{E}_x\mathrm{E}_z \\ \\ 0 & -2\mathrm{E}_x\mathrm{E}_y & -\mathrm{E}_y^2 + \mathrm{E}_x^2 + \mathrm{E}_z^2 & -2\mathrm{E}_y\mathrm{E}_z \\ \\ 0 & -2\mathrm{E}_x\mathrm{E}_z & -2\mathrm{E}_y\mathrm{E}_z & -\mathrm{E}_z^2 + \mathrm{E}_x^2 + \mathrm{E}_y^2 \end{bmatrix} \cdot \vartheta \tag{4.2}$$

For the vacuum tensor, the strain portion of $\widehat{\mathbf{T}}_{\mathbf{\Lambda}}$ can be symmetrized since

$$\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = 0 \qquad\qquad \beta = 0 \tag{4.3}$$

The time dependence of the scalar potential also vanishes and it is then appropriate to write

$$\mathrm{T}^{ij}_{\Lambda} = \frac{\sigma_e}{2}\left[-\frac{1}{2}\left(\frac{\partial A_i}{\partial x_j} + \frac{\partial A_j}{\partial x_i}\right) + \delta^{ij}\boldsymbol{\nabla}\cdot\boldsymbol{A}\right] \qquad\qquad \beta = 0 \tag{4.4}$$

The strain portion of $\widehat{\mathbf{T}}_{\Lambda}$ has a restoring character as is evidenced by the minus sign, while the diagonal term has an admissibility character already alluded to.

Spacelike Integration of Components: Using $(c\tau, \boldsymbol{r})$ as rest frame coordinates, the integrated total stress-energy $\mathcal{E}^{\mu\nu}(\tau)$ at some time τ is

$$\mathcal{E}^{\mu\nu}(\tau) = \int \mathcal{T}^{\mu\nu}\cdot\vartheta(c\tau - r + r_e)d^3r = \int \Theta^{\mu\nu}\cdot\vartheta\, d^3r + \int \Lambda^{\mu\nu}\cdot\vartheta\, d^3r \tag{4.5}$$

In the first integral on the right, the coordinate r must have a lower limit to prevent the integral from diverging. The implication is a lower bound on proper time which is assumed to be $c\tau \geq 0$. If this condition is not met then the first integrand vanishes. No such requirement is needed for the second integral where the time coordinate can (and must) be extended to $c\tau = -r_e$ allowing the field to propagate from $r = 0$. Figure 4 is included for reference and shows an upper limit of spatial integration $c\tau + r_e$ for both integrals.

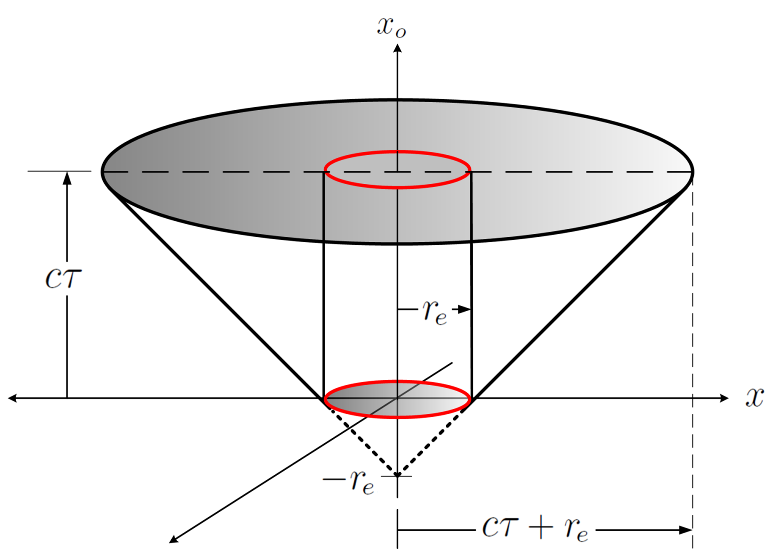

Figure 4: *Spacetime diagram indicating the expansion of the radial step $\vartheta(c\tau - r + r_e)$ in the rest frame. The radius r_e defines two regions on the inside and outside of the electron.*

Integrated components are as follows:

$\mathfrak{T}^{\tau\tau}$:
****

$$\mathfrak{T}^{\tau\tau} = \varTheta^{\tau\tau} + \Lambda^{\tau\tau}$$

$$= \frac{1}{8\pi}\mathrm{E}^2 - \frac{1}{2}\sigma_e \boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{e^2}{8\pi}\left[\frac{1}{r^4} - \frac{1}{r_e^2 r^2}\right]$$

Integrating as required by (4.5) produces

$$\mathcal{E}^{\tau\tau}(\tau) = \int \mathfrak{T}^{\tau\tau} \cdot \vartheta\, r^2\, dr d\Omega = \frac{e^2}{2}\left[\frac{1}{r_e} - \frac{1}{c\tau + r_e}\right] - 2\pi e \sigma_e (c\tau + r_e) \rightarrow -\frac{1}{2}\dot{\varrho} c\tau$$

where $\dot{\varrho} = 4\pi\sigma_e\, e > 0$ is the total momentum per unit time radiated.

$\mathfrak{T}^{\tau x}, \mathfrak{T}^{\tau y}, \mathfrak{T}^{\tau z}$:
****
There is no contribution from $\varTheta^{\tau i}$ here and it is appropriate to use the vector

$$\frac{1}{c}\mathbf{S}_{V} = \Lambda^{tj}\boldsymbol{e}_j = -\frac{1}{2}\sigma_e \boldsymbol{\nabla} A$$

The integral is

$$\frac{1}{c}\int \mathbf{S}_{V} \cdot \vartheta\, d^3 r = \mathbf{0}$$

which means that $\mathcal{E}^{\tau i} = 0$, for $i = x, y, z$. These terms must be zero since the flow of canonical energy is symmetric about the source.

$\mathfrak{T}^{xx}$, $\mathfrak{T}^{yy}$, $\mathfrak{T}^{zz}$:
****

$$\mathfrak{T}^{xx} = \varTheta^{xx} + \Lambda^{xx}$$

$$= \frac{1}{8\pi}(-\mathrm{E}_x^2 + \mathrm{E}_y^2 + \mathrm{E}_z^2) + \frac{1}{2}\sigma_e\left[\partial_\alpha A^\alpha - \frac{\partial A_x}{\partial x}\right]$$

$$= \frac{e^2}{8\pi}\left[\frac{(-x^2 + y^2 + z^2)}{r^6} + \frac{1}{r^2 r_e^2} + \frac{(x^2 - y^2 - z^2)}{r^4 r_e^2}\right]$$

The electromagnetic stress on the third line is removed by the vacuum strain at $r = r_e$. Integrating produces

$$\mathcal{E}^{xx}(\tau) = \int \mathcal{T}^{xx} \cdot \vartheta\, d^3r = \frac{1}{3}\frac{e^2}{2}\left[\frac{1}{r_e} - \frac{1}{c\tau + r_e}\right] + \frac{1}{3}4\pi e\sigma_e(c\tau + r_e)$$

$$\to mc^2 + \frac{1}{3}\dot{\varrho}c\tau$$

where use has been made of $e^2/2r_e = mc^2$. Similarly $\mathcal{E}^{yy} = \mathcal{E}^{zz} = \mathcal{E}^{xx}$.

$\mathcal{T}^{xy}$, $\mathcal{T}^{xz}$, $\mathcal{T}^{yz}$:
****

$$\mathcal{T}^{xy} = \Theta^{xy} + \Lambda^{xy} = -\frac{1}{4\pi}\mathrm{E}_x\mathrm{E}_y - \frac{1}{4}\sigma_e\left[\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\right] = \frac{e^2}{4\pi}\left[-\frac{xy}{r^6} + \frac{xy}{r_e^2 r^4}\right]$$

Here again the (symmetrized) vacuum strain removes the electromagnetic stress at $r = r_e$. It is easy to see that

$$\mathcal{E}^{xy} = \int \mathcal{T}^{xy} \cdot \vartheta\, d^3r = 0$$

and from similar calculations $\mathcal{E}^{xz} = \mathcal{E}^{yz} = 0$.

