Vacuum Gauge Electron in the Spherical Basis

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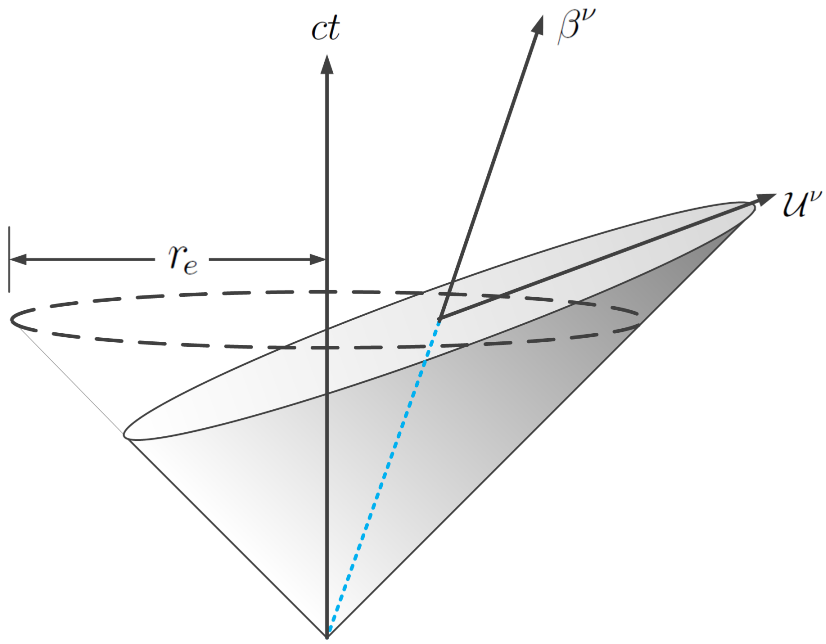

"To get a consistent picture, we must imagine that something holds the electron together. The charges must be held to the sphere by some kind of rubber bands—something that keeps the charges from flying off."

R.P. Feynman, *The Feynman Lectures*, chapter 28

Abstract

A covariant theory of the Maxwell-Lorentz electron can be established in a spherical four-coordinate system having basis vectors describing a spacetime with a distorted solid angle as viewed by a moving observer. Although the spacetime remains flat, its implementation requires the use of ten independent Christoffel symbols generated by the coordinate transformation. Velocity and acceleration field strength tensors occupy independent subspaces and are easily constructed in the spherical system. Symmetric and total stress tensors are also more tractable and can be derived from a basis-independent approach. The spherically based system is highly efficient and complimentary to a description of the classical electron in the vacuum gauge.

Contents

List of Figures

List of Tables

1 Spherical Four-Coordinates

A mathematical introduction precedes the description of the vacuum gauge electron in the spherically based four-coordinate system. The coordinate transformation derives four mutually orthogonal basis vectors constructed from coordinates relative to the retarded position of the particle.

1.1 Coordinate Transformation

Two coordinate systems in Minkowski space are linked by a four-coordinate transformation parameterized by the quantity $\boldsymbol{\beta}$ representing the velocity of one frame relative to another. In the frame $\mathcal{S}_o$ the coordinates of the spacetime event P shown in figure 1 can be written as the sum of purely timelike and purely spacelike components

$$x_o^\nu = x_\tau^\nu + x_s^\nu \tag{1.1}$$

The same event viewed in the frame $\mathcal{S}$—and connected by a homogeneous Lorentz

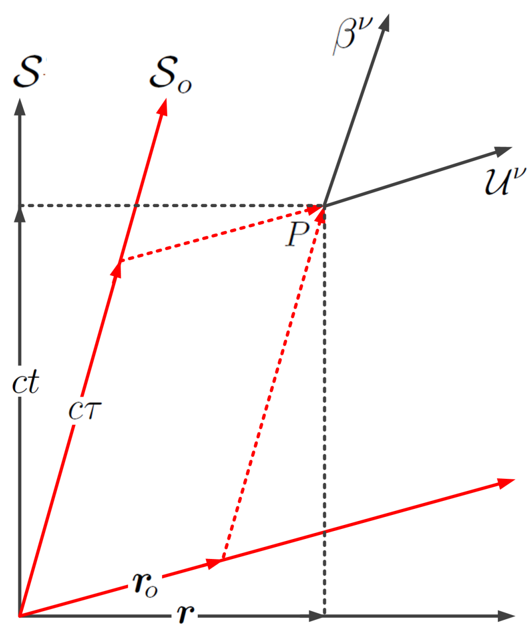

Figure 1: *Spacetime event x^ν analyzed in terms of timelike and spacelike components.*

transformation—can be represented by the expansion

$$x^\nu = (x^\lambda \beta_\lambda)\beta^\nu - (x^\lambda \mathcal{U}_\lambda)\mathcal{U}^\nu \tag{1.2}$$

where the timelike and spacelike unit vectors β^ν and $\mathcal{U}^\nu$ are Lorentz transformations of their rest frame counterparts having norms of 1 and -1, respectively. Component lengths along each direction are easily determined to be the Lorentz scalars

$$x^\lambda \beta_\lambda = c\tau \qquad \text{and} \qquad x^\lambda \mathcal{U}_\lambda = -\rho \tag{1.3}$$

and the transformation law follows from the observation that $\mathcal{U}^\nu$ can be written as a function of two appropriately chosen angular coordinates (θ, ϕ) allowing the present time coordinates to be written $x^\nu = x^\nu(c\tau, \rho, \theta, \phi)$ or

$$\boxed{x^\nu = c\tau\beta^\nu + \rho\,\mathcal{U}^\nu} \tag{1.4}$$

The quantities ρ and $c\tau$ are assumed to be independent of each other and also independent of the angular coordinates since it can be shown that

$$\frac{\partial\tau}{\partial\theta} = \frac{\partial\tau}{\partial\phi} = 0 \qquad\qquad \frac{\partial\rho}{\partial\theta} = \frac{\partial\rho}{\partial\phi} = 0 \tag{1.5}$$

In general, it will not be necessary to consider points x^ν outside the particles causal light cone which will follow as long as $\rho \leq c\tau$.

Now suppose the world line of the particle is described by the four-vector w^ν. This can also be projected onto spacelike and timelike vectors via

$$w^\nu = (w^\lambda\beta_\lambda)\beta^\nu - (w^\lambda\mathcal{U}_\lambda)\mathcal{U}^\nu \tag{1.6}$$

and the light cone condition in the new coordinate system is

$$R^\nu = (c\tau - w^\lambda\beta_\lambda)\beta^\nu + (\rho + w^\lambda\mathcal{U}_\lambda)\mathcal{U}^\nu \tag{1.7}$$

Contracting on both sides of the equation with β^ν and $\mathcal{U}^\nu$ then shows that

$$w^\lambda\beta_\lambda = c\tau - \rho \tag{1.8a}$$

$$w^\lambda\mathcal{U}_\lambda = 0 \tag{1.8b}$$

This means that for any point along the world line of the particle, the transformation in (1.4) requires the position of the particle to point instantaneously along the direction of its four-velocity. Equations (1.8) may now be re-inserted into (1.7) to derive the general form of the null vector.

Basis Vectors and Unit Vectors: Basis vectors with lowered indices are easily derived from the coordinate transformation in (1.4). As an intermediate step one can calculate angular derivatives of $\mathcal{U}^\nu$

$$\frac{\partial\mathcal{U}^\nu}{\partial\theta} = \frac{R}{\rho}\,\theta^\nu \qquad\qquad \frac{\partial\mathcal{U}^\nu}{\partial\phi} = \frac{R\sin\theta}{\rho}\,\phi^\nu \tag{1.9}$$

leading to the four basis vectors

$$\vec{\mathrm{e}}_\tau = \frac{\partial x_\nu}{\partial c\tau} = \beta_\nu \qquad\qquad \vec{\mathrm{e}}_\rho = \frac{\partial x_\nu}{\partial\rho} = \mathcal{U}_\nu \tag{1.10a}$$

$$\vec{\mathrm{e}}_\theta = \frac{\partial x_\nu}{\partial\theta} = R\,\theta_\nu \qquad\qquad \vec{\mathrm{e}}_\phi = \frac{\partial x_\nu}{\partial\phi} = R\sin\theta\,\phi_\nu \tag{1.10b}$$

These vectors include the scale factors

$$h_\tau = 1 \qquad h_\rho = 1 \qquad h_\theta = R \qquad h_\phi = R\sin\theta \tag{1.11}$$

which can be divided out to give an orthonormal set of unit four-vectors written explicitly as functions of an accompanying set of three-vectors. Raising indices for convenience yields

$$\beta^\nu = \begin{bmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{bmatrix} \tag{1.12a}$$

$$\mathcal{U}^\nu = \frac{1}{\gamma(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})}\begin{bmatrix} 1 \\ \hat{\boldsymbol{n}} \end{bmatrix} - \begin{bmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{bmatrix} \tag{1.12b}$$

$$\theta^\nu = \frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})}\begin{bmatrix} \boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\theta}}+\boldsymbol{\beta}\times\hat{\boldsymbol{\phi}} \end{bmatrix} \tag{1.12c}$$

$$\phi^\nu = \frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})}\begin{bmatrix} \boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\phi}}-\boldsymbol{\beta}\times\hat{\boldsymbol{\theta}} \end{bmatrix} \tag{1.12d}$$

In an arbitrary frame of reference the spacelike vectors may be difficult to visualize, especially θ^ν and ϕ^ν which generally have non-zero time components. Regardless, figure 2 makes an attempt to show them at a present time event. Timelike and spacelike unit magnitudes consistent with the spacetime metric are

$$\beta^\nu\beta_\nu = 1 \qquad \mathcal{U}^\nu\mathcal{U}_\nu = -1 \qquad \theta^\nu\theta_\nu = -1 \qquad \phi^\nu\phi_\nu = -1 \tag{1.13}$$

and the six different orthogonality relations can also be verified.

The total differential calculated from the coordinate transformation is

$$ds^2 = dx^\nu dx_\nu = \frac{dx^\nu}{dq_\lambda}\frac{dx_\nu}{dq_\alpha}\,dq^\lambda dq^\alpha \tag{1.14}$$

and this produces the specific form of the interval

$$ds^2 = c^2 d\tau^2 - d\rho^2 - \frac{\rho^2}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2}\left[d\theta^2 + \sin^2\theta\,d\phi^2\right] \tag{1.15}$$

From here the diagonal metric tensor can be easily constructed. Using R as a function of the spatial coordinates (ρ, θ, ϕ) an appropriate matrix representation is

$$\mathrm{g}_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2\sin^2\theta \end{bmatrix} \tag{1.16}$$

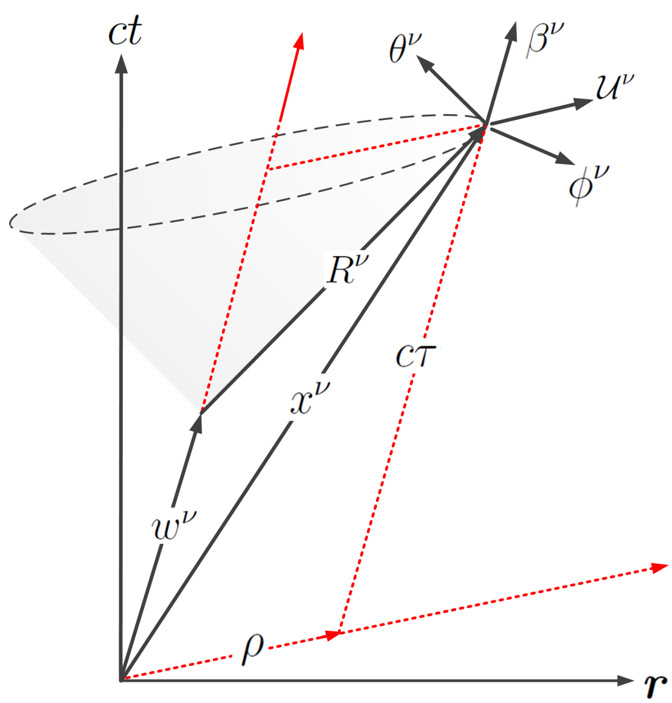

Figure 2: *Orthonormal set of unit four-vectors at the point* $(ct, \boldsymbol{r})$ *defined relative to the retarded position of an electron moving along its world line.*

Inverse Transformation: The inversion of (1.4) can be written $q^\mu = q^\mu(x^\nu)$. Considering each of the components (R_x, R_y, R_z) as functions of the present time coordinates a simple representation of the inverse transformation is

$$c\tau = ct_r/\gamma + \rho \qquad \rho = \gamma R - \gamma \boldsymbol{R}\cdot\boldsymbol{\beta}$$

$$\theta = \tan^{-1}\left[\frac{(R_x^2 + R_y^2)^{1/2}}{R_z}\right] \qquad \phi = \tan^{-1}\left[\frac{R_y}{R_x}\right] \tag{1.17}$$

