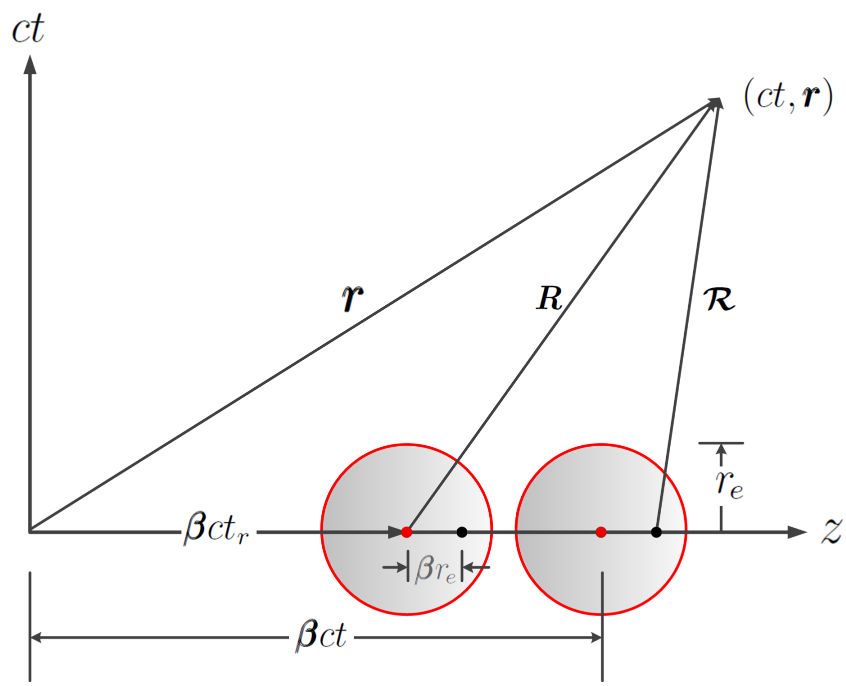
The New Radiation Reaction Force

March 23, 2017



"...This tremendous edifice which is such a beautiful success in explaining so many phenomena, ultimately falls on its face."

R.P. Feynman, *The Feynman Lectures*, chapter 28

Abstract

Physics in the spherically-based four-coordinate system is extended to address accelerated motions of the electron. Timelike and spacelike volume integrals of the total electromagnetic stress tensor are performed and the divergence theorem is applied. In addition, particle accelerations with moving velocity fields provide a new perspective by which the well known problem of the radiation reaction can be addressed. The violation of energy conservation forces out the Lorentz-Dirac equation and generates solutions for straight line and circular motions without the difficulties associated with the conventional theory.

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1 Spherical Four-Coordinates

A spherically based-coordinate system can be constructed in Minkowski space to address problems associated with the vacuum gauge electron. The usefulness of the coordinate system mainly derives from it ability to facilite complex integrations associated with the particle radius.

1.1 Coordinate Transformation:

A practical method for deriving the required transformation begins with the light cone condition

$$\begin{aligned} x^\nu &= R^\nu + w^\nu \\ &= \rho\beta^\nu + \rho\mathcal{U}^\nu + w^\nu \end{aligned} \tag{1.1}$$

Now expand w^ν along timelike and spacelike directions as

$$w^\nu = (w^\lambda\beta_\lambda)\beta^\nu - (w^\lambda\mathcal{U}_\lambda)\mathcal{U}^\nu \tag{1.2}$$

and insert above to give

$$x^\nu = (\rho + w^\lambda\beta_\lambda)\beta^\nu + (\rho - w^\lambda\mathcal{U}_\lambda)\mathcal{U}^\nu \tag{1.3}$$

In the proper frame of an electron, one finds

$$w^\lambda\beta_\lambda = c\tau - \rho \tag{1.4a}$$

$$w^\lambda\mathcal{U}_\lambda = 0 \tag{1.4b}$$

so that the appropriate coordinate transformation is

$$\boxed{x^\nu = c\tau\beta^\nu + \rho\mathcal{U}^\nu} \tag{1.5}$$

More specifically, this transformation is of the form

$$(ct, x, y, z) = f(c\tau, \rho, \theta, \phi) \tag{1.6}$$

where the angular dependence is determined from the spacelike vector $\mathcal{U}^\nu = \mathcal{U}^\nu(\theta, \phi)$ and both angles are measured from the retarded position of the particle. Of importance here is the fact that all of the curvi-linear coordinates are to be treated as independent variables[1].

[1] Rohrlich differentiates ρ with respect to proper time in chapter 4 based on the defining equation $\rho \equiv R^\nu\beta_\nu$. Then

$$\frac{d\rho}{d\tau} = -1 + \rho\, a_{\mathcal{U}} \tag{1.7}$$

In the spherically based system ρ is determined by a Lorentz transformation of the proper frame variable.

$$\frac{dR^\nu}{d\tau} = \frac{dx^\nu}{d\tau} - \frac{dw^\nu}{d\tau} = 0 \qquad\qquad \frac{d\rho}{d\tau} = \frac{dR^\nu}{d\tau}\beta_\nu = 0 \tag{1.8}$$

This is not completely understood.