$\mathcal{T}^{x\tau}, \mathcal{T}^{y\tau}, \mathcal{T}^{z\tau}$:
****
For the last three components, use the vector

$$\frac{1}{c}\mathbf{S}_A = \Lambda^{jt}\boldsymbol{e}_j = \frac{1}{2}\sigma_e\frac{\partial \boldsymbol{A}}{\partial c\tau} = \mathbf{0}$$

These terms represent canonical energy flux based on explicit changes of the vacuum deformation with time and they vanish for the stationary electron.
****

In a matrix format the previously integrated terms are

$$\mathcal{E}^{\mu\nu}(\tau) = \begin{bmatrix} -\frac{1}{2}\dot{\varrho}c\tau & 0 & 0 & 0 \\ 0 & mc^2 + \frac{1}{3}\dot{\varrho}c\tau & 0 & 0 \\ 0 & 0 & mc^2 + \frac{1}{3}\dot{\varrho}c\tau & 0 \\ 0 & 0 & 0 & mc^2 + \frac{1}{3}\dot{\varrho}c\tau \end{bmatrix} \tag{4.6}$$

The energy component should have yielded a term mc^2 from an integral of the electromagnetic energy density but this term was obliterated by the presence of vacuum stresses inside the particle radius. In addition to this, mc^2 terms in the remaining diagonal components are an admixture of electromagnetic and vacuum stresses. But the dilemma created by the integration can be easily disentangled with the inclusion of an integration constant added to the total energy tensor

$$\mathcal{E}^{\mu\nu} \longrightarrow \mathcal{E}^{\mu\nu} + \mathrm{g}^{\mu\nu} mc^2 \equiv \mathcal{E}^{\mu\nu}_{part} + \mathcal{E}^{\mu\nu}_{vac} \tag{4.7}$$

where

$$\mathcal{E}^{\mu\nu}_{part} = \begin{bmatrix} mc^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{E}^{\mu\nu}_{vac} = \begin{bmatrix} -\frac{1}{2}\dot{\varrho}c\tau & 0 & 0 & 0 \\ 0 & \frac{1}{3}\dot{\varrho}c\tau & 0 & 0 \\ 0 & 0 & \frac{1}{3}\dot{\varrho}c\tau & 0 \\ 0 & 0 & 0 & \frac{1}{3}\dot{\varrho}c\tau \end{bmatrix} \tag{4.8}$$

On the left, $\mathcal{E}^{\mu\nu}_{part}$ is determined by a structureless constant—completely free of all electromagnetic stresses. On the right, the tensor $\mathcal{E}^{\mu\nu}_{vac}$ represents an unstable continuum, with properties defined by the particle radius. Of particular interest is the negative sign appearing in the energy component. In the language of deformable media it represents the total canonical stress imparted to the vacuum in a time $c\tau$ by the canonical momentum. Equivalently, it represents the total work done on the vacuum by an admissibility force constant $\lambda \sim \sigma_e^2$. Nevertheless, this should be interpreted as positive energy density surrounding the particle which is necessarily one-half the momentum density times a factor of c—and requiring a quantum of the theory to satisfy

$$u = \tfrac{1}{2}|\boldsymbol{p}|c \tag{4.9}$$

It is assumed that the factor of 1/2 reflects that the associated longitudinal wave has only one polarization state—as opposed to two polarization states for transverse waves. The ***inertial power*** P_{in} has already been determined in equation (2.46) and follows by differentiating the energy component $\mathcal{E}^{\mu\nu}_{vac}$ with respect to time.

An accurate physical interpretation of $\mathcal{E}^{\mu\nu}_{vac}$ is complicated. As a first guess it may be supposed that this energy has no mass equivalent and bears no functional relationship to any other established law of physics. This doesn't seem unreasonable since the overt violation of energy conservation is determined by the presence of the gauge field A^{ν}_{ℓ}.

Cast in a different light, the local deformation of the vacuum, the radius of the particle, and the generation of canonical field energy are all intimately connected properties in the abstract space provided by the vacuum gauge potentials. Since potentials are of no tangible substance, it can only be concluded that all of these properties lie beyond the possibility of any experimental verification. If this discussion is correct then the classical vacuum gauge electron bears a strong similarity with its quantum mechanical counterpart in the sense that there exists well defined limitations on measurable information which can be obtained about the particle.

Timelike Integration: Integrating over causal space is not the only way to arrive at the tensors $\mathcal{E}^{\mu\nu}_{part}$ and $\mathcal{E}^{\mu\nu}_{vac}$ in the rest frame. The three-volume element

$$dV_s = r_e^2 \, d\Omega c d\tau \tag{4.10}$$

represents a differential surface element of the spacetime cylinder in figure 4. Restricting the proper time interval to $[0, \tau_e]$, the integral over components of $\mathcal{T}^{\mu\nu}$ is

$$\mathcal{E}^{\mu\nu} = \int_{\tau_e} \mathcal{T}^{\mu\nu} dV_s \tag{4.11}$$

This is a flux integral equivalent to an evaluation of the total vacuum energy contained within the particle radius. The resulting tensor reads

$$\mathcal{E}^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & mc^2 & 0 & 0 \\ 0 & 0 & mc^2 & 0 \\ 0 & 0 & 0 & mc^2 \end{bmatrix} \tag{4.12}$$

and re-produces $\mathcal{E}^{\mu\nu}_{part}$ with the inclusion of the integration constant. The quantity $\mathcal{E}^{\mu\nu}_{vac}$ results from replacing the integrand in equation (4.11) with $\Lambda^{\mu\nu}$ and extending to an arbirtrary time τ.

4.2 Moving Frame Solution

A covariant solution to the stability problem can be established by first generalizing the total stress tensor in equation (3.35) to include acceleration components

$$\mathcal{T}^{\mu\nu} = \Theta_1^{\mu\nu} + \Theta_2^{\mu\nu} + \Theta_3^{\mu\nu} + \Lambda^{\mu\nu} \tag{4.13}$$

While this equation has not been adequately justified yet it is easy to show that magnitudes of the components of $\Theta_1^{\mu\nu}$ and $\Lambda^{\mu\nu}$ are the same in the neighborhood $\rho \sim r_e$. On the other hand, the ratios of individual terms of the symmetric tensor are

$$\left| \frac{\Theta_2^{\mu\nu}}{\Theta_1^{\mu\nu}} \right| \sim \frac{a^\nu R_\nu}{c^2} \qquad\qquad \left| \frac{\Theta_3^{\mu\nu}}{\Theta_1^{\mu\nu}} \right| \sim \left[\frac{a^\nu R_\nu}{c^2} \right]^2 \tag{4.14}$$

Near the radius of the particle, these terms are utterly neglible unless accelerations are on the order of $10^{31}\, m/s^2$. This means that the stability of the particle will be determined exclusively by the velocity tensors only.

Covariant Theory of Stability: To address the stability problem first define the unitless quantities

$$\mathcal{G}_1^{\mu\nu} \equiv 2\beta^\mu \beta^\nu - 2\mathcal{U}^\mu \mathcal{U}^\nu - \mathrm{g}^{\mu\nu} \tag{4.15a}$$

$$\mathcal{G}_2^{\mu\nu} \equiv \beta^\mu \beta^\nu + \beta^\mu \mathcal{U}^\nu - \mathrm{g}^{\mu\nu} = -\partial^\nu R^\mu \tag{4.15b}$$

The symmetric stress tensor and the vacuum tensor are then

$$\Theta_1^{\mu\nu} = \frac{1}{8\pi}\eta^2 \mathcal{G}_1^{\mu\nu} \qquad\qquad \Lambda^{\mu\nu} = -\frac{1}{2}\sigma_e\eta\,[\mathcal{G}_1^{\mu\nu} - \mathcal{G}_2^{\mu\nu}] \tag{4.16}$$

and the total stress may be written

$$\mathcal{T}^{\mu\nu} = -(\mathcal{L}_o + \mathcal{L}_\Lambda)\mathcal{G}_1^{\mu\nu} + \mathcal{L}_\Lambda \mathcal{G}_2^{\mu\nu} \tag{4.17}$$

Stability is determined by the first term which vanishes at $\rho = r_e$. What remains is the last term representing radiated stress in the form of canonical energy flux. This tensor only has real meaning at the electron radius and is just a portion of the vacuum tensor at any other radius. It is still an energy density though and deserves its own name

$$\mathcal{E}_{rad}^{\mu\nu} \equiv \mathcal{L}_\Lambda \mathcal{G}_2^{\mu\nu}\bigg|_{\rho=r_e} = -\frac{e^2}{8\pi r_e^4}\,\partial^\nu R^\mu \tag{4.18}$$

It may be readily integrated over the particle radius to determine the four-vector vacuum radiation rate. The integral could be constructed as a sum over Fourier modes similar to previous integrals. Instead, a simplified variation is:

$$P^\mu = c\int_{\rho=r_e} \mathcal{E}_{rad}^{\mu\nu}\,\mathcal{U}_\nu R^2 d\Omega = P_{in}\beta^\mu \tag{4.19}$$

As a final calculation it is easy to recover the original vacuum Lagrangian through contractions of $\mathcal{T}^{\mu\nu}$ with the four-velocity:

$$\mathcal{L}_{vac} = -\beta_\mu \mathcal{T}_{new}^{\mu\nu}\beta_\nu \tag{4.20}$$

This eliminates the radiation term $\mathcal{E}_{rad}^{\mu\nu}$–either at the electron radius or anywhere else—and renders the correct Lagrangian after application of the vacuum gauge condition in equation (3.19).