Each spherical coordinate here can be set to a constant value which will define a three-dimensional hypersurface in Minkowski space. Once again, these surfaces may be difficult to visualize except in the rest frame where the spacelike surfaces become identical to those of the standard spherical-polar coordinate system. Dual basis vectors, similar to those in equation (1.10) except with raised indices, follow from the inverse transformation as:

$$\vec{\omega}^{\,\tau} = \partial^\nu c\tau = \beta^\nu \qquad \vec{\omega}^{\,\rho} = \partial^\nu \rho = -\mathcal{U}^\nu \tag{1.18}$$

$$\vec{\omega}^{\,\theta} = \partial^\nu \theta = -\frac{1}{R}\theta^\nu \qquad \vec{\omega}^{\,\phi} = \partial^\nu \phi = -\frac{1}{R\sin\theta}\phi^\nu \tag{1.19}$$

In this case the scale factors appear in the denominators and a general relation connecting the two sets of basis vectors is given by

$$\vec{\mathrm{e}}_\mu \cdot \vec{\omega}^{\,\nu} = \delta_\mu^{\ \nu} \tag{1.20}$$

Christoffel Symbols: Derivatives of the basis vectors define ten independent Christoffel symbols thru the relations

$$\frac{\partial\, \vec{\mathrm{e}}_\mu}{\partial q^\nu} = \Gamma^\lambda_{\mu\nu}\, \vec{\mathrm{e}}_\lambda \qquad\qquad \frac{\partial\, \vec{\omega}^{\,\mu}}{\partial q^\nu} = -\Gamma^\mu_{\lambda\nu}\, \vec{\omega}^{\,\lambda} \tag{1.21}$$

which can also be determined directly from the metric tensor:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\mathrm{g}^{\lambda\alpha}\left[\mathrm{g}_{\lambda\mu,\nu} + \mathrm{g}_{\lambda\nu,\mu} - \mathrm{g}_{\mu\nu,\lambda}\right] \tag{1.22}$$

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1. $\Gamma^\phi_{\rho\phi} = \dfrac{1}{\rho}$
2. $\Gamma^\theta_{\rho\theta} = \dfrac{1}{\rho}$
3. $\Gamma^\rho_{\theta\theta} = -\dfrac{\rho}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2}$
4. $\Gamma^\rho_{\phi\phi} = -\dfrac{\rho\sin^2\theta}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2}$
5. $\Gamma^\theta_{\theta\theta} = \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}$
6. $\Gamma^\phi_{\theta\theta} = -\dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}$
7. $\Gamma^\phi_{\theta\phi} = \cot\theta + \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}$
8. $\Gamma^\theta_{\theta\phi} = \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}\sin\theta$
9. $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta - \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}\sin^2\theta$
10. $\Gamma^\phi_{\phi\phi} = \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}\sin\theta$

The Christoffel symbols will naturally be useful for future calculations, but it will be unnecessary to use them to calculate components of a curvature tensor since we are working exclusively in a flat spacetime. If $\beta \to 0$, several of the symbols vanish while the remaining non-zero symbols reduce to those of a 3D spherical-polar coordinate system.

1.2 Tensors and Differential Operations

Vectors: In general, a four-vector $\vec{V}$ in the spherical basis can be written in terms of either set of dual basis vectors:

$$\vec{V} = V^\nu \vec{\mathrm{e}}_\nu \qquad \vec{V} = V_\nu \,\vec{\omega}^{\,\nu} \tag{1.23}$$

However, like the well known 3D curvi-linear coordinate systems, it is frequently easier to write vectors in terms of the corresponding unit four-vectors. In particular, the components V^ν of an arbitrary vector generally has an immediate expansion

$$V^\nu = (V^\alpha \beta_\alpha)\beta^\nu - (V^\alpha \mathcal{U}_\alpha)\mathcal{U}^\nu - (V^\alpha \theta_\alpha)\theta^\nu - (V^\alpha \phi_\alpha)\phi^\nu \tag{1.24}$$

If the vector is radial from the retarded position of the charge then components along θ^ν and ϕ^ν are automatically zero. As an example, the vacuum gauge velocity potentials have a repesentation in terms of the ordered quadruplet

$$A^\nu_v = \left[\frac{e}{\rho},\, \frac{e}{\rho},\, 0,\, 0\right] \tag{1.25}$$

The set of four-space cartesian unit vectors is rarely included in the literature to adequately represent a four vector, but its transformation to the spherical system is important since it generates a rotation matrix defined by the matrix equation

$$\begin{bmatrix} \mathrm{e}^o \\ \mathrm{e}^x \\ \mathrm{e}^y \\ \mathrm{e}^z \end{bmatrix} = \begin{bmatrix} \mathrm{e}^o\beta_\mu & -\mathrm{e}^o\mathcal{U}_\mu & -\mathrm{e}^o\theta_\mu & -\mathrm{e}^o\phi_\mu \\ \mathrm{e}^x\beta_\mu & -\mathrm{e}^x\mathcal{U}_\mu & -\mathrm{e}^x\theta_\mu & -\mathrm{e}^x\phi_\mu \\ \mathrm{e}^y\beta_\mu & -\mathrm{e}^y\mathcal{U}_\mu & -\mathrm{e}^y\theta_\mu & -\mathrm{e}^y\phi_\mu \\ \mathrm{e}^z\beta_\mu & -\mathrm{e}^z\mathcal{U}_\mu & -\mathrm{e}^z\theta_\mu & -\mathrm{e}^z\phi_\mu \end{bmatrix} \cdot \begin{bmatrix} \beta^\nu \\ \mathcal{U}^\nu \\ \theta^\nu \\ \phi^\nu \end{bmatrix} \tag{1.26}$$

Labeling the 4×4 matrix as $\mathsf{U}^{\mu\nu}$ it is easy to see that its four columns are the components of the unit coordinate vectors. Moreover, one finds

$$\det\left[\mathsf{U}^{\mu\nu}\right] = 1 \tag{1.27}$$

which is entirely appropriate for a rotation matrix.

Second Rank Tensors: The outer product symbol is $\otimes$ and it can be used to define basis vectors for a rank 2 tensor

$$\vec{\mathrm{e}}_{\mu\nu} \equiv \vec{\mathrm{e}}_\mu \otimes \vec{\mathrm{e}}_\nu \qquad \vec{\omega}^{\,\mu\nu} \equiv \vec{\omega}^{\,\mu} \otimes \vec{\omega}^{\,\nu} \tag{1.28}$$

Sidestepping mixed basis vectors for now, tensors with raised and lowered indices are:

$$\widehat{T} \equiv T^{\mu\nu}\vec{\mathrm{e}}_{\mu\nu} \qquad \widehat{T} \equiv T_{\mu\nu}\,\vec{\omega}^{\,\mu\nu} \tag{1.29}$$

Once again however, it is possible to write individual components of a cartesian tensor $T^{\mu\nu}_{cart}$ in terms of spherical components which serve as unit coordinate vectors. Using the rotation tensor $\mathsf{U}^{\mu\nu}$ the appropriate transformation is

$$T^{\mu\nu}_{sph} = \mathsf{U}^{\mu}{}_{\alpha}\, T^{\alpha\lambda}_{cart}\, \mathsf{U}_{\lambda}{}^{\nu} \tag{1.30}$$

A matrix format for each of the sixteen components of $T^{\mu\nu}_{sph}$ is shown in figure 3 for convenience.

	β^ν	$\mathcal{U}^\nu$	θ^ν	ϕ^ν
β^μ	×	×	×	×
$\mathcal{U}^\mu$	×	×	×	×
θ^μ	×	×	×	×
ϕ^μ	×	×	×	×

Figure 3: *Showing the sixteen components of an arbitrary tensor $X^{\mu\nu}_{sph}$ in the spherical basis.*

An example of a second rank tensor using equation (1.29) is the metric tensor given by

$$\widehat{\mathrm{g}} = \mathrm{g}_{\tau\tau}\, \vec{\omega}^{\,\tau\tau} + \mathrm{g}_{\rho\rho}\, \vec{\omega}^{\,\rho\rho} + \mathrm{g}_{\theta\theta}\, \vec{\omega}^{\,\theta\theta} + \mathrm{g}_{\phi\phi}\, \vec{\omega}^{\,\phi\phi} \tag{1.31}$$

But the scale factors in the components of the tensor cancel with scale factors in the basis vectors and this allows for a simple representation of the metric in terms of the spherical unit vectors

$$\mathrm{g}^{\mu\nu} = \beta^\mu\beta^\nu - \mathcal{U}^\mu\mathcal{U}^\nu - \theta^\mu\theta^\nu - \phi^\mu\phi^\nu \tag{1.32}$$

In other words, the metric has identical components when referenced to unit vectors in the cartesian or spherical coordinate system.

The spherical basis is also well-suited for the construction of anti-symmetric tensors. Using vectors in (1.12) six possible tensor brackets are

$$\begin{array}{lll} \mathcal{H}^{\mu\nu}_1 = [\beta^\mu, \mathcal{U}^\nu] & \mathcal{H}^{\mu\nu}_2 = [\beta^\mu,\, \theta^\nu] & \mathcal{H}^{\mu\nu}_3 = [\beta^\mu,\, \phi^\nu] \\ \mathcal{H}^{\mu\nu}_4 = [\mathcal{U}^\mu,\, \theta^\nu] & \mathcal{H}^{\mu\nu}_5 = [\mathcal{U}^\mu,\, \phi^\nu] & \mathcal{H}^{\mu\nu}_6 = [\theta^\mu,\, \phi^\nu] \end{array} \tag{1.33}$$

and any anti-symmetric tensor follows by summing over independent components

$$X^{\mu\nu}_{[AS]} = \sum_{k=1}^{6} C_k \cdot \mathcal{H}^{\mu\nu}_k \tag{1.34}$$

Two other brackets having units of length are the linear combinations

$$\mathsf{H}^{\mu\nu} = [R^\mu,\, \theta^\nu] \qquad\qquad \mathsf{H}^{\mu\nu}_{D} = [R^\mu,\, \phi^\nu] \tag{1.35}$$

The importance of these can be understood by writing them in the cartesian basis where the three-space unit vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ separate into individual components:

$$\mathsf{H}^{\mu\nu} = R\begin{bmatrix} 0 & \hat{\theta}_x & \hat{\theta}_y & \hat{\theta}_z \\ -\hat{\theta}_x & 0 & \hat{\phi}_z & -\hat{\phi}_y \\ -\hat{\theta}_y & -\hat{\phi}_z & 0 & \hat{\phi}_x \\ -\hat{\theta}_z & \hat{\phi}_y & -\hat{\phi}_x & 0 \end{bmatrix} \qquad \mathsf{H}^{\mu\nu}_{D} = R\begin{bmatrix} 0 & \hat{\phi}_x & \hat{\phi}_y & \hat{\phi}_z \\ -\hat{\phi}_x & 0 & -\hat{\theta}_z & \hat{\theta}_y \\ -\hat{\phi}_y & \hat{\theta}_z & 0 & -\hat{\theta}_x \\ -\hat{\phi}_z & -\hat{\theta}_y & \hat{\theta}_x & 0 \end{bmatrix} \tag{1.36}$$

These are zero norm dual tensors to each other related by the Levi-Civita permutation tensor:

$$\mathsf{H}^{\mu\nu}_{D} = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\lambda}\mathsf{H}_{\mu\nu} \tag{1.37}$$

2 Fields in the Spherical Basis

The dynamical variables of the electron are determined from the particles world line $w^\nu = w^\nu(\tau)$. The four-velocity also serves as the timelike coordinate vector in the spherical basis, but the four-acceleration should have a general expansion along several of the four coordinate vectors. Since $a^\nu\beta_\nu = 0$ it will be composed of purely spacelike vectors. In terms of basis vectors it can be written

$$\vec{a} = (a^\lambda\mathcal{U}_\lambda)\,\vec{\omega}^{\,\rho} + (Ra^\lambda\theta_\lambda)\,\vec{\omega}^{\,\theta} + (R\sin\theta\, a^\lambda\phi_\lambda)\,\vec{\omega}^{\,\phi} \tag{2.1}$$

On the other hand, with the definitions

$$a_u \equiv a^\nu\mathcal{U}_\nu \qquad\qquad a_\theta \equiv a^\nu\theta_\nu \qquad\qquad a_\phi \equiv a^\nu\phi_\nu \tag{2.2}$$

then components a^ν can be written

$$a^\nu = -a_u\,\mathcal{U}^\nu - a_\theta\theta^\nu - a_\phi\phi^\nu \tag{2.3}$$

The norm of this vector along with the norm of the vector $a^\nu_\perp$ orthogonal to $\mathcal{U}^\nu$ are easily determined to be

$$-a^\nu a_\nu = a_u^2 + a_\theta^2 + a_\phi^2 \qquad\qquad -a^\nu_\perp a^\perp_\nu = a_\theta^2 + a_\phi^2 \tag{2.4}$$

As an example of the usefulness of these expressions, the power formula for transverse radiation can be written

$$P_{acc} = \frac{2}{3}\frac{e^2}{c^3}\left[a_u^2 + a_\theta^2 + a_\phi^2\right] \tag{2.5}$$

2.1 Velocity Fields

Dropping the subscript label, the vacuum gauge velocity potentials in the spherical basis can be written

$$\vec{A} = A_\nu \, \vec{\omega}^{\,\nu} \tag{2.6}$$

As noted earlier, both angular coordinates are zero while the non-zero coordinates are only functions of ρ. The covariant derivative follows from

$$\vec{\nabla}\vec{A} = \left[A_{\nu,\,\mu} - A_\lambda \Gamma^\lambda_{\nu\mu}\right] \vec{\omega}^{\,\nu\mu} = A_{\nu;\,\mu}\, \vec{\omega}^{\,\nu\mu} \tag{2.7}$$

In general, $A_{\nu;\,\mu}$ has sixteen components but the simple functional form of $\vec{A}$ reduces this number to four components which are all proportional to the vacuum dilation η. Using the rules of covariant differentiation and keeping careful track of minus signs shows that

$$A_{\tau;\,\rho} = A_{\tau,\,\rho} = -\eta \qquad\qquad A_{\rho\,;\,\rho} = A_{\rho,\,\rho} = \eta \tag{2.8a}$$

$$A_{\theta\,;\,\theta} = -A_\rho \Gamma^\rho_{\theta\theta} = -\eta R^2 \qquad\qquad A_{\phi\,;\,\phi} = -A_\rho \Gamma^\rho_{\phi\phi} = -\eta R^2 \sin^2\theta \tag{2.8b}$$

The vacuum strain tensor can then be written

$$\widehat{\eta} = A_{\tau;\,\rho}\, \vec{\omega}^{\,\rho\tau} + A_{\rho\,;\,\rho}\, \vec{\omega}^{\,\rho\rho} + A_{\theta\,;\,\theta}\, \vec{\omega}^{\,\theta\theta} + A_{\phi\,;\,\phi}\, \vec{\omega}^{\,\phi\phi} \tag{2.9}$$

In terms of unit coordinate vectors components of this tensor are

$$\eta^{\mu\nu} = \eta\left[\mathcal{U}^\mu \beta^\nu + \mathcal{U}^\mu \mathcal{U}^\nu - \theta^\mu\theta^\nu - \phi^\mu\phi^\nu\right] \tag{2.10}$$

Both the field strength tensor and the vacuum tensor can now be easily calculated using the formulas

$$\underset{v}{F}^{\mu\nu} = \eta^{\mu\nu} - \eta^{\nu\mu} \qquad\qquad \varDelta^{\mu\nu} = \eta^{\mu\nu} - \mathrm{g}^{\mu\nu}\eta \tag{2.11}$$

Neither tensor has any component along angular coordinates and are best represented in a 2×2 subspace of the matrix format in figure 3:

$$\underset{v}{F}^{\mu\nu} = \begin{bmatrix} 0 & -\eta \\ \eta & 0 \end{bmatrix} \qquad\qquad \varDelta^{\mu\nu} = \begin{bmatrix} -\eta & 0 \\ \eta & 2\eta \end{bmatrix} \tag{2.12}$$

These tensors can also be constructed thru a simple change of variables from Cartesian to spherical coordinates. A divergence on the first index of either tensor follows from a differential operation on an arbitrary second rank tensor $T^{\mu\nu}$ leading to

$$T^{\mu\nu}{}_{;\,\alpha} = T^{\mu\nu}{}_{,\,\alpha} + T^{\lambda\nu}\,\Gamma^\mu_{\lambda\alpha} + T^{\mu\lambda}\,\Gamma^\nu_{\lambda\alpha} \tag{2.13}$$

For the case where $\alpha \to \mu$ one finds

$$F^{\mu\nu}{}_{;\,\mu} = F^{\mu\nu}{}_{,\,\mu} + F^{\lambda\nu}\,\Gamma^\mu_{\lambda\mu} + F^{\mu\lambda}\,\Gamma^\nu_{\lambda\mu} \tag{2.14}$$

$$\varDelta^{\mu\nu}{}_{;\,\mu} = \varDelta^{\mu\nu}{}_{,\,\mu} + \varDelta^{\lambda\nu}\,\Gamma^\mu_{\lambda\mu} + \varDelta^{\mu\lambda}\,\Gamma^\nu_{\lambda\mu} \tag{2.15}$$

When $\rho > 0$ all components of both tensors which require a calculation can ultimately be reduced to

$$\frac{\partial \eta}{\partial \rho} + \eta[\Gamma^{\theta}_{\rho\theta} + \Gamma^{\phi}_{\rho\phi}] = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 \frac{e}{\rho^2}\right] = 0 \tag{2.16}$$

The case where $\rho \to 0$ is more complicated but it may be assumed that the point charge delta function arises in both cases so that

$$F^{\mu\nu}{}_{;\,\mu} = \frac{4\pi}{c} j_e^{*\nu} \qquad\qquad \varDelta^{\mu\nu}{}_{;\,\mu} = \frac{4\pi}{c} j_e^{*\nu} \tag{2.17}$$

2.2 Acceleration Fields

One way to determine the acceleration strain and acceleration field strength tensor is by differentiating the vacuum gauge acceleration potentials. Since acceleration fields fall like $1/\rho$ the acceleration potential will not have any dependence on ρ and this is clearly visible in the spherical basis by writing

$$\vec{A}_a = -e a^{\lambda} \mathcal{U}_{\lambda} (\vec{\mathrm{e}}_{\tau} + \vec{\mathrm{e}}_{\rho}) \tag{2.18}$$

Acceleration Strain Tensor: The covariant derivative of $\vec{A}_a$ is somewhat difficult to calculate in the spherical basis and it is probably easier to begin with the cartesian tensor

$$\partial^{\mu} A^{\nu}_a = -\partial^{\mu}\chi \cdot A^{\nu} - \chi \eta^{\mu\nu} + O(a^2) \tag{2.19}$$

where $\chi = a^{\lambda} R_{\lambda}$, and then transform components to the spherical basis. The result is

$$\partial^{\mu} A^{\nu}_a \longrightarrow -\frac{e}{\rho}\begin{bmatrix} \dot{a}^{\lambda} R_{\lambda} & \dot{a}^{\lambda} R_{\lambda} & 0 & 0 \\ \dot{a}^{\lambda} R_{\lambda} & \dot{a}^{\lambda} R_{\lambda} & 0 & 0 \\ a^{\lambda}\theta_{\lambda} & a^{\lambda}\theta_{\lambda} & -a^{\lambda}\mathcal{U}_{\lambda} & 0 \\ a^{\lambda}\phi_{\lambda} & a^{\lambda}\phi_{\lambda} & 0 & -a^{\lambda}\mathcal{U}_{\lambda} \end{bmatrix} \tag{2.20}$$

Labeling the six symmetric elements of this tensor as $s^{\mu\nu}$ leads to the extraction of the strain tensor thru the relation

$$\partial^{\mu} A^{\nu}_a = s^{\mu\nu} + \epsilon^{\nu\mu} \tag{2.21}$$

Acceleration strain is actually a bi-linear quantity which can be written as a vector outer product

$$\widehat{\epsilon} = \vec{A}_v \otimes \vec{a}_{\perp} \tag{2.22}$$

and having an explicit representation

$$\widehat{\epsilon} = -\frac{e}{\rho}\left[\frac{a_{\theta}}{R}\,\vec{\mathrm{e}}_{\tau\theta} + \frac{a_{\theta}}{R}\,\vec{\mathrm{e}}_{\rho\theta} + \frac{a_{\phi}}{R\sin\theta}\,\vec{\mathrm{e}}_{\tau\phi} + \frac{a_{\phi}}{R\sin\theta}\,\vec{\mathrm{e}}_{\rho\phi}\right] \tag{2.23}$$

In the basis of unit vectors define the 2×2 matrix

$$\mathcal{F}_a \equiv -\frac{e}{\rho}\begin{bmatrix} a_\theta & a_\phi \\ a_\theta & a_\phi \end{bmatrix} \tag{2.24}$$

so that two forms of the strain tensor can be identified as

$$\epsilon^{\mu\nu} = \begin{bmatrix} 0 & \mathcal{F}_a \\ 0 & 0 \end{bmatrix} \qquad \epsilon^{\nu\mu} = \begin{bmatrix} 0 & 0 \\ \mathcal{F}_a^\dagger & 0 \end{bmatrix} \tag{2.25}$$

The requirement $\epsilon = 0$ follows by inspection along with the equation $\det[\epsilon^{\mu\nu}] = 0$. Moreover, since $\det[\mathcal{F}_a] = 0$ this implies the vanishing determinants of all individual two-by-two minors in the cartesian basis as well.

The divergence applied to the first index of the acceleration strain is

$$\epsilon^{\mu\nu}{}_{;\,\mu} = \epsilon^{\mu\nu}{}_{,\,\mu} + \epsilon^{\lambda\nu}\,\Gamma^{\mu}_{\lambda\mu} + \epsilon^{\mu\lambda}\,\Gamma^{\nu}_{\lambda\mu} \tag{2.26}$$

The first two components of the resulting vector are automatically zero while the remaining components are determined from

$$\epsilon^{\mu\theta}{}_{;\,\mu} = \frac{\partial \epsilon^{\rho\theta}}{\partial \rho} + \epsilon^{\rho\theta}\left[2\Gamma^{\theta}_{\rho\theta} + \Gamma^{\phi}_{\rho\phi}\right] = \frac{\epsilon^{\rho\theta}}{\rho} \tag{2.27a}$$

$$\epsilon^{\mu\phi}{}_{;\,\mu} = \frac{\partial \epsilon^{\rho\phi}}{\partial \rho} + \epsilon^{\rho\phi}\left[\Gamma^{\theta}_{\rho\theta} + 2\Gamma^{\phi}_{\rho\phi}\right] = \frac{\epsilon^{\rho\phi}}{\rho} \tag{2.27b}$$

These are components of the strain current which allows the divergence equation to be written

$$\epsilon^{\mu\nu}{}_{;\,\mu} = \eta a_\perp^\nu \tag{2.28}$$

Acceleration Field Strength Tensor: A fundamental representation of the acceleration field strength tensor in the spherical basis can be derived by writing the tensors in equation (1.35) as

$$\mathsf{H}^{\mu\nu} = \rho\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathsf{H}_D^{\mu\nu} = \rho\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \tag{2.29}$$

Couple this with equations (2.25) and the acceleration fields become

$$F_a^{\mu\nu} = [A^\mu,\, a_\perp^\nu] = -\eta a_\theta[R^\mu,\, \theta^\nu] - \eta a_\phi[R^\mu,\, \phi^\nu] \tag{2.30}$$

This looks like a separation of the fields into independent polarization states with unitless terms multiplying the dilatation η. In fact, all three polarization states can be written together in the unit coordinate basis since velocity and acceleration fields belong to their own independent subspaces. Defining $\mathcal{F}_v$ the total field is

$$F^{\mu\nu} = \begin{bmatrix} \mathcal{F}_v & \mathcal{F}_a \\ -\mathcal{F}_a^\dagger & 0 \end{bmatrix} \tag{2.31}$$

and an extension of (2.30) is

$$F^{\mu\nu} = \eta[\mathcal{U}^\mu,\, \beta^\nu] - \eta a_\theta[R^\mu,\, \theta^\nu] - \eta a_\phi[R^\mu,\, \phi^\nu] \tag{2.32}$$

3 Total Stress Tensor in the Spherical Basis

The velocity portion of the symmetric stress tensor is $\Theta_1^{\mu\nu}$ and can be derived beginning with

$$\Delta^{\mu\lambda}_{\ \ ;\,\mu}\,\eta^\nu{}_\lambda = \frac{4\pi}{c} j_e^{*\lambda}\,\eta^\nu{}_\lambda \tag{3.1}$$