Total Energy Tensor in a Moving Frame: A Lorentz transformation of $\mathcal{E}_{vac}^{\mu\nu}$ in equation (4.8) can be implemented by considering the timelike energy term and spacelike pressure terms separately:

$$\mathcal{E}_{vac}^{\mu\nu} = -\tfrac{1}{2}\dot{\varrho}c\tau \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad\qquad \mathcal{P}_{vac}^{\mu\nu} = \tfrac{1}{3}\dot{\varrho}\tau \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.21}$$

The general Lorentz transformation is $\mathcal{X}'^{\mu\nu} = L^\mu{}_\alpha\, \mathcal{X}^{\alpha\beta}\, L_\beta{}^\nu$ which leads to the transformed quantities

$$\mathcal{E}'^{\mu\nu}_{vac} = -\tfrac{1}{2}\dot{\varrho}c\tau\beta^\mu\beta^\nu \tag{4.22a}$$

$$\mathcal{P}'^{\mu\nu}_{vac} = \tfrac{1}{3}\dot{\varrho}\tau\cdot(\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu}) \tag{4.22b}$$

One can also arrive at these results by integrating Fourier modes of the vacuum tensor over the invariant particle radius. Begin with the integral

$$\mathcal{E}'^{\mu\nu}_{vac} + \mathcal{P}'^{\mu\nu}_{vac} c = \int \left[\oint_{\rho=r_e} c\Lambda^{\mu\nu}_{\omega} R^2 d\Omega \right] d\omega \tag{4.23}$$

The vacuum tensor can be expanded similar to equations (3.8) and (3.10),

$$\Lambda^{\mu\nu} = \Lambda^{\mu\nu}_{\ell} + \Lambda^{\mu\nu}_{e} \tag{4.24}$$

but the integration over $\Lambda^{\mu\nu}_{e}$ is zero while the remaining symmetric timelike and spacelike terms can be separated as

$$\mathcal{E}'^{\mu\nu}_{vac} = -\frac{c}{2a_e} \int \left[\oint_{\rho=r_e} A^{\mu}_{e:\omega} A^{\nu}_{e:\omega} R^2 d\Omega \right] d\omega = -\tfrac{1}{2}\dot{\varrho} c\tau \beta^{\mu}\beta^{\nu} \tag{4.25a}$$

$$\mathcal{P}'^{\mu\nu}_{vac} = \frac{1}{a_e} \int \left[\oint_{\rho=r_e} A^{\mu}_{\ell:\omega} A^{\nu}_{\ell:\omega} R^2 d\Omega \right] d\omega = \tfrac{1}{3}\dot{\varrho}\tau (\beta^{\mu}\beta^{\nu} - \mathrm{g}^{\mu\nu}) \tag{4.25b}$$

Contracting both terms with the metric tensor (while removing primes and subscripts) then derives the general scalar relation

$$\boxed{\mathcal{E} = \tfrac{1}{2}\mathcal{P} c} \tag{4.26}$$

where both $\mathcal{E}$ and $\mathcal{P}$ are seen to be scalar invariants. It is important to understand that $\Lambda^{\mu\nu}_{e}$ plays no role in the integration implying that both tensors in equation (4.25) are determined exclusively from the gauge field A^{ν}_{ℓ}. The gauge field is the propagator of vacuum energy in the causal theory.

The sum of the two contributions in (4.25) define a quantity resembling the relativistic perfect fluid stress tensor. The comparison seems to have limitations though since each term is a linear function of proper time. Moreover, the four velocity β^{ν} is the motion of the source while the actual 'fluid' is radiating outward at speed c. Keeping the energy and momentum terms separate seems to be a more useful approach. As an example, a divergence operation applied to $\mathcal{E}'^{\mu\nu}_{vac}$ with a minus sign leads to the four-vector inertial power formula already derived in (4.19). Regardless, the Lorentz transformation of $\mathcal{E}^{\mu\nu}_{part}$ in equation (4.8) may now be added to the vacuum terms to determine the general total energy tensor.

$$\mathcal{E}'^{\mu\nu}_{total} = mc^2\beta^{\mu}\beta^{\nu} - \tfrac{1}{2}\dot{\varrho} c\tau \beta^{\mu}\beta^{\nu} + \tfrac{1}{3}\dot{\varrho} c\tau \left(\beta^{\mu}\beta^{\nu} - \mathrm{g}^{\mu\nu}\right) \tag{4.27}$$

Contracting on both indices with the four-velocity will then determine the invariant Hamiltonian[7]

$$\boxed{\mathcal{H} = mc^2 - \tfrac{1}{2}\dot{\varrho} c\tau} \tag{4.28}$$

This Hamiltonian can be shown to be independent of particle accelerations.

[7] A more rigorous calculation of $\mathcal{H}$ is available in the Appendix and includes the operator $\vartheta(\tau)$.

4.3 Dirac Electron

In terms of timelike and spacelike four-vectors, radiated stress in (4.18) can be written

$$\mathcal{E}^{\mu\nu}_{rad} = \frac{mc^2}{4\pi r_e^3}\left[\mathcal{U}^\nu\beta^\mu + \beta^\mu\beta^\nu - \beta^\lambda(\beta_\lambda + \mathcal{U}_\lambda)\mathrm{g}^{\mu\nu}\right] \tag{4.29}$$

The terms multiplying the metric might be considered superfluous but they are necessary to enforce the overall form of the tensor as a quantity $X^{\mu\nu} - X\mathrm{g}^{\mu\nu}$. The scalar contraction is an easily recognizable quantity

$$\mathcal{E}_{rad} = -\frac{mc^2}{\mathcal{V}_e} \tag{4.30}$$

where $\mathcal{V}_e$ is the volume inside the electron radius.

To provide a link with Dirac electron theory replace the four-velocity in equation (4.29) with the four-momentum using $p^\nu = mc\beta^\nu$ and make the quantum mechanical substitution

$$\boxed{\mathcal{U}^\nu \longrightarrow \pm\gamma^\nu} \tag{4.31}$$

where γ^ν are the Dirac matrices. Choosing the minus sign, the result is a new tensor

$$\mathcal{E}^{\mu\nu}_{Dirac} = \frac{c}{4\pi r_e^3}\left[-\gamma^\nu p^\mu + \frac{1}{mc}p^\mu p^\nu - mc\,\mathrm{g}^{\mu\nu} + \gamma^\lambda p_\lambda \mathrm{g}^{\mu\nu}\right] \tag{4.32}$$

The term inside the brackets of (4.32) already contains the Dirac energy-momentum tensor which becomes apparent by operating on the left and right with Dirac spinor fields rendering

$$\frac{1}{mc}p^\mu p^\nu \bar{\psi}\psi - \mathsf{T}^{\dagger\mu\nu}_{Dirac} = 0 \tag{4.33}$$

where

$$\mathsf{T}^{\mu\nu}_{Dirac} = \bar{\psi}[\gamma^\mu p^\nu - \mathrm{g}^{\mu\nu}(\gamma^\lambda p_\lambda - mc)]\psi \tag{4.34}$$

One can also derive the Dirac Lagrangian directly from the contraction

$$\mathcal{L}_{Dirac} = \bar{\psi}\left[\frac{1}{c}\mathcal{E}_{Dirac}\cdot\mathcal{V}_e\right]\psi \tag{4.35}$$

The simplicity by which Dirac electron theory has emerged from the vacuum gauge electron is impressive. Of importance is the fact that $\mathcal{E}^{\mu\nu}_{Dirac}$ vanishes when operated on by a set of Dirac spinors. We interpret this to mean that stress associated with the flow of the vacuum field cannot be known to the quantum mechanical particle. It is also important to mention that the connection in equation (4.31) is determined exclusively by the gauge field A^ν_ℓ as was the case in the previous section. This fact supports all other calculations in vacuum gauge electrodynamics which requires the gauge field alone to be the vehicle for propagation of the vacuum. According to calculations presented here the gauge field also provides the link to the quantum mechanical electron—re-interpreted as a manifestation of the relentless proliferation of vacuum energy.