This equation is just the second equation in (2.17) multiplied by the velocity strain. Operations on the left side are

$$\begin{aligned} \Delta^{\mu\lambda}{}_{;\,\mu}\,\eta^\nu{}_\lambda &= (\Delta^{\mu\lambda}\eta^\nu{}_\lambda)_{;\,\mu} - \Delta^{\mu\lambda}\,\eta^\nu{}_{\lambda\,;\,\mu} \\ &= (\eta^{\mu\lambda}\eta^\nu{}_\lambda - \eta\eta^{\nu\mu})_{\,;\,\mu} - \eta^{\mu\lambda}\eta^\nu{}_{\lambda\,;\,\mu} + \eta\eta^{\nu\mu}{}_{;\,\mu} \end{aligned} \tag{3.2}$$

but each of the last two terms can be written as total derivatives

$$\eta^{\mu\lambda}\eta^\nu{}_{\lambda\,;\,\mu} = \tfrac{1}{2}\mathrm{g}^{\mu\nu}(\eta^{\alpha\lambda}\eta_{\alpha\lambda})_{\,;\,\mu} \qquad\qquad \eta\eta^{\nu\mu}{}_{;\,\mu} = \tfrac{1}{2}\mathrm{g}^{\mu\nu}(\eta^2)_{\,;\,\mu} \tag{3.3}$$

so the first equation can be re-arranged to read

$$\frac{1}{4\pi}\left[\eta^{\mu\lambda}\eta^\nu{}_\lambda - \eta\eta^{\nu\mu} - \tfrac{1}{2}\mathrm{g}^{\mu\nu}(\eta^{\mu\lambda}\eta_{\mu\lambda} - \eta^2)\right]_{\,;\,\mu} = \frac{1}{c} j_e^{*\lambda}\,\eta^\nu{}_\lambda \tag{3.4}$$

The free term vacuum Lagrangian is

$$\mathcal{L}_o = -\frac{1}{8\pi}\Delta^{\mu\nu}\eta_{\mu\nu} \tag{3.5}$$

which easily finds its way into (3.4) with the inclusion of an overall minus sign to read

$$\left[\frac{\partial\mathcal{L}_o}{\partial\eta_{\mu\lambda}}\eta^\nu{}_\lambda - \mathrm{g}^{\mu\nu}\mathcal{L}_o\right]_{;\,\mu} = -\frac{1}{c} j_e^{*\lambda}\,\eta^\nu{}_\lambda \tag{3.6}$$

The term in brackets is the vacuum form of the canonical stress tensor. Like the electromagnetic theory it must be symmetrized by removing the superfluous term with zero divergence:

$$(\eta\eta^{\nu\mu})_{\,;\,\mu} = 0 \tag{3.7}$$

What remains is the symmetric stress tensor

$$\Theta^{\mu\nu}_{vac} = \frac{1}{4\pi}\left[\tfrac{1}{2}\mathrm{g}^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^\nu{}_\lambda\right] \tag{3.8}$$

obeying the divergence relation

$$\Theta^{\mu\nu}_{vac\ \ ;\,\mu} = -\frac{1}{c} j_e^{*\lambda}\,\eta^\nu{}_\lambda \tag{3.9}$$

3.1 Solution to the Stability Problem

To complete the velocity theory it will be necessary to propagate the velocity fields. This can be done with the addition of a term linear in the vacuum dilatation. The complete Lagrangian for the velocity theory inclusive of an interaction term is then

$$\begin{aligned}\mathcal{L}_{vac} &= -\frac{1}{8\pi}\Delta^{\mu\nu}\eta_{\mu\nu} + \frac{1}{2}\sigma_e\,\eta - \frac{1}{c}j_e^{*\mu}A_\mu \\ &= \mathcal{L}_o + \mathcal{L}_\Lambda + \mathcal{L}_{int}\end{aligned} \tag{3.10}$$

The total stress tensor associated with this Lagrangian can be written

$$\mathcal{T}^{\mu\nu} = \Theta_1^{\mu\nu} + \Lambda^{\mu\nu} \tag{3.11}$$

and is easily transformed to the spherical basis by writing

$$\begin{aligned}\mathcal{T}^{\mu\nu}_{sph} &= \mathsf{U}^\mu{}_\alpha\, T^{\alpha\lambda}\, \mathsf{U}_\lambda{}^\nu \\ &= \mathsf{U}^\mu{}_\alpha\, \Theta_1^{\alpha\lambda}\, \mathsf{U}_\lambda{}^\nu + \mathsf{U}^\mu{}_\alpha\, \Lambda^{\alpha\lambda}\, \mathsf{U}_\lambda{}^\nu = \Theta^{\mu\nu}_{1\,sph} + \Lambda^{\mu\nu}_{sph}\end{aligned} \tag{3.12}$$

Matrix representation of both tensors are

$$\Theta^{\mu\nu}_{1\,sph} = \frac{1}{8\pi}\begin{bmatrix}\eta^2 & 0 & 0 & 0\\ 0 & -\eta^2 & 0 & 0\\ 0 & 0 & \eta^2 & 0\\ 0 & 0 & 0 & \eta^2\end{bmatrix} \qquad \Lambda^{\mu\nu}_{sph} = \frac{\sigma_e}{2}\begin{bmatrix}-\eta & \eta & 0 & 0\\ 0 & 2\eta & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{3.13}$$

The symmetric stress tensor is diagonal implying that the spherical basis vectors are also eigenvectors. This is not true of the vacuum tensor which has a single off diagonal element and no components along θ^ν or ϕ^ν. To show stability of the particle, conveniently write the sum of these two tensors as

$$\mathcal{T}^{\mu\nu}_{sph} = \left[\frac{1}{8\pi}\eta^2 - \frac{1}{2}\sigma_e\eta\right]\mathcal{G}_1^{\mu\nu} + \frac{1}{2}\sigma_e\eta\,\mathcal{G}_2^{\mu\nu} \tag{3.14}$$

where $\mathcal{G}_1^{\mu\nu}$ and $\mathcal{G}_2^{\mu\nu}$ are matrices of ones and zeros only and defined by

$$\mathcal{G}_1^{\mu\nu} \equiv \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \mathcal{G}_2^{\mu\nu} \equiv \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{3.15}$$

In terms of components of the orignal Lagrangian, (3.14) may be written more concisely as

$$\mathcal{T}^{\mu\nu}_{sph} = -(\mathcal{L}_o + \mathcal{L}_\Lambda)\,\mathcal{G}_1^{\mu\nu} + \mathcal{L}_\Lambda\,\mathcal{G}_2^{\mu\nu} \tag{3.16}$$

and stability will be determined by the condition $\rho = r_e$ for which the value of the vacuum Lagrangian is zero. Meanwhile, the remaining tensor is canonical energy flux thru the radius and not associated with local stress on the particle. It can also be written

$$\mathcal{E}^{\mu\nu}_{rad} = -\mathcal{L}_{\Lambda} \cdot \partial^{\nu} R^{\mu} \tag{3.17}$$

where the tensor on the right indicates directly, the dispersion of radiation from the retarded position of the charge. As a final step, vacuum gauge theory can be brought full circle by re-deriving the vacuum Lagrangian from the contraction

$$\mathcal{L}_{vac} = -\beta_{\mu} \mathcal{T}^{\mu\nu}_{sph} \beta_{\nu} \tag{3.18}$$

and subsequent application of the vacuum gauge condition.

Symmetric Stress Tensor for Accelerated Motions: Accelerated motions of the electron can be accommodated by applying a perturbation to the vacuum strain of the form

$$\eta^{\mu\nu} \longrightarrow \eta^{\mu\nu} - \epsilon^{\mu\nu} \tag{3.19}$$

The dilatation is unaltered by the perturbation since $\mathrm{g}_{\mu\nu}\epsilon^{\mu\nu} = 0$, but the stress tensor will acquire terms linear and quadratic in the acceleration strain:

$$\Theta^{\mu\nu}_{vac} = \frac{1}{4\pi}\left[\tfrac{1}{2}\mathrm{g}^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^{\nu}{}_{\lambda} + \eta^{\mu\lambda}\epsilon^{\nu}{}_{\lambda} - \epsilon^{\mu\lambda}\epsilon^{\nu}{}_{\lambda} + \epsilon^{\mu\lambda}\eta^{\nu}{}_{\lambda}\right] \tag{3.20}$$

It is useful to construct this tensor in the spherical basis where its matrix representation is given by

$$\Theta^{\mu\nu}_{vac} \rightarrow \frac{\eta^2}{4\pi}\begin{bmatrix} \frac{1}{2} + \rho^2(a_\theta^2 + a_\phi^2) & \rho^2(a_\theta^2 + a_\phi^2) & -\rho a_\theta & -\rho a_\phi \\ \rho^2(a_\theta^2 + a_\phi^2) & -\frac{1}{2} + \rho^2(a_\theta^2 + a_\phi^2) & -\rho a_\theta & -\rho a_\phi \\ -\rho a_\theta & -\rho a_\theta & \frac{1}{2} & 0 \\ -\rho a_\phi & -\rho a_\phi & 0 & \frac{1}{2} \end{bmatrix} \tag{3.21}$$

This tensor should be compared with its cumbersome and lengthy cartesian counterpart found in many textbooks. The matrix above is not only concise, but each of the two components linear and quadratic in the quantities a_θ and a_ϕ is easily discernable by inspection. More precisely, one can write

$$\Theta^{\mu\nu}_{2} = \frac{e\eta}{4\pi}\begin{bmatrix} 0 & \mathcal{F}_a \\ \mathcal{F}_a^{\dagger} & 0 \end{bmatrix} \qquad\qquad \Theta^{\mu\nu}_{3} = \frac{e\eta}{4\pi}(a_\theta^2 + a_\phi^2)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3.22}$$

3.2 Independent Derivation of the Total Stress Tensor

The tensor $\mathcal{T}^{\mu\nu}$ can be constructed from a theory of vacuum stresses in either a cartesian or spherical coordinate system without appealing to a Lagrangian formulation. This derivation enjoys the added benefit of producing the symmetric stress tensor $\Theta^{\mu\nu}$ with no additional requirement to eliminate a superfluous term like $T_D^{\mu\nu}$.

For general motions of the electron, the vacuum tensor is

$$\Delta^{\mu\nu} = \zeta^{\mu\nu} - \mathrm{g}^{\mu\nu}\zeta \quad \text{where} \quad \zeta^{\mu\nu} = \eta^{\mu\nu} - \epsilon^{\mu\nu} \tag{3.23}$$

A fourth rank Maxwell-Lorentz deformation tensor is generated from $\Delta^{\mu\nu}$ by the bi-linear construction

$$\mathcal{R}^{\mu\lambda\alpha\nu} \equiv \frac{1}{4\pi}\left[\Delta^{\mu\lambda}\Delta^{\alpha\nu} - \Delta^{\lambda\mu}\Delta^{\alpha\nu} + \Delta^{\lambda\mu}\Delta^{\nu\alpha} - \Delta^{\mu\lambda}\Delta^{\nu\alpha}\right] \tag{3.24a}$$

$$= \frac{1}{4\pi}\left[\zeta^{\mu\lambda}\zeta^{\alpha\nu} - \zeta^{\lambda\mu}\zeta^{\alpha\nu} + \zeta^{\lambda\mu}\zeta^{\nu\alpha} - \zeta^{\mu\lambda}\zeta^{\nu\alpha}\right] \tag{3.24b}$$

Clearly an undistorted, 'flat' vacuum is associated with $\mathcal{R}^{\mu\lambda\alpha\nu} = 0$. It is useful to write $\mathcal{R}^{\mu\lambda\alpha\nu}$ in the simplified form

$$\mathcal{R}^{\mu\lambda\alpha\nu} \equiv \frac{1}{4\pi}\mathcal{F}^{\mu\lambda}\mathcal{F}^{\alpha\nu} \tag{3.25}$$

Although $\mathcal{F}^{\mu\lambda}$ is identical to the field strength tensor, it has a role in terms of independent strain tensors which warrants the new descriptive variable. Contracting on the second and third indices produces the second rank tensor

$$\mathcal{R}^{\mu\nu} \equiv \mathrm{g}_{\lambda\alpha}\mathcal{R}^{\mu\lambda\alpha\nu} = \mathcal{R}^{\mu\lambda}{}_{\lambda}{}^{\nu} \tag{3.26}$$

but this is not the only possible contraction. There are a total of five others shown in Table 1. The fact that each of the contractions produces only variations of a single tensor $\mathcal{R}^{\mu\nu}$ is related to symmetry properties of $\mathcal{R}^{\mu\lambda\alpha\nu}$ which can be written

$$\mathcal{R}^{\mu\lambda\alpha\nu} = -\mathcal{R}^{\lambda\mu\alpha\nu} = -\mathcal{R}^{\mu\lambda\nu\alpha} = \mathcal{R}^{\alpha\nu\mu\lambda} \tag{3.27}$$

along with

$$\mathcal{R}^{\mu\lambda\alpha\nu} + \mathcal{R}^{\mu\nu\lambda\alpha} + \mathcal{R}^{\mu\alpha\nu\lambda} = 0 \tag{3.28}$$