5 Accelerated Motions of the Electron

It is well known that an accelerating charge emits transverse electromagnetic waves over a range of frequencies. For the vacuum gauge particle, such waves can be interpreted as resulting from induced variations of the surface charge density, in addition to the velocity configuration $\sigma_e(\theta,\phi)$. A description of these variations will naturally require the inclusion of an acceleration strain tensor $\epsilon^{\mu\nu}$. One essential property of $\epsilon^{\mu\nu}$ is immediately available by requiring that it may not be a source of additional vacuum dilatation so that $\epsilon = 0$ only. This requirement is consistent with classical radiation theory which utilizes only transverse potentials for the description of propagating photons. In other words, classical radiation theory de-couples from the vacuum theory of the electron.[8]

Naturally, $\epsilon^{\mu\nu}$ is expected to be functionally dependent on the four-acceleration of the particle, and as a preliminary calculation it will be useful to define the quantity $a^\nu_\perp$ orthogonal to the spacelike vector $\mathcal{U}^\nu$ and given by

$$a^\nu_\perp \equiv a^\nu + (a^\lambda \mathcal{U}_\lambda)\mathcal{U}^\nu \tag{5.1}$$

A link between $a^\nu_\perp$ and the velocity theory can be immediately established from the contractions

$$\varDelta^{\mu\nu} a^\perp_\nu = \varDelta^{\mu\nu} a^\perp_\mu = 0 \tag{5.2}$$

which are the same as the two eigenvalue equations:

$$\eta^{\mu\nu} a^\perp_\nu = \eta a^\mu_\perp \qquad\qquad \eta^{\mu\nu} a^\perp_\mu = \eta a^\nu_\perp \tag{5.3}$$

It is also important to observe that $a^\nu_\perp$ is orthogonal to the four-vectors R^ν and β^ν in addition to its defined orthogonality with $\mathcal{U}^\nu$.

5.1 Theory of Acceleration Strain

The vacuum gauge condition allows the velocity and acceleration fields of the electron to be addressed as independent theories. An initial conjecture for the form of an acceleration strain tensor might therefore be to simply follow the velocity theory and differentiate the acceleration potentials. Employing equation (3.5) allows this tensor to be written

$$\varepsilon^{\mu\nu} \equiv \partial^\mu A^\nu_a = -\chi(\eta^{\mu\nu} + J^{\mu\nu}_a) - \partial^\mu\chi \cdot A^\nu \tag{5.4}$$

A problem already presents itself though because the covariant derivative in this equation contains a term proportional to the velocity strain. Nevertheless, an acceleration Lagrangian can still be written as

$$\mathcal{L}_a = \frac{1}{8\pi}\left[\varepsilon^{\mu\nu}\varepsilon_{\mu\nu} - \varepsilon^2\right] + \frac{1}{c} J^\nu_a A^a_\nu \tag{5.5}$$

[8] This section relies heavily on various contractions of the tensors $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu}$. These contractions have been evaluated explicitly in separate tables in Appendix B serving as a useful calculational tool.

for which correct equations of motion are easily verified. However, it is evident that both ε and $\mathcal{L}_a$ both have the non-zero values

$$\varepsilon = -2\chi\eta \qquad\qquad \mathcal{L}_a = \chi^2\eta^{\mu\nu}\eta_{\mu\nu} \tag{5.6}$$

and this violates the premise that acceleration strain may not dilate the vacuum.

A clue to resolving the problem can be found from a review of equation (3.14) which makes no specification as to what theory of acceleration strain is used to produce the current density J_a^ν. This important observation is suggestive of a strategy to 'cleanse' $\varepsilon^{\mu\nu}$ of unwanted terms leaving an appropriate tensor $\epsilon^{\mu\nu}$ with the required property $\epsilon = 0$. As an initial guess, suppose all symmetric terms are separated from $\varepsilon^{\mu\nu}$ by

$$\varepsilon^{\mu\nu} = s^{\mu\nu} - \epsilon^{\nu\mu} \tag{5.7}$$

This relation can be inserted into $\mathcal{L}_a$ producing the quadratic form

$$\mathcal{L}_a = \frac{1}{8\pi}\left[s^{\mu\nu}s_{\mu\nu} - 2\epsilon^{\mu\nu}s_{\nu\mu} + \epsilon^{\mu\nu}\epsilon_{\mu\nu} - s^2 + 2s\epsilon - \epsilon^2\right] + \frac{1}{c}J_a^\nu A_\nu^a \tag{5.8}$$

Viewing this equation in terms independent field quantities $s^{\mu\nu}$ and $\epsilon^{\mu\nu}$, and associating the current J_a^ν with $\epsilon^{\mu\nu}$, a legimate set of Euler-Lagrange equations is:

$$\begin{aligned}\partial_\mu\epsilon^{\mu\nu} - \partial_\mu(s^{\nu\mu} - \mathrm{g}^{\mu\nu}s) &= \frac{4\pi}{c}J_a^\nu \\ \partial_\mu(s^{\mu\nu} - \mathrm{g}^{\mu\nu}s) - \partial_\mu\epsilon^{\nu\mu} &= 0\end{aligned} \tag{5.9}$$

The tensor $s^{\mu\nu}$ is now easily eliminated resulting in the equation

$$\partial_\mu[\epsilon^{\mu\nu} - \epsilon^{\nu\mu}] = \frac{4\pi}{c}J_a^\nu \tag{5.10}$$

The task of removing the symmetric terms is tricky business and depends critically on the expansion of the acceleration four-vector in terms of $a_\perp^\nu$. The suprisingly simple result is

$$\boxed{\epsilon^{\mu\nu} \equiv A^\mu a_\perp^\nu} \tag{5.11}$$

There are many interesting properties of this bi-linear quantity including $\epsilon = 0$ and vanishing determinants of all individual two-by-two minors which implies $\det \epsilon^{\mu\nu} = 0$. Divergences on each index of $\epsilon^{\mu\nu}$ are

$$\partial_\mu\epsilon^{\nu\mu} = \frac{4\pi}{c}(j_a^\nu - J_a^\nu) \qquad\qquad \partial_\mu\epsilon^{\mu\nu} = \frac{4\pi}{c}j_a^\nu \tag{5.12}$$

and the second equation is the acceleration analog of equation (3.15) producing the current density j_a^ν given by

$$j_a^\nu \equiv \frac{1}{4\pi c}\eta\, a_\perp^\nu \tag{5.13}$$

Properties of the Acceleration Current: A Comparison with equation (5.3) shows that j_a^ν results from an interaction of $a_\perp^\nu$ with the vacuum strain. Although j_a^ν is not a conserved current it is not associated with any additional charge density either as can be seen in the instantaneous rest frame where its time component vanishes. On the other hand, it can be evaluated at the electron radius taking the form

$$j_a^\nu \bigg|_{\rho=r_e} = \sigma_e \dot{\beta}_\perp^\nu \tag{5.14}$$

This may be interpreted as a modulation of the charge density during accelerated motions and functioning to generate acceleration strain waves. An important integral of the strain current using the hyper-surface element in (2.43) is

$$\int j_a^\nu \mathcal{U}_\nu R^2 d\Omega\, d\tau = 0 \tag{5.15}$$

indicating that j_a^ν is not radiated.

5.2 Generalized Vacuum Lagrangian

While the acceleration potentials have no direct connection with an acceleration strain tensor, they are still useful—and necessary—for the development of a generalized vacuum Lagrangian which accommodates accelerations of the particle. Hamilton's principle for the velocity fields is

$$\delta \mathcal{S} = \delta \int \mathcal{L}_{vac}(\partial^\mu A^\nu, A^\nu, x^\nu) d^4x = 0 \tag{5.16}$$

and the correct potentials A^ν which satisfy this condition follow by considering variations of the form

$$A^\nu \longrightarrow A^\nu + \alpha \xi^\nu \tag{5.17}$$

where ξ^ν is an arbitrary function of the coordinates and α is a scalar. However, comparison of (5.17) with the general form of the vacuum gauge potentials in equation (1.14) suggests that the acceleration potentials are to be used as variable functions with the replacements

$$\alpha \longrightarrow e \qquad\qquad \xi^\nu \longrightarrow -\frac{a^\lambda R_\lambda}{\rho^2} R^\nu \tag{5.18}$$

In other words, the velocity theory is a stationary value relative to all possible accelerations of the particle and the Lagrange equations (with a slight change of notation) are then derivable from the condition

$$\frac{d\mathcal{S}}{de} = \int \left[\frac{\partial \mathcal{L}_{vac}}{\partial A_\lambda} \frac{\partial A_\lambda}{\partial e} + \frac{\partial \mathcal{L}_{vac}}{\partial A_{\lambda,\,\mu}} \frac{\partial A_{\lambda,\,\mu}}{\partial e} \right] d^4x = 0 \tag{5.19}$$

There is no injustice here in choosing e as a vanishing scalar parameter since previous calculations require the charge to be associated with velocity fields only. Performing an integration by parts gives

$$\frac{d\mathcal{S}}{de} = \int \left[\frac{\partial \mathcal{L}_{vac}}{\partial A_\lambda} - \frac{d}{dx_\mu} \frac{\partial \mathcal{L}_{vac}}{\partial A_{\lambda,\mu}} \right] \frac{\partial A_\lambda}{\partial e} \, d^4x + \int \frac{d}{dx_\mu} \left[\frac{\partial \mathcal{L}_{vac}}{\partial A_{\lambda,\mu}} \frac{\partial A_\lambda}{\partial e} \right] d^4x = 0 \tag{5.20}$$

The second integral might be shown to be zero for acceleration potentials which vanish at endpoints x_1^ν and x_2^ν; however no such requirement is necessary based on the divergence calculation

$$\partial_\mu \left[\Delta^{\mu\lambda} \xi_\lambda \right] = 0 = \tfrac{1}{2} \partial_\mu J_a^\mu \tag{5.21}$$

This is not the whole story though because both terms in (5.20) allow for the inclusion of additional terms in the integrand without changing the stationary condition. Specifically, any vector field which is orthogonal to the null vector R^λ is fair game leading to possible current densities

$$\mathcal{J}^\lambda = h_1 \cdot a_\perp^\lambda + h_2 \cdot R^\lambda \tag{5.22}$$

Both current densities in equation (5.12) are exactly of this form.

Suppose therefore that the explicit form of the velocity Lagrangian[9] is written

$$\mathcal{L}_{vac} = -\frac{1}{8\pi} \left[\partial^\mu A^\nu \partial_\mu A_\nu - (\partial_\nu A^\nu)^2 \right] - \frac{1}{c} j_e^{*\nu} A_\nu \tag{5.23}$$

and assume that accelerated motions of the electron follow from a first order correction to the strain tensor accompanied by the causal appearance of an acceleration four-current:

$$\partial^\mu A^\nu \longrightarrow \partial^\mu A^\nu - A^\mu a_\perp^\nu \qquad\qquad j_e^{*\nu} \longrightarrow j_e^{*\nu} - j_a^\nu \tag{5.24}$$

Inserting these variations and keeping only terms first order in accelerations results in

$$\mathcal{L}_{vac} = -\frac{1}{8\pi} \left[\partial^\mu A^\nu \partial_\mu A_\nu - 2\partial^\mu A^\nu A_\mu a_\nu^\perp - (\partial_\nu A^\nu)^2 + 2\partial_\nu A^\nu A^\mu a_\mu^\perp \right] - \frac{1}{c} (j_e^{*\nu} - j_a^\nu) A_\nu \tag{5.25}$$

The form of this Lagrangian motivates (briefly) a definition for an acceleration stress tensor

$$\Delta_a^{\mu\nu} \equiv A^\mu a_\perp^\nu - \mathrm{g}^{\mu\nu} A_\lambda a_\perp^\lambda \tag{5.26}$$

The Lagrange equations will then be determined from

$$\frac{\partial \mathcal{L}_{vac}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4\pi} \left[\Delta^{\mu\nu} - \Delta_a^{\mu\nu} \right] \qquad \frac{\partial \mathcal{L}_{vac}}{\partial A_\nu} = \frac{1}{4\pi} \Delta^{\mu\nu} a_\mu^\perp - \frac{1}{c} \left(j_e^{*\nu} - j_a^\nu \right) \tag{5.27}$$

[9]Use lowercase $j_e^{*\nu}$ now for the point current density.

Making the connection $\mathit{\Delta}_a^{\mu\nu} \to \epsilon^{\mu\nu}$, equations of motion are now easily identified with the help of (5.2) rendering

$$\partial_\mu \mathit{\Delta}^{\mu\nu} = \frac{4\pi}{c} j_e^{*\nu} \qquad\qquad \partial_\mu \epsilon^{\mu\nu} = \frac{4\pi}{c} j_a^\nu \tag{5.28}$$

One can argue that a proper generalized Lagrangian should lead to both strain equations in (5.12). The inclusion of the conjugate tensor is not difficult and properly accounted for in Appendix C.

5.3 Symmetric and Total Stress Tensors

The most general symmetric stress tensor for accelerated motions of the electron is derivable from two different approaches.

Stress Tensor from the Velocity Theory: The velocity portion of the symmetric stress tensor has already been derived in section 3.3:

$$\mathit{\Theta}_{vac}^{\mu\nu} = \frac{1}{4\pi} \left[\tfrac{1}{2} \mathrm{g}^{\mu\nu} \eta^2 - \eta^{\mu\lambda} \eta^\nu{}_\lambda \right] \tag{5.29}$$

The generalization of this tensor follows immediately by inserting the variation

$$\eta^{\mu\nu} \longrightarrow \eta^{\mu\nu} - \epsilon^{\mu\nu} \tag{5.30}$$

The resulting stress tensor is then

$$\mathit{\Theta}_{vac}^{\mu\nu} = \frac{1}{4\pi} \left[\tfrac{1}{2} \mathrm{g}^{\mu\nu} \eta^2 - \eta^{\mu\lambda} \eta^\nu{}_\lambda + \eta^{\mu\lambda} \epsilon^\nu{}_\lambda - \epsilon^{\mu\lambda} \epsilon^\nu{}_\lambda + \epsilon^{\mu\lambda} \eta^\nu{}_\lambda \right] \tag{5.31}$$

which is exactly equation (1.35a).

Stress Tensor Using the Total Strain: The generalized stress tensor can also be determined by using the quantity $\zeta^{\mu\nu}$ as the field quantity and j^ν as an appropriately chosen total current density, which casts the Lagrangian into the compact form

$$\mathcal{L}_{vac} = \frac{1}{8\pi} [\zeta^2 - \zeta^{\mu\nu} \zeta_{\mu\nu}] - \frac{1}{c} j^\nu A_\nu \tag{5.32}$$

This Lagrangian produces both acceleration strain equations of motion in addition to the velocity equation. Invoking translational invariance as usual leads to the Noether current

$$T_{vac}^{\mu\nu} = \frac{\partial \mathcal{L}_{vac}}{\partial \zeta_{\mu\lambda}} \zeta^\nu{}_\lambda - \mathrm{g}^{\mu\nu} \mathcal{L}_{vac} = \frac{1}{4\pi} \left[\tfrac{1}{2} \mathrm{g}^{\mu\nu} \zeta^2 - \zeta^{\mu\lambda} \zeta^\nu{}_\lambda + \zeta \zeta^{\nu\mu} \right] \tag{5.33}$$

To extract the symmetric stress tensor one can simply write

$$T_{vac}^{\mu\nu} = \mathit{\Theta}_{vac}^{\mu\nu} + \zeta \zeta^{\nu\mu} \tag{5.34}$$

where

$$\Theta^{\mu\nu}_{vac} = \frac{1}{4\pi}\left[\tfrac{1}{2}g^{\mu\nu}\zeta^2 - \zeta^{\mu\lambda}\zeta^{\nu}{}_{\lambda}\right] \tag{5.35}$$

After crossing off superfluous null terms this tensor re-produces (5.31). If the tensor $\mathcal{R}^{\mu\nu}$ is defined by

$$\mathcal{R}^{\mu\nu} \equiv \frac{1}{4\pi}\zeta^{\mu\lambda}\zeta^{\nu}{}_{\lambda} \tag{5.36}$$

then a concise representation of the total stress tensor, inclusive of the requirement to propagate the velocity field, is

$$\boxed{\mathcal{T}^{\mu\nu} = \tfrac{1}{4}g^{\mu\nu}\mathcal{R} - \mathcal{R}^{\mu\nu} + \Lambda^{\mu\nu}} \tag{5.37}$$