Aside from symmetry properties, a Lorentz scalar can be formed by contracting on $\mathcal{R}^{\mu\nu}$ so that

$$\mathcal{R} = \mathrm{g}_{\mu\nu}\mathcal{R}^{\mu\nu} = -\frac{1}{4\pi}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} \tag{3.29}$$

Now consider a useful property of $\mathcal{F}^{\mu\nu}$ in the form

$$\mathcal{F}_{\lambda\nu;\,\mu} + \mathcal{F}_{\nu\mu;\,\lambda} + \mathcal{F}_{\mu\lambda;\,\nu} = 0 \tag{3.30}$$

Contraction Indices	Result	Contraction Indices	Result
2 and 3	$\mathcal{R}^{\mu\nu}$	1 and 4	$\mathcal{R}^{\lambda\alpha}$
1 and 2	0	1 and 3	$-\mathcal{R}^{\nu\lambda}$
3 and 4	0	2 and 4	$-\mathcal{R}^{\lambda\alpha}$

Table 1: Contractions on $\mathcal{R}^{\mu\lambda\alpha\nu}$

An equation similar to the Bianchi identities of General Relativity follows by multiplying this tensor equation by $\mathcal{F}_{\sigma\alpha}$:

$$\mathcal{F}_{\sigma\alpha}\mathcal{F}_{\lambda\nu;\,\mu} + \mathcal{F}_{\sigma\alpha}\mathcal{F}_{\nu\mu;\,\lambda} + \mathcal{F}_{\sigma\alpha}\mathcal{F}_{\mu\lambda;\,\nu} = 0 \tag{3.31}$$

Two contractions on (3.31) result by multiplying through by $\mathrm{g}^{\alpha\mu}\mathrm{g}^{\sigma\lambda}$ leading to

$$\mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\nu;\,\mu} + \mathcal{F}^{\lambda\mu}\mathcal{F}_{\nu\mu;\,\lambda} + \mathcal{F}^{\lambda\mu}\mathcal{F}_{\mu\lambda;\,\nu} = 0 \tag{3.32}$$

The first two terms can be combined using the anti-symmetry of $\mathcal{F}^{\mu\nu}$ and a change of indices. Writing the last term as a total derivative, the previous equation then follows as

$$2\mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\nu;\,\mu} - \tfrac{1}{2}(\mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\mu})_{;\,\nu} = 0 \tag{3.33}$$

The first term can also be written as a total derivative at the expense of introducing a point source. For a Big Bang cosmology this source can be labeled $j_e^{*\lambda}$ where the asterisk is representative of a time ct_o before which the particle does not exist. The appropriate divergence relation is

$$\mathcal{F}^{\lambda\mu}{}_{;\,\mu} = -\frac{4\pi}{c}j_e^{*\lambda} \tag{3.34}$$

then

$$2(\mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\nu})_{;\,\mu} - \tfrac{1}{2}(\mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\mu})_{;\,\nu} = -\frac{8\pi}{c}\mathcal{F}_{\lambda\nu}j_e^{*\lambda} \tag{3.35}$$

The stress tensor follows from writing each term on the left in terms of contractions on $\mathcal{R}^{\mu\lambda\alpha\nu}$

$$\mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\nu} = -4\pi\mathcal{R}^{\mu}{}_{\nu} \quad \text{and} \quad \mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\mu} = -4\pi\mathcal{R} \tag{3.36}$$

Introducing the metric tensor and raising the index ν to a contravariant status leads to the source equation

$$(\mathcal{R}^{\mu\nu} - \tfrac{1}{4}\mathrm{g}^{\mu\nu}\mathcal{R})_{;\,\mu} \equiv \Theta^{\mu\nu}_{vac\,;\,\mu} = \frac{1}{c}\mathcal{F}^{\lambda\nu}j^{*}_{e\,\lambda} \tag{3.37}$$

This is a gauge invariant derivation of $\Theta^{\mu\nu}_{em}$ even though vacuum gauge potentials have been assumed throughout. On the other hand, gauge invariance must be broken to propagate the vacuum—made possible by adding a single instance of the vacuum tensor—leading to the total stress tensor

$$\mathcal{T}^{\mu\nu} = \mathcal{R}^{\mu\nu} - \tfrac{1}{4}\mathrm{g}^{\mu\nu}\mathcal{R} + \Lambda^{\mu\nu} \tag{3.38}$$

For accelerated motions $\mathcal{R}^{\mu\nu}$ is the sum of sixteen components determined from bi-linear combinations of velocity and acceleration tensors $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu}$. This is an over-specfication of the tensor which can be minimized to four components thru the definition

$$\mathcal{R}^{\mu\nu} \equiv \zeta^{\mu\lambda}\zeta^{\nu}{}_{\lambda} \tag{3.39}$$

Components of the stress tensor can then be re-organized as

$$\mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda}{}^{\nu} = 4\pi\left(\mathcal{R}^{\mu\nu} - \tfrac{1}{2}\mathrm{g}^{\mu\nu}\mathcal{R}\right) \qquad \text{and} \qquad \mathcal{F}^{\lambda\mu}\mathcal{F}_{\lambda\mu} = -4\pi\mathcal{R} \tag{3.40}$$

and equations for a stable classical particle become

$$\boxed{\tfrac{1}{4}\mathrm{g}^{\mu\nu}\mathcal{R} - \mathcal{R}^{\mu\nu} = \mathcal{T}^{\mu\nu} - \Lambda^{\mu\nu}} \tag{3.41}$$

This set of equations, while designed specifically for the classical particle, can also be applied to other macroscopic phenomena in electromagnetism. One such application is a more general formulation the theory of transverse electromagnetic waves as a vacuum gauge theory. In this case the vacuum gauge condition is

$$|\partial_\nu A^\nu| = \sqrt{\mathrm{E}^2 - \mathrm{B}^2} = 0 \tag{3.42}$$

implying that such waves are not to be associated with vacuum dilatation. According to equation (3.38), the theory follows by removing source and the propagation terms leaving

$$\mathcal{T}^{\mu\nu} = -\mathcal{R}^{\mu\nu} = -\frac{1}{4\pi}\eta^{\mu\lambda}\eta^{\nu}{}_{\lambda} \tag{3.43}$$

For waves propagating in the direction $\boldsymbol{k}$, the appropriate form of $\boldsymbol{A}$ must be inserted by hand along with the requirement $\boldsymbol{A}\cdot\boldsymbol{k} = 0$. A vacuum strain tensor can then be constructed satisfying

$$\Box^2\eta^{\mu\nu} = 0 \tag{3.44}$$

4 Integrals in the Spherical Basis

A primary advantage of using the spherically based four-coordinate system is to faciliate the integration of scalars, vectors, and second rank tensors. One may also apply the divergence theorem over the causal light cone to derive relations among integrals.

4.1 Volume and Surface Elements

Scale factors are immediately available from equation (1.11) in section 1 to construct both volume and surface elements. The four-volume element is

$$d^4\mathcal{V} = R^2 d\rho\, d\Omega\, cd\tau = \rho^2 d\rho\, d\Omega'\, cd\tau \tag{4.1}$$

and the invariant volume of the light cone follows almost immediately by placing appropriate integration limits on the variable ρ:

$$\mathcal{V}_{light\,cone} = \tfrac{1}{3}\pi\, c^4\tau^4 \tag{4.2}$$

Spacelike and timelike 3-surface elements having the same Doppler factor as $d^4\mathcal{V}$ are

$$d^3\vec{\sigma}_s = [R^2 d\rho\, d\Omega]\, \vec{\mathrm{e}}_\tau \tag{4.3a}$$
$$d^3\vec{\sigma}_\tau = [R^2 cd\tau\, d\Omega]\, \vec{\mathrm{e}}_\rho \tag{4.3b}$$

Dropping the superscript 3, these may be written more usefully in terms of contravariant unit vectors

$$d\sigma_s^\mu = [R^2 d\rho\, d\Omega]\, \beta^\mu \tag{4.4a}$$
$$d\sigma_\tau^\mu = [R^2 cd\tau\, d\Omega]\, \mathcal{U}^\mu \tag{4.4b}$$

Equating increments of $d\rho$ and $cd\tau$ the combination of both 3-elements in (4.4) leads to the light cone 3-surface element

$$d\sigma_\imath^\mu = -\frac{1}{\rho}[R^2 d\rho\, d\Omega]\, R^\mu \tag{4.5}$$

where the sign has been introduced for simple applications of the divergence theorem.

4.2 Integral of the Total Stress Tensor

From the matrix representation of equation (3.13), the symmetric stress tensor and the vacuum tensor can be immediately written

$$\Theta_1^{\mu\nu} = \frac{1}{8\pi}\eta^2\left[\beta^\mu\beta^\nu - \mathcal{U}^\mu\mathcal{U}^\nu + \theta^\mu\theta^\nu + \phi^\mu\phi^\nu\right] \tag{4.6}$$

$$\Lambda^{\mu\nu} = \frac{1}{2}\sigma_e\eta\left[2\mathcal{U}^\mu\mathcal{U}^\nu - \beta^\mu\beta^\nu + \beta^\mu\mathcal{U}^\nu\right] \tag{4.7}$$

Four-volume integrals of both tensors can be performed by separating the timelike energy terms from spacelike pressure terms. For the symmetric stress tensor radial integrals extend from the radius r_e to a large time $c\tau + r_e$ which can be approximated by infinity

$$d\mathcal{S}_{[1\tau]}^{\mu\nu} = \frac{1}{8\pi}\int \eta^2\beta^\mu\beta^\nu d^4\mathcal{V} = mc^2\beta^\mu\beta^\nu \cdot cd\tau \tag{4.8}$$

$$d\mathcal{S}_{[1s]}^{\mu\nu} = \frac{1}{8\pi}\int \eta^2\left[-\mathcal{U}^\mu\mathcal{U}^\nu + \theta^\mu\theta^\nu + \phi^\mu\phi^\nu\right] d^4\mathcal{V} = \tfrac{1}{3}mc^2\left[\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu}\right] \cdot cd\tau \tag{4.9}$$

A similar procedure for the vacuum tensor follows by first observing that only the symmetric terms make contributions to the integral. Moreover, the lower limit to the radial integration must be extended to zero so that

$$d\mathcal{S}^{\mu\nu}_{[2\tau]} = -\frac{\sigma_e}{2}\int_0^{r_e+c\tau}\int_\Omega \eta\beta^\mu\beta^\nu d^4\mathcal{V} = -\tfrac{1}{2}\dot{\rho}[r_e+c\tau]\beta^\mu\beta^\nu\cdot cd\tau \tag{4.10}$$

$$d\mathcal{S}^{\mu\nu}_{[2s]} = \sigma_e\int_0^{r_e+c\tau}\int_\Omega \eta\mathcal{U}^\mu\mathcal{U}^\nu d^4\mathcal{V} = \tfrac{1}{3}\dot{\varrho}[r_e+c\tau]\left[\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu}\right]\cdot cd\tau \tag{4.11}$$

Summing individual terms with the inclusion of a necessary integration constant leads to

$$d\mathcal{S}^{\mu\nu} = d\mathcal{S}^{\mu\nu}_{[1\tau]} + d\mathcal{S}^{\mu\nu}_{[1s]} + d\mathcal{S}^{\mu\nu}_{[2\tau]} + d\mathcal{S}^{\mu\nu}_{[2s]} + \mathrm{g}^{\mu\nu}mc^2\cdot cd\tau \tag{4.12}$$

A second rank total energy tensor written exclusively in terms of spherical basis unit vectors is

$$\frac{1}{c}\frac{d\mathcal{S}^{\mu\nu}}{d\tau} = mc^2\beta^\mu\beta^\nu - \tfrac{1}{2}\dot{\rho}c\tau\beta^\mu\beta^\nu + \tfrac{1}{3}\dot{\varrho}c\tau\left(\mathcal{U}^\mu\mathcal{U}^\nu + \theta^\mu\theta^\nu + \phi^\mu\phi^\nu\right) \tag{4.13}$$

$$= \mathcal{E}^{\mu\nu}_{part} + \mathcal{E}^{\mu\nu}_{vac} + \mathcal{P}^{\mu\nu}_{vac}c \tag{4.14}$$

The invariant hamiltonian can be extracted immediately from the single timelike component

$$\boxed{\mathcal{H} = mc^2 - \tfrac{1}{2}\dot{\rho}c\tau} \tag{4.15}$$

Using $\mathcal{H}$, a diagonal matrix representation of a total energy tensor $\mathcal{E}^{\mu\nu}_{tot}$ valid in the spherical basis is easily shown to be

$$\mathcal{E}^{\mu\nu}_{tot} = \begin{bmatrix} \mathcal{H} & 0 & 0 & 0 \\ 0 & \frac{1}{3}\mathcal{P}c & 0 & 0 \\ 0 & 0 & \frac{1}{3}\mathcal{P}c & 0 \\ 0 & 0 & 0 & \frac{1}{3}\mathcal{P}c \end{bmatrix} \tag{4.16}$$

To complete the calculation it is also useful to form scalar contractions of the vacuum terms deriving the general relation

$$\boxed{\mathcal{E}_{vac} = \tfrac{1}{2}\mathcal{P}_{vac}c} \tag{4.17}$$

These calculations are quite general and should apply independent of accelerations of the particle.

4.3 Applications of the Divergence Theorem

The three hyper-surfaces in equations (4.3) and (4.4) can be combined to form a closed surface $\mathcal{S}^\nu$ on the casual light cone. The divergence theorem for vectors and second rank tensors is then

$$\int_{\mathcal{V}} \partial_\nu X^\nu \, d^4\mathcal{V} = \oint_{\mathcal{S}} X^\nu d^3\mathcal{S}_\nu \tag{4.18}$$

$$\int_{\mathcal{V}} \partial_\nu Y^{\mu\nu} \, d^4\mathcal{V} = \oint_{\mathcal{S}} Y^{\mu\nu} d^3\mathcal{S}_\nu \tag{4.19}$$

Four-Volume: A spacetime diagram showing the four-volume inside the light cone for an electron moving at speed β^ν is shown in figure 4. The radius of the central hyper-cylinder is given by $\rho = r_e$ and the world line of the particle travels through its center. Probably the simplest example of the use of the divergence theorem is a calculation of the surrounding four-volume. This can be accomplished begininning with the function

$$f = \frac{1}{3}\partial_\lambda R^\lambda = 1 \tag{4.20}$$

Limits for ρ and $c\tau$ are identical and may be written

$$r_e \le \rho \le c\tau_o \tag{4.21a}$$

$$r_e \le c\tau \le c\tau_o \tag{4.21b}$$

The volume of 4-solid is then

$$\mathcal{V}_4 = \int d^4\mathcal{V} = \frac{1}{3}\pi c^4 \tau_o^4 - \frac{1}{3}\pi r_e^4 - \frac{4}{3}\pi r_e^3 (c\tau_o - r_e) \tag{4.22}$$

This volume may also be determined by integrating over the spacelike plane and the timelike tubular hyper-surface

$$\mathcal{V}_4 = \int_{\mathcal{S}_1} \frac{1}{3} R_\nu d\sigma_s^\nu + \int_{\mathcal{S}_2} \frac{1}{3} R_\nu d\sigma_\tau^\nu \tag{4.23}$$

Velocity Potentials: The vacuum gauge condition can also be integrated over the volume using the same limits given in (4.21):

$$\int \partial_\nu A^\nu R^2 d\rho \, d\Omega \, cd\tau = 2\pi e (c\tau_o - r_e)^2$$

Instead, one may also integrate the velocity potentials over the spacelike and time like surfaces rendering

$$\int \partial_\nu A^\nu d^4\mathcal{V} = \Sigma_1 + \Sigma_2 \tag{4.24}$$

where the two contributing surface integrals are given by

$$\Sigma_1 = \int_{\mathcal{S}_1} A_\nu \cdot d\sigma_s^\nu = 2\pi e \left[c^2\tau^2 - r_e^2\right] \tag{4.25a}$$

$$\Sigma_2 = \int_{\mathcal{S}_2} A_\nu \cdot d\sigma_\tau^\nu = 4\pi \left[r_e^2 - c\tau r_e\right] \tag{4.25b}$$

and the integration over $\mathcal{S}_3$ is zero.

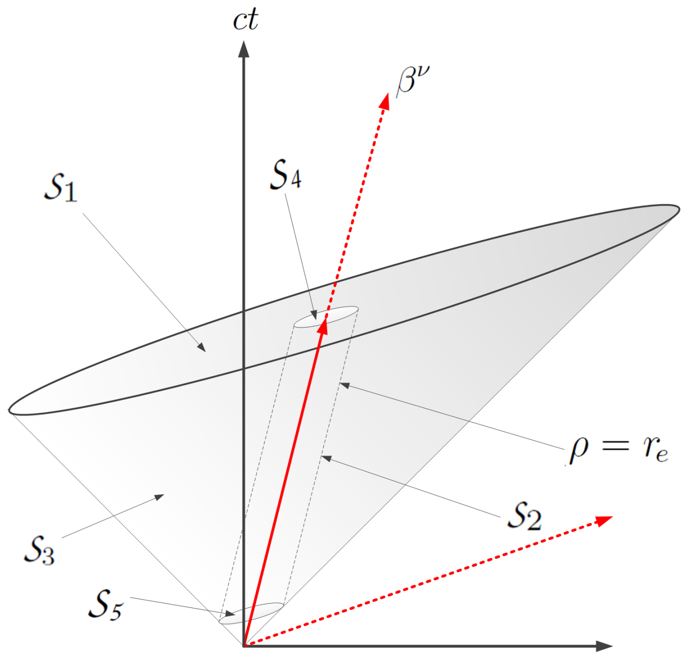

Figure 4: *Causal light cone surrounding a constant velocity electron.*

Vacuum Tensor: A more challenging problem for constant velocity motion is the application of the divergence theorem to the vacuum tensor. Once again, we refer to figure 4 but this time the theorem will be applied to both the central hyper-cylinder and also to the causal light cone less the hyper-cylinder.

For the first problem, three surface integrals replace the four dimensional volume integral producing

$$\int_{\mathcal{V}_{cyl}} \partial_\mu \Lambda^{\mu\nu} d^4\mathcal{V} = -\int_{\mathcal{S}_4} \Lambda^{\mu\nu} d\sigma_\mu^s + \int_{\mathcal{S}_5} \Lambda^{\mu\nu} d\sigma_\mu^s - \int_{\mathcal{S}_2} \Lambda^{\mu\nu} d\sigma_\mu^\tau \tag{4.26}$$

But the divergence on the left side of the equation is zero along with the surface integral over $\mathcal{S}_2$. This implies that integrals over the two hyper-ellipses are the same to within a sign. If the radius is $\rho = r_e$ these integrals also determine the energy-momentum four-vector of the particle:

$$\int_{\mathcal{S}_{4,5}} \Lambda^{\mu\nu} d\sigma^s_\mu = \pm mc^2\beta^\mu \tag{4.27}$$

A slightly more complicated problem is to re-write equation (4.26) with contractions on the second index. In this case the divergence on the left side of the equation is a delta function so the volume integral is non-zero. The integrals over the ends of the cylinder still cancel and this requires the integral over $\mathcal{S}_2$ to be non-zero. In fact, to within a sign, each of the remaining integrals is equal to the energy-momentum four-vector of the electron multiplied by an appropriate time interval:

$$mc^2\beta^\mu \cdot \frac{\tau}{\tau_e} = \int_{\mathcal{V}_{cyl}} \partial_\nu \Lambda^{\mu\nu} d^4\mathcal{V} = -\int_{\mathcal{S}_2} \Lambda^{\mu\nu} d\sigma^\tau_\mu \tag{4.28}$$

This equation relates a point source to a flux integral over $\mathcal{S}_2$ which determines the inertia of the particle.

For the second problem, the vacuum tensor is integrated over the three surfaces $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. The analysis is similar to the hyper-cylinder and includes contractions on either index. Choosing the second index renders the equation

$$\int_{\mathcal{V}_{lc}} \partial_\nu \Lambda^{\mu\nu} d^4\mathcal{V} = \int_{\mathcal{S}_1} \Lambda^{\mu\nu} d\sigma^s_\nu + \int_{\mathcal{S}_2} \Lambda^{\mu\nu} d\sigma^\tau_\nu + \int_{\mathcal{S}_3} \Lambda^{\mu\nu} d\sigma^l_\nu \tag{4.29}$$

On the left, the four-volume avoids the delta function produced by the integrand so the integral is zero. On the right, integrals over the spacelike and timelike surfaces are identical. This means the integral over the lightlike surface is twice the value of either one except for a sign change:

$$\frac{1}{2}\int_{\mathcal{S}_3} \Lambda^{\mu\nu} d\sigma^l_\nu = -\int_{\mathcal{S}_1} \Lambda^{\mu\nu} d\sigma^s_\nu = mc^2\beta^\mu \left[1 - \frac{\tau}{\tau_e}\right] \tag{4.30}$$

This verifies the divergence theorem but it is also important to exhibit a general relation among the timelike and spacelike surface integrals. If the initial and final proper times are such that $\Delta\tau = \tau - \tau_o$ then

$$\boxed{-\int_{\Delta\tau} \Lambda^{\mu\nu} d\sigma^\tau_\nu = -\int_{\tau} \Lambda^{\mu\nu} d\sigma^s_\nu} \tag{4.31}$$

Equation (4.31) says that the total flux of vacuum energy through the particle radius over a proper time $\Delta\tau$ is equal to the energy density in the vacuum at time τ integrated over the space which it occupies.

A Classical Action

To calculate the action for the vacuum gauge particle assume that a differential change in the vacuum strain $d\eta_{\mu\nu}$, is accompanied by a differential change in the action $d\mathcal{S}$ through the relation

$$d\mathcal{S} = -\frac{1}{4\pi c}\int \varDelta^{\mu\nu} d\eta_{\mu\nu}\, d^4\mathcal{V} \tag{A.1}$$

where the volume element is given in equation (4.1). It is straightforward to show that the integrand can be written as a total differential

$$\varDelta^{\mu\nu} d\eta_{\mu\nu} = \frac{1}{2} d\left[\varDelta^{\mu\nu}\eta_{\mu\nu}\right] = \frac{1}{2}\frac{d}{dR}\left[\varDelta^{\mu\nu}\eta_{\mu\nu}\right] dR \tag{A.2}$$

where R is a variable representing intermediate states of the electron radius as it expands towards its value of r_e. The action follows as

$$\mathcal{S} = -\frac{1}{8\pi c}\int\int_0^{r_e} \frac{d}{dR}\left[\varDelta^{\mu\nu}\eta_{\mu\nu}\right] dR\, d^4\mathcal{V} \tag{A.3}$$

But the integration over the radius is trivial so that

$$\mathcal{S} = -\frac{1}{8\pi}\int_{r_e} \varDelta^{\mu\nu}\eta_{\mu\nu}\, d^4\mathcal{V} \tag{A.4}$$

The propagation of the field follows from the symmetry operation

$$\varDelta^{\mu\nu} \longrightarrow \varDelta^{\mu\nu} - 4\pi\sigma_e\, \mathrm{g}^{\mu\nu} \tag{A.5}$$

and the transformed action becomes $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ with individual contributions given by

$$\mathcal{S}_1 = -\frac{1}{8\pi c}\int_{r_e} \varDelta^{\mu\nu}\eta_{\mu\nu}\, d^4\mathcal{V} \qquad\qquad \mathcal{S}_2 = \frac{1}{2c}\int \sigma_e\eta\, d^4\mathcal{V} \tag{A.6}$$

Both integrals are Lorentz scalars and constrained by the causality step $\vartheta(\tau - \rho/c + r_e)$. The value of $\mathcal{S}_1$ can be determined using a lower integration limit at the classical electron radius leading to

$$d\mathcal{S}_1 \longrightarrow -mc^2 d\tau \cdot \vartheta \tag{A.7}$$

for large times.