Unfortunately, this derivation neglects the offending term which can be written

$$\zeta\zeta^{\nu\mu} = \frac{1}{4\pi}\eta\eta^{\nu\mu} - \Theta^{\mu\nu}_{2} \tag{5.38}$$

The problem of removing the velocity term has already been addressed. The presence of $\Theta^{\mu\nu}_{2}$ is assumed to be a consequence of neglecting acceleration terms to formulate the velocity theory. More specifically, a divergence operation on equation (5.29) is prohibited from carrying acceleration terms which rightfully belong to the acceleration theory. Relaxing this requirement gives rise to a small but significant modification to the velocity theory

$$\partial_\mu\Theta^{\mu\nu}_{1} \longrightarrow \partial_\mu\Theta^{\mu\nu}_{1} + \frac{1}{c}\eta J^{\nu}_{a} \tag{5.39}$$

Enforcement of a zero divergence for all regions of space excluding the location of the particle will then require the appearance of an interaction term satisfying

$$\partial_\mu\Theta^{\mu\nu}_{int} = -\frac{1}{c}\eta J^{\nu}_{a} \tag{5.40}$$

To solve this equation for $\Theta^{\mu\nu}_{int}$ write

$$\begin{aligned} 4\pi\cdot\partial_\mu\Theta^{\mu\nu}_{int} &= -\eta\partial_\mu(\epsilon^{\mu\nu} - \epsilon^{\nu\mu}) \\ &= \partial_\mu(\eta\epsilon^{\mu\nu} + \eta\epsilon^{\nu\mu}) = 4\pi\cdot\partial_\mu\Theta^{\mu\nu}_{2} \end{aligned} \tag{5.41}$$

For a final calculation it will be necessary to adequately address the problem of the divergence of the symmetric stress tensor in vacuum gauge electrodynamics. Since quadratic stresses are not defined inside the vacuum boundary, the point source interaction term on the right side of (3.37) will not be necessary. However, if velocity and acceleration theories are truely independent then the acceleration current must be included so that divergences on both indices are

$$\partial_\mu\mathcal{T}^{\mu\nu} = -\frac{1}{c}\eta J^{\nu}_{a} \tag{5.42a}$$

$$\partial_\nu\mathcal{T}^{\mu\nu} = -\frac{1}{c}\eta J^{\mu}_{a} + \frac{1}{c}2\pi\sigma_e\, j^{\mu}_{e} \tag{5.42b}$$

A Derivatives of the Null Vector

The covariant derivative of R^ν is

$$\partial^\mu R^\nu = \mathrm{g}^{\mu\nu} - \frac{R^\mu \beta^\nu}{\rho} \tag{A.1}$$

The trace of the resulting matrix gives the 4-divergence $\partial_\nu R^\nu = 3$. In terms of individual components–and with the inclusion of a sign–a useful construction is:

$$-\partial^\mu R^\nu = \begin{bmatrix} -\dfrac{\partial R}{\partial t} & -\dfrac{\partial \boldsymbol{R}}{\partial t} \\ \boldsymbol{\nabla} R & \boldsymbol{\nabla}\boldsymbol{R} \end{bmatrix} \tag{A.2}$$

where individual components are given by

$$\frac{\partial R}{\partial t} = 1 - \frac{\gamma R}{\rho} \qquad\qquad \frac{\partial \boldsymbol{R}}{\partial t} = \frac{-\gamma R \boldsymbol{\beta}}{\rho} \tag{A.3}$$

$$\boldsymbol{\nabla} R = \frac{\gamma \boldsymbol{R}}{\rho} \qquad\qquad \boldsymbol{\nabla}\boldsymbol{R} = \mathbf{1} + \frac{\gamma \boldsymbol{R}\boldsymbol{\beta}}{\rho} \tag{A.4}$$

The determinant of (A.2) can be written $\det[\partial^\mu R^\nu] = 0$. The divergence of $\boldsymbol{R}$ follows from $Tr[\boldsymbol{\nabla}\boldsymbol{R}]$ and has a value

$$\boldsymbol{\nabla} \cdot \boldsymbol{R} = 3 + \frac{\gamma \boldsymbol{R} \cdot \boldsymbol{\beta}}{\rho} \tag{A.5}$$

Let $\boldsymbol{w}(ct_r)$ be the retarded position of a charged particle at time ct_r. The light cone condition is defined by

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{w}(ct_r) \qquad\qquad R \equiv ct - ct_r \tag{A.6}$$

Suppose that retarded coordinates are viewed collectively as $x_r^\mu = (ct_r, \boldsymbol{R})$. A transformation to present time coordinates $x^\mu = (ct, \boldsymbol{r})$ is then $x_r^\nu = x_r^\nu(x^\mu)$ and it follows that

$$dx_r^\nu = \frac{\partial x_r^\nu}{\partial x^\mu}\, dx^\mu \tag{A.7}$$

The matrix generated by this transformation can be written

$$\frac{\partial x_r^\nu}{\partial x^\mu} = \begin{bmatrix} \dfrac{\partial ct_r}{\partial t} & \dfrac{\partial \boldsymbol{R}}{\partial t} \\ -\boldsymbol{\nabla} ct_r & -\boldsymbol{\nabla}\boldsymbol{R} \end{bmatrix}$$

Derivatives of R and $\boldsymbol{R}$ have already been evaluated while derivatives of the retarded time are

$$\frac{\partial t_r}{\partial t} = \frac{\gamma R}{\rho} \qquad\qquad \boldsymbol{\nabla} ct_r = \frac{-\gamma \boldsymbol{R}}{\rho} \tag{A.8}$$

B Determinants of the Vacuum Tensor

A fundamental property of the vacuum tensor is its vanishing determinant

$$\det \mathit{\Delta}^{\mu\nu} = \sum_{i,j,k,l} \varepsilon_{ijkl} \mathit{\Delta}^{1i} \mathit{\Delta}^{2j} \mathit{\Delta}^{3k} \mathit{\Delta}^{4l} = 0 \tag{B.1}$$

However, this is only part of the story because for each of 16 three-by-three minors defined by $\mathcal{M}^{[\mu\nu]}$, one finds

$$\det \mathcal{M}^{[\mu\nu]} = \sum_{i,j,k} \varepsilon_{ijk} \mathcal{M}^{1i} \mathcal{M}^{2j} \mathcal{M}^{3k} = 0 \tag{B.2}$$

These determinants are interesting but nothing new since the field strength tensor exhibits the same properties. In fact, it can be shown that values for the nine sub-minors associated with each minor are proportional to those of $F^{\mu\nu}$. Mathematically,

$$\mathit{\Delta}^{ij} \mathit{\Delta}^{kl} - \mathit{\Delta}^{kj} \mathit{\Delta}^{il} = -2(F^{ij} F^{kl} - F^{kj} F^{il}) \tag{B.3}$$

On the other hand, by evaluating each $\det \mathcal{M}^{[\mu\nu]}$, a component of the velocity field (E_i or B_j) can be factored from the resulting homogeneous equation resulting in linear relations among the components of $\mathit{\Delta}^{\mu\nu}$ and $F^{\mu\nu}$. Two classes of covariant equations emerge from calculating along rows or columns yielding

$$\mathit{\Delta}^{\mu\lambda} F^{\alpha\nu} - \mathit{\Delta}^{\alpha\lambda} F^{\mu\nu} - \mathit{\Delta}^{\nu\lambda} F^{\alpha\mu} = 0 \tag{B.4a}$$
$$\mathit{\Delta}^{\mu\lambda} F^{\alpha\nu} - \mathit{\Delta}^{\mu\alpha} F^{\lambda\nu} - \mathit{\Delta}^{\mu\nu} F^{\alpha\lambda} = 0 \tag{B.4b}$$

The structure of each individual equation makes it possible to derive the covariant expression for the homogeneous Maxwell-Lorentz equations simply by making the replacement $\mathit{\Delta}^{\alpha\lambda} \to \partial_\alpha$ where α is the unrepeated index in each occurence of the vacuum tensor. Making the replacement $\mathit{\Delta}^{\mu\nu} = F^{\mu\nu} + \mathit{\Delta}^{\nu\mu}$ in both equations eliminates the vacuum tensor and leaves the covariant $\mathbf{E} \cdot \mathbf{B} = 0$ expression

$$F^{\mu\lambda} F^{\alpha\nu} + F^{\mu\nu} F^{\lambda\alpha} + F^{\mu\alpha} F^{\nu\lambda} = 0 \tag{B.5}$$