Unfortunately, the term $\mathcal{S}_2$ will diverge unless causality forces an upper limit to the integral. Furthermore, it will be mandatory to extend the lower integration limit to include $\rho = 0$. Distinct portions of $\mathcal{S}_2$ corresponding to volumes inside an outside the electrons' vacuum boundary are

$$\mathcal{S}_2 = \frac{1}{2c}\int_{\mathcal{V}_<} \sigma_e\eta\, d^4\mathcal{V} + \frac{1}{2c}\int_{\mathcal{V}_>} \sigma_e\eta\, d^4\mathcal{V} = \mathcal{S}_< + \mathcal{S}_> \tag{A.8}$$

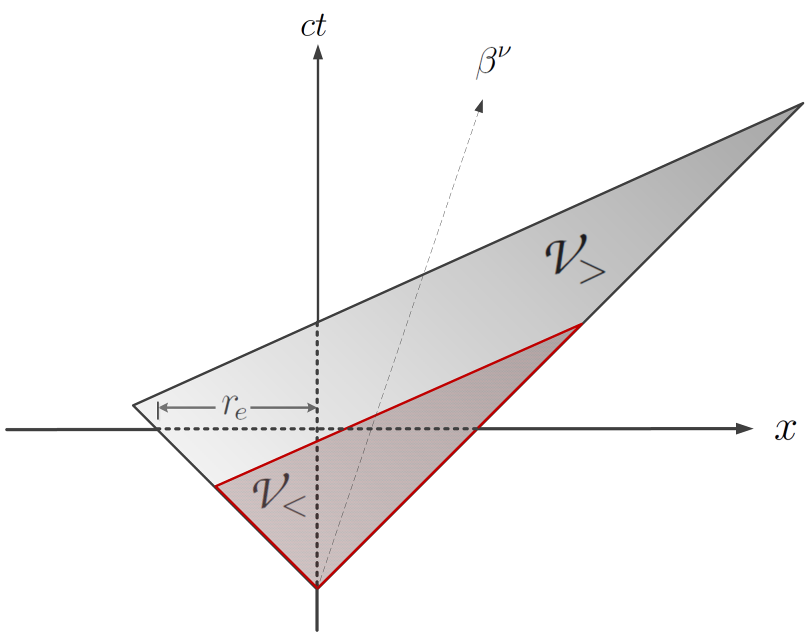

Figure 5: *Spacetime diagram showing four-volumes $\mathcal{V}_<$ and $\mathcal{V}_>$ associated with the causal light cone.*

The relevant volumes are shown in figure 5 in an abitrary Lorentz frame. Evaluating the integrals over the spatial variables shows that

$$d\mathcal{S}_< = mc^2 \cdot d\tau \qquad\qquad d\mathcal{S}_> = \frac{1}{2}\dot{\varrho}c\tau \cdot d\tau \tag{A.9}$$

Individual conributions to the action can now be added together. For consistency an integration constant can be included in the form $d\mathcal{S}_0 = -mc^2 d\tau$ to show that the total rate at which the action changes with respect to proper time is given by

$$\boxed{\frac{d\mathcal{S}}{d\tau} = -\mathcal{H}} \tag{A.10}$$

where $\mathcal{H}$ is the Hamiltonian of equation (4.15). Of particular interest is a slice of the action from the radiation term. If $\delta\tau_e$ is the time required for light to traverse the radius of the particle then

$$\delta\mathcal{S}_> = \frac{d\mathcal{S}_>}{d\tau} \cdot \delta\tau_e = mc^2\tau \tag{A.11}$$

A Dirac plane wave in moving frame coordinates is easily constructed from

$$\psi(ct, \boldsymbol{r}) = \psi_o e^{i\,\delta\mathcal{S}_>/\hbar} = \psi_o e^{i(Et-\boldsymbol{p}\cdot\boldsymbol{r})/\hbar} \tag{A.12}$$

where ψ_o is the appropriate spinor field.

B Present Position Coordinates

The vector $\boldsymbol{R}$ causally connects the retarded position of the electron to the present time event where the field is evaluated. However, a theory of the electron can also be rationalized using the vector $\boldsymbol{\mathcal{R}}$ from the present position to the field point. For constant velocity motion $\boldsymbol{\mathcal{R}}$ is defined by

$$\boldsymbol{\mathcal{R}} \equiv \boldsymbol{r} - \boldsymbol{\beta} ct \tag{B.1}$$

Its relationship to the vector $\boldsymbol{R}$ derives from the light cone condition and is illustrated in figure 6 for the simplified case of z-directed motion. The resulting formula reads

$$\boldsymbol{\mathcal{R}} = \boldsymbol{R} - R\boldsymbol{\beta} \tag{B.2}$$

It is important to determine the inverse transformation $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{\mathcal{R}})$ which can be

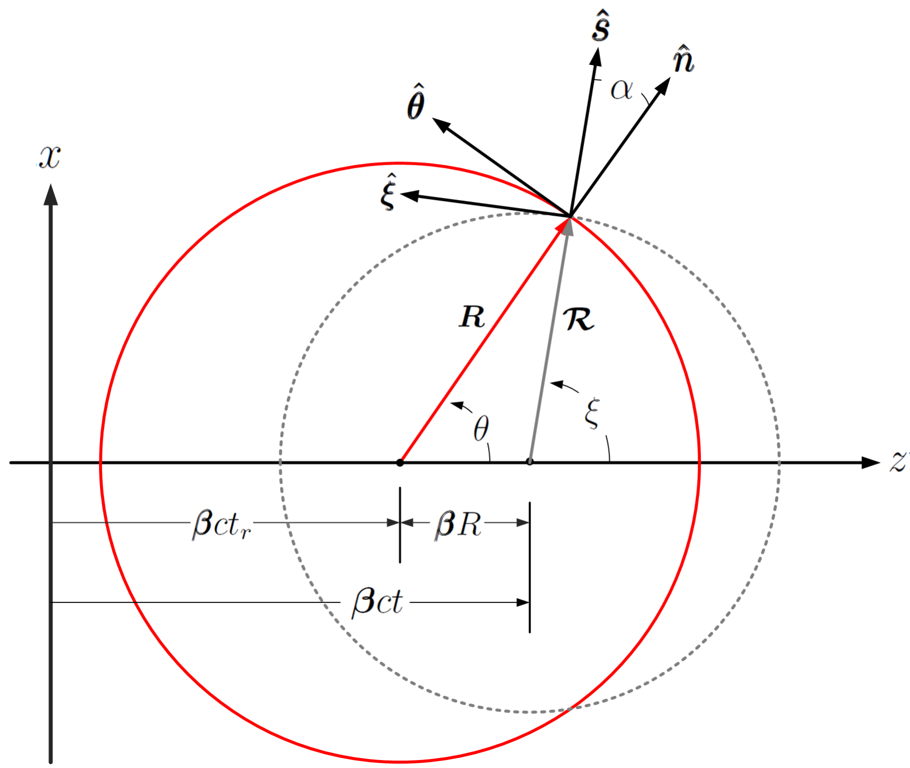

Figure 6: *A sphere of radius R in grey and a sphere of radius $\mathcal{R}$ in red. Associated coordinate systems are related by a rotation through angle $\alpha = \xi - \theta$.*

accomplished by squaring (B.2) and using the quadratic formula to yield

$$R = \gamma^2 \boldsymbol{\mathcal{R}} \cdot \boldsymbol{\beta} + \gamma[\mathcal{R}^2 + \gamma^2(\boldsymbol{\mathcal{R}} \cdot \boldsymbol{\beta})^2]^{1/2} \equiv f(\boldsymbol{\mathcal{R}}) \tag{B.3}$$

Inserting this result back into (B.2) gives the vector relation

$$\boldsymbol{R} = \boldsymbol{\mathcal{R}} + f(\boldsymbol{\mathcal{R}})\boldsymbol{\beta} \tag{B.4}$$

Also of importance is a formula for ρ in terms of present position coordinates. This can be determined beginning with (B.2) and dotting both sides with $\hat{\boldsymbol{n}}$:

$$\rho = \gamma\boldsymbol{\mathcal{R}} \cdot \hat{\boldsymbol{n}} = [\mathcal{R}^2 + \gamma^2(\boldsymbol{\mathcal{R}} \cdot \boldsymbol{\beta})^2]^{1/2} \tag{B.5}$$

Present Position Potentials: There are probably several ways to derive moving frame vacuum gauge velocity potentials in terms of the present position vector. The most straight forward approach is to Lorentz transform the rest frame potentials and coordinates, and then substitute equation (B.1). An easier method is to simply insert the transformations in (B.3) and (B.4) into the retarded vacuum gauge velocity potentials:

$$A = \frac{e\gamma}{\rho} + \frac{e\gamma^2}{\rho^2}\boldsymbol{\beta} \cdot \boldsymbol{\mathcal{R}} \tag{B.6a}$$

$$\boldsymbol{A} = \frac{e\gamma\boldsymbol{\beta}}{\rho} + \frac{e}{\rho^2}\left[\boldsymbol{\mathcal{R}} + \gamma^2(\boldsymbol{\beta} \cdot \boldsymbol{\mathcal{R}})\boldsymbol{\beta}\right] \tag{B.6b}$$

The Liénard-Weichert potentials are easily discernable as the first term on the right of both equations. This requires the remaining terms to be representations of the gauge field. In fact, since the gauge field is proportional to the spacelike vector $\mathcal{U}^\nu$, then the remaining terms can be multiplied by ρ/e to give the present position formula

$$\mathcal{U}^\nu = \frac{1}{\rho}\left[\gamma^2\boldsymbol{\beta} \cdot \boldsymbol{\mathcal{R}},\, \boldsymbol{\mathcal{R}} + \gamma^2(\boldsymbol{\beta} \cdot \boldsymbol{\mathcal{R}})\boldsymbol{\beta}\right] \tag{B.7}$$

A useful graphic is provided by figure 7 showing the congruency between the potentials and coordinate vectors. In the moving frame, $\boldsymbol{A}$ moves with the particle. Although it does not point radially from the present position, it is still radial with respect to the retarded position and its magnitude and direction are independent of which set of coordinates are used.

Spherical Coordinates at the Present Position: For the present time problem the coordinate transformation is still

$$x^\nu = c\tau\beta^\nu + \rho\,\mathcal{U}^\nu \tag{B.8}$$

with the provision that the four-vector $\mathcal{U}^\nu$ must be written in terms of present time angles. Defining present time unit vectors by the set $(\hat{\boldsymbol{s}}, \hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\phi}})$ then the components of $\mathcal{U}^\nu$ are

$$\mathcal{U}^\nu = \frac{1}{[1 + \gamma^2(\boldsymbol{\beta} \cdot \hat{\boldsymbol{s}})^2]^{1/2}}\left[\gamma^2\boldsymbol{\beta} \cdot \hat{\boldsymbol{s}},\, \hat{\boldsymbol{s}} + \gamma^2(\boldsymbol{\beta} \cdot \hat{\boldsymbol{s}})\boldsymbol{\beta}\right] \tag{B.9}$$

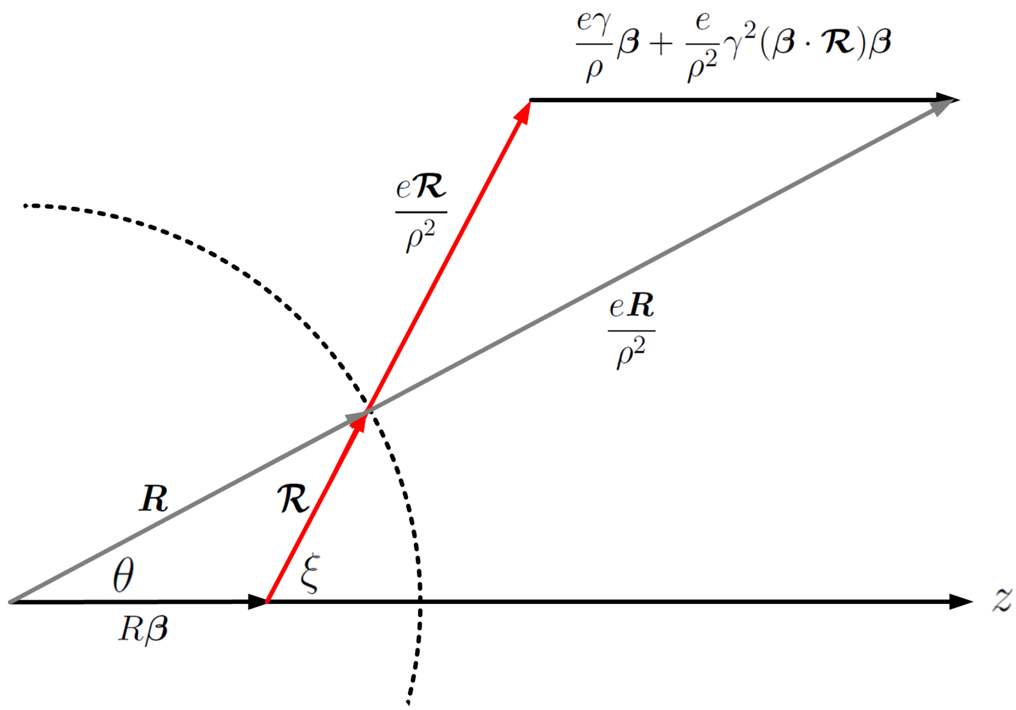

Figure 7: *Vacuum gauge vector potential* $\boldsymbol{A}$ *decomposed into components along the vectors* $\boldsymbol{\mathcal{R}}$, *and* $\boldsymbol{\beta}$.