One can also eliminate the field strength tensor in (B.4) in favor of an expression among components of the vacuum tensor

$$\mathit{\Delta}^{\alpha\lambda} \mathit{\Delta}^{\nu\mu} - \mathit{\Delta}^{\nu\lambda} \mathit{\Delta}^{\alpha\mu} + \mathit{\Delta}^{\mu\alpha} \mathit{\Delta}^{\lambda\nu} - \mathit{\Delta}^{\mu\nu} \mathit{\Delta}^{\lambda\alpha} = 0 \tag{B.6}$$

Other properties of equations (B.4) can be ascertained by considering individual components. Of the several relations among vacuum gauge potentials and fields, the most interesting are the electric equations,

$$(\boldsymbol{\nabla} \cdot \boldsymbol{A})\mathbf{E} - \boldsymbol{\nabla} \boldsymbol{A} \cdot \mathbf{E} - \partial \boldsymbol{A}/\partial ct \times \mathbf{B} = 2\eta \mathbf{E} \tag{B.7a}$$
$$(\boldsymbol{\nabla} \cdot \boldsymbol{A})\mathbf{E} - \mathbf{E} \cdot \boldsymbol{\nabla} \boldsymbol{A} + \boldsymbol{\nabla} A \times \mathbf{B} = 2\eta \mathbf{E} \tag{B.7b}$$

and their magnetic counterparts

$$\boldsymbol{\nabla} \boldsymbol{A} \cdot \mathbf{B} = \eta \mathbf{B} \qquad\qquad \mathbf{B} \cdot \boldsymbol{\nabla} \boldsymbol{A} = \eta \mathbf{B} \tag{B.7c}$$

The structure inherent in (B.7) compels a formal definition of the space-space components of the vacuum tensor which can be written

$$\widehat{\mathbf{T}}_{\Lambda} \equiv -\boldsymbol{\nabla} \boldsymbol{A} + \mathbf{1}\, \partial_\nu A^\nu \tag{B.8}$$

The magnetic equations are simply $\widehat{\mathbf{T}}_{\Lambda} \cdot \mathbf{B} = \mathbf{B} \cdot \widehat{\mathbf{T}}_{\Lambda} = 0$ but more importantly is the fundamental expression determining the Poynting vector by subtracting equations (B.7b) and (B.7a):

$$\mathbf{S} = \frac{c}{4\pi} [\widehat{\mathbf{T}}_{\Lambda} \cdot \mathbf{E} - \mathbf{E} \cdot \widehat{\mathbf{T}}_{\Lambda}] \tag{B.9}$$

To re-iterate, this is flux associated with velocity fields only.

Determinant of the Strain Tensor: Calculating the determinant of the strain tensor is complicated by the presence of additional terms in the diagonal entries. Like the vacuum tensor, $\det \eta^{\mu\nu} = 0$. The simplest way to see this is in the rest frame where all entries along the top row vanish. A more formal analysis shows that three-by-three minors are not zero, but the entire determinant can be written as a 4th order polynomial in η:

$$\det \eta^{\mu\nu} = \det \Delta^{\mu\nu} + \sum_{k=0}^{3} (-1)^k \mathcal{M}^{[k\nu]} \cdot \eta + 2(B^2 - E^2) \cdot \eta^2 - g_{\alpha\lambda} \Delta^{\alpha\lambda} \cdot \eta^3 - \eta^4 = 0 \tag{B.10}$$

In this equation the first two terms drop out as a result of properties of the vacuum tensor, while the last three terms cancel each other out.

C Lagrange Equations for Particle Accelerations

Beginning with A^ν, the interacting charged particle velocity Lagrangian is:

$$\mathcal{L}_{vac} = -\frac{1}{8\pi}\left[\partial^\mu A^\nu \partial_\mu A_\nu - (\partial_\nu A^\nu)^2\right] - \frac{1}{c} j_e^{*\nu} A_\nu \tag{C.1}$$

Under conditions where accelerated motions occur assume that $\partial^\mu A^\nu$ receives a small anti-symmetric perturbation first order in the quantity $a_\perp^\nu$ with the appearance of an acceleration four-current:

$$\partial^\mu A^\nu \longrightarrow \partial^\mu A^\nu - A^\mu a_\perp^\nu + A^\nu a_\perp^\mu \qquad\qquad j_e^{*\nu} \longrightarrow j_e^{*\nu} - J_a^\nu \tag{C.2}$$

Still keeping only first order corrections in $a_\perp^\nu$ the modified velocity theory can be written

$$\begin{aligned}\mathcal{L}_{vac} = -\frac{1}{8\pi}\left[\partial^\mu A^\nu \partial_\mu A_\nu - (\partial_\nu A^\nu)^2\right] - \frac{1}{c} j_e^{*\nu} A_\nu \\ + \frac{1}{4\pi}\left[\partial^\mu A^\nu A_\mu a_\nu^\perp - \partial^\mu A^\nu A_\nu a_\mu^\perp\right] + \frac{1}{c} J_a^\nu A_\nu\end{aligned} \tag{C.3}$$

Derivatives with respect to the field quantity and its derivative are

$$\frac{\partial \mathcal{L}_{vac}}{\partial(\partial_\mu A_\lambda)} = -\frac{1}{4\pi}\left[\partial^\mu A^\lambda - \mathrm{g}^{\mu\lambda}\partial_\nu A^\nu - A^\mu a_\perp^\lambda + A^\lambda a_\perp^\mu\right] \tag{C.4}$$

$$\frac{\partial \mathcal{L}_{vac}}{\partial A_\lambda} = \frac{1}{4\pi}\left[\partial^\lambda A^\mu - \partial^\mu A^\lambda\right] a_\mu^\perp - \frac{1}{c}\left[j_e^{*\lambda} - J_a^\lambda\right] \tag{C.5}$$

Appealing to equation (5.3), the second equation produces all the correct current densities necessary to write

$$\partial_\mu\left[\partial^\mu A^\lambda - \mathrm{g}^{\mu\lambda}\partial_\nu A^\nu\right] = \frac{4\pi}{c} j_e^{*\lambda} = \partial_\mu \varDelta^{\mu\lambda} \tag{C.6a}$$

$$\partial_\mu\left[A^\mu a_\perp^\lambda\right] = \frac{4\pi}{c} j_a^\lambda = \partial_\mu \epsilon^{\mu\lambda} \tag{C.6b}$$

$$\partial_\mu\left[A^\lambda a_\perp^\mu\right] = \frac{4\pi}{c}\left[j_a^\lambda - J_a^\lambda\right] = \partial_\mu \epsilon^{\lambda\mu} \tag{C.6c}$$

These results may also be obtained beginning with the symmetric variation

$$\partial^\mu A^\nu \longrightarrow \partial^\mu A^\nu - A^\mu a_\perp^\nu - A^\nu a_\perp^\mu \tag{C.7}$$

In this case however it is necessary to use acceleration current densities derived in equations (C.6) for interaction terms instead of J_a^ν.

D Contractions of Vacuum Strain Tensors

$$\eta^{\mu\nu} = \frac{e}{\rho^2}\mathrm{g}^{\mu\nu} - \frac{eR^\mu\beta^\nu}{\rho^3} - \frac{2eR^\nu\beta^\mu}{\rho^3} + \frac{2eR^\mu R^\nu}{\rho^4} \tag{D.1}$$

$$\epsilon^{\mu\nu} = \frac{e}{\rho^2}R^\mu a^\nu - \frac{e}{\rho^3}\chi R^\mu\beta^\nu + \frac{e}{\rho^4}\chi R^\mu R^\nu \tag{D.2}$$

$\eta^{\mu\lambda}\eta_\lambda{}^\nu = \frac{e^2}{\rho^4}\mathrm{g}^{\mu\nu} - \frac{e^2}{\rho^5}R^\mu\beta^\nu$	$\eta^{\mu\lambda}\eta^\nu{}_\lambda = \frac{e^2}{\rho^4}\mathrm{g}^{\mu\nu} - \frac{e^2}{\rho^5}(R^\mu\beta^\nu + R^\nu\beta^\mu) + \frac{e^2}{\rho^6}R^\mu R^\nu$
$\eta^{\lambda\mu}\eta^\nu{}_\lambda = \frac{e^2}{\rho^4}\mathrm{g}^{\mu\nu} - \frac{e^2}{\rho^5}\beta^\mu R^\nu$	$\eta^{\lambda\mu}\eta_\lambda{}^\nu = \frac{e^2}{\rho^4}\mathrm{g}^{\mu\nu} - \frac{e}{\rho^3}(R^\mu\beta^\nu + \beta^\mu R^\nu)$

Table 1: Contractions of $\eta^{\mu\nu}$.