and angular basis vectors $\vec{e}_\xi$ and $\vec{e}_\phi$ can be determined from

$$\vec{e}_\xi = \frac{\partial \mathcal{U}^\nu}{\partial \xi} \qquad\qquad \vec{e}_\phi = \frac{\partial \mathcal{U}^\nu}{\partial \phi} \tag{B.10}$$

A problem arises however because orthogonality $\vec{e}_\xi \cdot \vec{e}_\phi = 0$ mandates an additional requirement that at least one of the three-space angles be orthogonal to $\boldsymbol{\beta}$. For this reason it is much easier to specify that $\boldsymbol{\beta}$ point along the z-axis with the requirement that $\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}} = 0$. The transformation (B.8) then simplifies to

$$ct = \left[\frac{\gamma\beta\cos\xi}{(1-\beta^2\sin^2\xi)^{1/2}}\right]\rho + \gamma c\tau \qquad x = \left[\frac{\sin\xi\cos\phi}{\gamma(1-\beta^2\sin^2\xi)^{1/2}}\right]\rho \tag{B.11a}$$

$$z = \left[\frac{\gamma\cos\xi}{(1-\beta^2\sin^2\xi)^{1/2}}\right]\rho + \gamma\beta c\tau \qquad y = \left[\frac{\sin\xi\sin\phi}{\gamma(1-\beta^2\sin^2\xi)^{1/2}}\right]\rho \tag{B.11b}$$

Basis vectors and unit vectors derived from this transformation are

$$\vec{\mathrm{e}}_\tau = \beta_\nu \qquad \beta^\nu = \begin{bmatrix} \gamma \\ 0 \\ 0 \\ \gamma\beta \end{bmatrix} \tag{B.12a}$$

$$\vec{\mathrm{e}}_\rho = \mathcal{U}_\nu \qquad \mathcal{U}^\nu = \frac{1}{\gamma(1-\beta^2\sin^2\xi)^{1/2}} \begin{bmatrix} \gamma^2\beta\cos\xi \\ \sin\xi\cos\phi \\ \sin\xi\sin\phi \\ \gamma^2\cos\xi \end{bmatrix} \tag{B.12b}$$

$$\vec{\mathrm{e}}_\xi = \frac{\rho}{\gamma(1-\beta^2\sin^2\xi)}\,\xi_\nu \qquad \xi^\nu = \frac{1}{(1-\beta^2\sin^2\xi)^{1/2}} \begin{bmatrix} -\beta\sin\xi \\ \cos\xi\cos\phi \\ \cos\xi\sin\phi \\ -\sin\xi \end{bmatrix} \tag{B.12c}$$

$$\vec{\mathrm{e}}_\phi = \frac{\rho\sin\xi}{\gamma(1-\beta^2\sin^2\xi)^{1/2}}\,\phi_\nu \qquad \phi^\nu = \begin{bmatrix} 0 \\ -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \tag{B.12d}$$

and the differential element of the interval is

$$ds^2 = c^2 d\tau^2 - d\rho^2 - \frac{\rho^2}{\gamma^2(1-\beta^2\sin^2\xi)}\left[\frac{d\xi^2}{1-\beta^2\sin^2\xi} + \sin^2\xi d\phi^2\right] \tag{B.13}$$

The metric tensor is still diagonal in the present position basis but not as clean as equation (1.16). In terms of the present position $\mathcal{R}$ it reads

$$\mathrm{g}_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\mathcal{R}^2/(1-\beta^2\sin^2\xi) & 0 \\ 0 & 0 & 0 & -\mathcal{R}^2\sin^2\xi \end{bmatrix} \tag{B.14}$$

Four-Volume and the Present Position: Using the present position scale factors the four-volume element is constructed as

$$d^4\mathcal{V}_p = \frac{\rho^2}{\gamma^2(1-\beta^2\sin^2\xi)^{3/2}} d\Omega_p\, d\rho\, cd\tau \tag{B.15}$$

The volume of the light cone is easy to calculate by separating out the angular integrations

$$\int_0^{c\tau}\left[\int_0^{c\tau'}\rho^2 d\rho\right]dc\tau'\cdot\left[\int\frac{d\Omega_p}{\gamma^2(1-\beta^2\sin^2\xi)^{3/2}}\right]=\frac{1}{3}\pi c^4\tau^4 \tag{B.16}$$

It is instructive to implement a change of variables $\rho=\rho(\mathcal{R})$ using equation (B.5) so that

$$d\rho = d\mathcal{R}\cdot\gamma(1-\beta^2\sin^2\xi)^{1/2} \tag{B.17}$$

This leads to a more easily recognizable volume element

$$d^4\mathcal{V}_p=\mathcal{R}^2 d\mathcal{R}d\Omega_p\, cdt \tag{B.18}$$

where $cdt=\gamma c d\tau$. The connection with the four-volume element of equation (4.1) can be immediately established by calculating the Jacobian determinant associated with the transformation of equation (B.2). One finds $J=\rho/\gamma R$ so that the radial integration can be replaced by

$$\mathcal{R}^2 d\mathcal{R}d\Omega_p \longrightarrow R^2 dR d\Omega(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}) \tag{B.19}$$

which re-derives (4.1). Now suppose the present position volume element is used to calculate the volume of the light cone. In this case the radial integral can still be performed but limits of integration will now depend on the polar angle. The rate of change of volume with proper time may be written

$$\frac{1}{c}\frac{d\mathcal{V}}{d\tau}=\gamma\int_\Omega\left[\int_0^{\mathcal{R}(c\tau,\xi)}\mathcal{R}^2 d\mathcal{R}\right]d\Omega_p \tag{B.20}$$

where the upper limit of integration is given by

$$\mathcal{R}(c\tau,\xi)=\frac{c\tau}{\gamma(1-\beta^2\sin^2\xi)^{1/2}} \tag{B.21}$$

The final result is (B.16).

Integral of the Null Current: A charged particle created at the origin of a coordinate system propagates a null delta current given by

$$J^\nu_{\scriptscriptstyle N}=-\frac{ec}{4\pi}\frac{R^\nu}{\rho^3}\delta(c\tau-\rho) \tag{B.22}$$

But the delta function is easily written in terms of present time coordinates

$$\delta(c\tau-\rho)\longrightarrow\delta[\gamma(ct-r)]=\frac{1}{\gamma}\delta(ct-r) \tag{B.23}$$

Use of the volume element in (B.18) generates the present position integral

$$\frac{1}{c}\frac{dQ^\nu}{d\tau} = -\frac{ec}{4\pi}\int \frac{1}{\rho^2}[\mathcal{U}^\nu + \beta^\nu]\cdot\delta(ct-r)\mathcal{R}^2 d\mathcal{R} d\Omega_p \tag{B.24}$$

To proceed further it will be necessary to write the delta function in terms of present posisition coordinates $\delta = \delta(ct, \mathcal{R}, \xi)$. Using equation (B.5) and the coordinate transformation in (B.11) it is easy to show that for motion in the z-direction

$$\delta(ct-r) = \delta\left[ct - \sqrt{\mathcal{R}^2 + 2\beta\mathcal{R}ct\cos\xi + \beta^2c^2t^2}\right] \tag{B.25}$$

The value of $\mathcal{R}$ which makes the argument zero is

$$\boxed{\mathcal{R}(ct,\xi) = ct[(1-\beta^2\sin^2\xi)^{1/2} - \beta\cos\xi]} \tag{B.26}$$

Now use the well known properties of delta functions to derive

$$\delta(ct-r) \longrightarrow \frac{1}{(1-\beta^2\sin^2\xi)^{1/2}}\cdot\delta\left\{\mathcal{R} - ct[(1-\beta^2\sin^2\xi)^{1/2} - \beta\cos\xi]\right\} \tag{B.27}$$

inserting this into the integral then leads to the simple formula

$$\frac{1}{c}\frac{dQ^\nu}{d\tau} = -\frac{ec}{4\pi}\beta^\nu\int\frac{d\Omega_p}{\gamma^2(1-\beta^2\sin^2\xi)^{1/2}} \tag{B.28}$$

A vector charge density now presents itself by multiplying and dividing this equation by the electron radius and using the surface element $d\mathbf{a} = \hat{\boldsymbol{s}} r_e^2 d\Omega_p$:

$$\frac{1}{c}\frac{dQ^\nu}{d\tau} = -c\beta^\nu\int\boldsymbol{\sigma_e}(\xi)\cdot d\mathbf{a} \tag{B.29}$$

Vacuum power in the Present Position Theory: Electric and magnetic energy flux vectors are particularly simple at the present position,

$$\mathbf{S}_E = \frac{ec}{2\mathcal{R}^2}\boldsymbol{\sigma_e}(\xi) \tag{B.30a}$$

$$\mathbf{S}_B = \frac{ec}{2\mathcal{R}^2}\boldsymbol{\beta}\times\boldsymbol{\sigma_e}(\xi) \tag{B.30b}$$

An equation for the associated surface can be determined by evaluating $\mathcal{R}(ct,\xi)$ in equation (B.26) at $ct = r_e$. Coordinates of the vector from the present position to the surface are then

$$\begin{aligned}\mathcal{R}_x &= \mathcal{R}(r_e,\xi)\sin\xi\cos\phi\\ \mathcal{R}_y &= \mathcal{R}(r_e,\xi)\sin\xi\sin\phi\\ \mathcal{R}_z &= \mathcal{R}(r_e,\xi)\cos\xi\end{aligned} \tag{B.31}$$

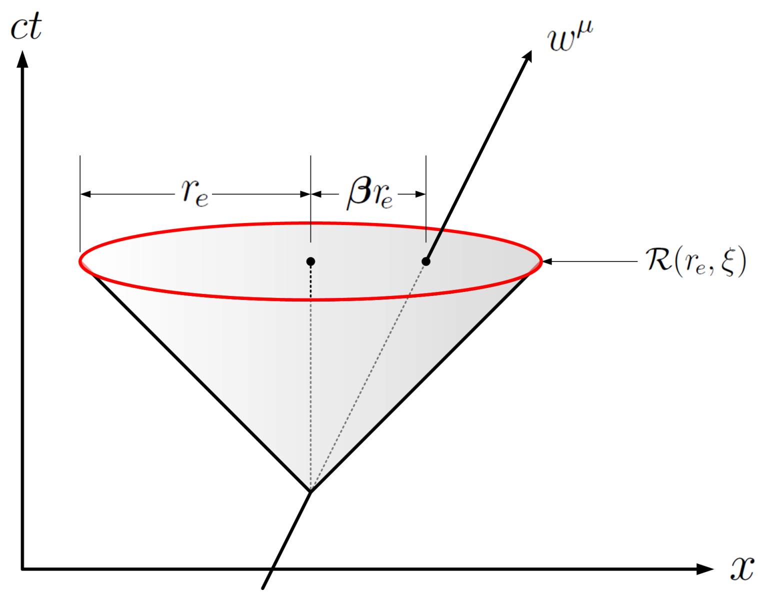

Figure 8: *Surface of the classical electron shown in red with particle present position shifted by* $\boldsymbol{\beta} r_e$ *from the center of the sphere.*

Two tangent vectors can be constructed from

$$\boldsymbol{T}_\xi = \left.\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \xi}\right|_{ct=r_e} \qquad \boldsymbol{T}_\phi = \left.\frac{\partial \boldsymbol{\mathcal{R}}}{\partial \phi}\right|_{ct=r_e} \tag{B.32}$$

so that

$$\boldsymbol{T}_\xi \times \boldsymbol{T}_\phi = \mathcal{R}^2(r_e, \xi) \sin\xi \left[\hat{\boldsymbol{s}} - \frac{\beta \sin\xi}{(1-\beta^2 \sin^2\xi)^{1/2}} \hat{\boldsymbol{\xi}}\right] \tag{B.33}$$

The area of the sphere follows from

$$S = \int \|\boldsymbol{T}_\xi \times \boldsymbol{T}_\phi\| d\xi\, d\phi = 4\pi r_e^2 \tag{B.34}$$

but the total charge is determined by integrating the vector charge density over the surface

$$e = \int \boldsymbol{\sigma}_{\mathbf{e}}(\xi) \cdot \boldsymbol{T}_\xi \times \boldsymbol{T}_\phi \, d\xi\, d\phi \tag{B.35}$$

Invariant vacuum power from both electric and magnetic energy flux will also be determined by similar surface integrals

$$P_E = \int \mathbf{S}_E \cdot \boldsymbol{T}_\xi \times \boldsymbol{T}_\phi \, d\xi\, d\phi = \frac{mc^2}{\tau_e} \tag{B.36}$$

$$P_B = \int \mathbf{S}_B \cdot \boldsymbol{T}_\xi \times \boldsymbol{T}_\phi \, d\xi\, d\phi = 0 \tag{B.37}$$