$\eta^{\mu\lambda}\epsilon^\nu{}_\lambda = \eta\epsilon^{\nu\mu}$	$\epsilon^{\mu\lambda}\eta^\nu{}_\lambda = \eta\epsilon^{\mu\nu}$	$\epsilon^{\mu\lambda}\epsilon^\nu{}_\lambda = -\Theta_3^{\mu\nu}$
$\eta^{\mu\lambda}\epsilon_\lambda{}^\nu = 0$	$\epsilon^{\mu\lambda}\eta_\lambda{}^\nu = \eta\epsilon^{\mu\nu}$	$\epsilon^{\mu\lambda}\epsilon_\lambda{}^\nu = 0$
$\eta^{\lambda\mu}\epsilon^\nu{}_\lambda = \eta\epsilon^{\nu\mu}$	$\epsilon^{\lambda\mu}\eta^\nu{}_\lambda = 0$	$\epsilon^{\lambda\mu}\epsilon^\nu{}_\lambda = 0$
$\eta^{\lambda\mu}\epsilon_\lambda{}^\nu = -\eta\epsilon^{\mu\nu}$	$\epsilon^{\lambda\mu}\eta_\lambda{}^\nu = -\eta\epsilon^{\nu\mu}$	$\epsilon^{\lambda\mu}\epsilon_\lambda{}^\nu = 0$

Table 2: Contractions of $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu}$.

E Fourier Modes in Electromagnetic and Vacuum Theories

A Fourier analysis can be applied to both electromagnetic and vacuum Lagrangians. For a given wave number k, a complex Fourier mode of the velocity potentials similar to (2.34) but without a phase can be written

$$A_k^\nu = A^\nu \cdot \frac{e^{ik(c\tau-\rho)}}{k} \tag{E.1}$$

Fourier modes of vacuum strain contain an extra complex and symmetric term:

$$\eta_k^{\mu\nu} = \frac{e^{ik(c\tau-\rho)}}{k}\left[\eta^{\mu\nu} + \frac{ikA^\nu R^\mu}{\rho}\right] \tag{E.2}$$

Contracting on indices produces a Fourier mode of the vacuum gauge condition and this implies that

$$\mathit{\Delta}_k^{\mu\nu} = \frac{e^{ik(c\tau-\rho)}}{k}\left[\mathit{\Delta}^{\mu\nu} + \frac{ikA^\nu R^\mu}{\rho}\right] \tag{E.3}$$

Now form associated electromagnetic and vacuum Lagrangians by considering distinct wave numbers k and p from which it can be shown that

$$(\mathcal{L}_{vac})_{kp} = -\frac{1}{8\pi}\mathit{\Delta}_k^{\mu\nu}(\eta_p)_{\mu\nu} = -\frac{1}{8\pi}\frac{e^{i(k+p)(c\tau-\rho)}}{kp}\mathit{\Delta}^{\mu\nu}\eta_{\mu\nu} \tag{E.4a}$$

$$(\mathcal{L}_{em})_{kp} = -\frac{1}{16\pi}F_k^{\mu\nu}(F_p)_{\mu\nu} = -\frac{1}{16\pi}\frac{e^{i(k+p)(c\tau-\rho)}}{kp}F^{\mu\nu}F_{\mu\nu} \tag{E.4b}$$

Two integratations over wave numbers k and p then reproduce the total Lagrangians. One may also differentiate Fourier modes of either the field strength tensor or the vacuum tensor. One finds $\partial_\mu \mathit{\Delta}_k^{\nu\mu} = 0$ but

$$\partial_\mu F_k^{\mu\nu} = \partial_\mu \mathit{\Delta}_k^{\mu\nu} = \frac{4\pi}{c}\left[J_e^\nu - \frac{ik}{4\pi\rho}A^\nu\right]\frac{e^{ik(c\tau-\rho)}}{k} \tag{E.5}$$

The second term on the right may be re-written using $A^\nu = -F^{\mu\nu}R_\mu$. Employing the relation

$$i\frac{R_\mu}{\rho}\cdot e^{ik(c\tau-\rho)} = \partial_\mu\left[\frac{e^{ik(c\tau-\rho)}}{k}\right] \tag{E.6}$$

then allows for integration over all wave numbers showing that

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J_e^\nu \cdot \vartheta + F^{\mu\nu}\cdot\partial_\mu\vartheta = \frac{4\pi}{c}J_e^{*\nu} \tag{E.7}$$

F Invariant Hamiltonian

A rigorous calculation of the invariant Hamiltonian from the velocity theory begins by writing the total stress tensor as

$$\mathcal{T}^{\mu\nu} = [\Theta_1^{\mu\nu} + \Lambda^{\mu\nu}] \cdot \vartheta(\tau - \rho/c + r_e) \tag{F.1}$$

where the causality sphere is given by $\vartheta = \vartheta(\tau - \rho/c + r_e)$. Integration over the space made available is then

$$\begin{aligned}\mathcal{E}^{\mu}_{total} &= \int \mathcal{T}^{\mu\nu} d\sigma^s_\nu \\ &= -\beta^\mu \int \mathcal{L}_{vac} \cdot \vartheta \rho^2 d\rho\, d\Omega' = \mathcal{E}^{\mu}_{part} + \mathcal{E}^{\mu}_{vac}\end{aligned} \tag{F.2}$$

The particle term is determined by the symmetric stress tensor. Following integration over solid angle it may be written

$$\mathcal{E}^{\mu}_{part} = \frac{e^2}{2}\beta^\mu \int_{r_e}^{c\tau + r_e} \frac{\vartheta(\tau - \rho/c + r_e)}{\rho^2} d\rho \tag{F.3}$$

Now implement an integration by parts

$$\mathcal{E}^{\mu}_{part} = mc^2\beta^\nu \cdot \vartheta(\tau) - \frac{e^2}{2}\beta^\mu \cdot \xi(\tau) \tag{F.4}$$

where $\xi(\tau)$ is given by

$$\begin{aligned}\xi(\tau) &= \int_{r_e}^{c\tau + r_e} \frac{\delta(\tau - \rho/c + \tau_e)}{\rho} d\rho \\ &= \frac{1}{c\tau + r_e} \int_{r_e}^{c\tau + r_e} \delta(\tau - \rho/c + \tau_e) d\rho \longrightarrow 0\end{aligned} \tag{F.5}$$

and vanishes for reasonably large times.

The vacuum term can also be immediately integrated over solid angle. The integration over the variable ρ must include the region inside the vacuum boundary so that

$$\mathcal{E}^{\mu}_{vac} = -\frac{1}{2}\sigma_e e \beta^\nu \int_0^{c\tau + r_e} \vartheta(\tau - \rho/c + r_e) d\rho \tag{F.6}$$

Writing the causality function as an integral over Fourier modes one then has the double integral

$$\begin{aligned}\mathcal{E}^{\mu}_{vac} &= -\frac{1}{2}\sigma_e e \beta^\mu \cdot \frac{2}{\pi} \iint \frac{\sin\omega(\tau - \rho/c + \tau_e)}{\omega} d\rho\, d\omega \\ &= -\frac{1}{2}\sigma_e e c \beta^\mu \cdot \frac{2}{\pi} \int_0^\infty \frac{1 - \cos\omega(\tau + \tau_e)}{\omega^2} d\omega \\ &= -mc^2\beta^\mu - \tfrac{1}{2}\dot{\varrho} c\tau \beta^\mu\end{aligned} \tag{F.7}$$

Adding in the integration constant, the complete energy four-vector of the particle is

$$\mathcal{E}^{\mu}_{total}(\tau) = \left[mc^2 \cdot \vartheta(\tau) - \tfrac{1}{2}\dot{\varrho}c\tau\right]\beta^{\mu} \tag{F.8}$$

This formula indicates that a particle was created at proper time $\tau = 0$ and then radiated vacuum energy at all frequencies for a time τ after it was created. contracting with the four velocity then tields the invariant Hamiltonian

$$\boxed{\mathcal{H}(\tau) = mc^2 \cdot \vartheta(\tau) - \tfrac{1}{2}\dot{\varrho}c\tau} \tag{F.9}$$

This formula may be compared with equation (4.28).