Basis Vectors: Covariant and contravariant basis vectors determined from (1.5) and its inverse are given in Table 1.1 along with unit vectors and their functional forms. It is easy to verify a magnitude of +1 for the timelike unit vector and magnitudes of −1 for the three spacelike unit vectors. Six orthogonality relations can also be verified.

$\vec{\mathrm{e}}_\nu$	$\vec{\omega}^\nu$	Unit
$\vec{\mathrm{e}}_\tau = \dfrac{\partial x_\nu}{\partial c\tau} = \beta_\nu$	$\vec{\omega}^\tau = \partial^\nu c\tau = \beta^\nu$	$\beta^\nu = \begin{bmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{bmatrix}$
$\vec{\mathrm{e}}_\rho = \dfrac{\partial x_\nu}{\partial \rho} = \mathcal{U}_\nu$	$\vec{\omega}^\rho = \partial^\nu \rho = -\mathcal{U}^\nu$	$\mathcal{U}^\nu = \dfrac{1}{\gamma(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})} \begin{bmatrix} 1 \\ \hat{\boldsymbol{n}} \end{bmatrix} - \begin{bmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{bmatrix}$
$\vec{\mathrm{e}}_\theta = \dfrac{\partial x_\nu}{\partial \theta} = R\,\theta_\nu$	$\vec{\omega}^\theta = \partial^\nu \theta = -\dfrac{1}{R}\theta^\nu$	$\theta^\nu = \dfrac{1}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})} \begin{bmatrix} \boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\theta}} + \boldsymbol{\beta}\times\hat{\boldsymbol{\phi}} \end{bmatrix}$
$\vec{\mathrm{e}}_\phi = \dfrac{\partial x_\nu}{\partial \phi} = R\sin\theta\,\phi_\nu$	$\vec{\omega}^\phi = \partial^\nu \phi = -\dfrac{1}{R\sin\theta}\phi^\nu$	$\phi^\nu = \dfrac{1}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})} \begin{bmatrix} \boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\phi}} - \boldsymbol{\beta}\times\hat{\boldsymbol{\theta}} \end{bmatrix}$

Table 1: *Covariant and contravariant basis vectors in the spherically-based spacetime coordinate system. All angles are defined relative to the retarded position of the particle.*

For convenience a graphic representation showing the operation of the spherical coordinate system is included in the spacetime diagram of figure 1. The position of an electron in the proper frame is always on the time axis as it moves along its world line. The position vector w^ν grows in length but the angle of the proper frame relative to moving frame will change as the particle accelerates.

Derivatives of the basis vectors define ten independent Christoffel symbols thru

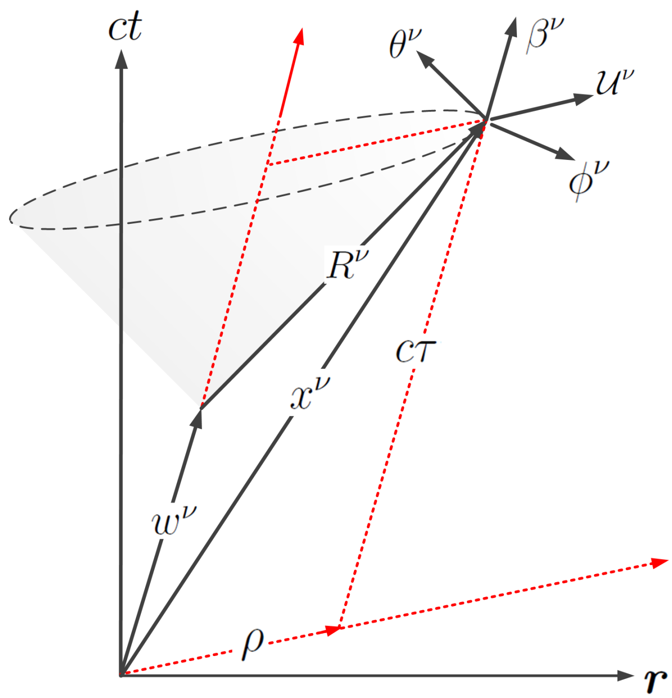

Figure 1: *Orthonormal set of unit four-vectors at the point* $(ct, \boldsymbol{r})$ *defined relative to the retarded position of an electron moving along its world line.*

the relations

$$\frac{\partial\,\vec{\mathrm{e}}_\mu}{\partial q^\nu} = \Gamma^\lambda_{\mu\nu}\,\vec{\mathrm{e}}_\lambda \qquad\qquad \frac{\partial\,\vec{\omega}^{\,\mu}}{\partial q^\nu} = -\Gamma^\mu_{\lambda\nu}\,\vec{\omega}^{\,\lambda} \tag{1.9}$$

Non-zero Christoffel symbols are available in the appendix which can be used for a wide range of calculations.

1.2 Tensors and Differential Operations

In general, a four-vector $\vec{V}$ in the spherical basis can be written in terms of either set of dual basis vectors:

$$\vec{V} = V^\nu \vec{\mathrm{e}}_\nu \qquad\qquad \vec{V} = V_\nu\,\vec{\omega}^{\,\nu} \tag{1.10}$$

However, like the well known 3D curvi-linear coordinate systems, it is frequently easier to write vectors in terms of the corresponding unit four-vectors. In particular, the components V^ν of an arbitrary vector generally have an immediate expansion

$$V^\nu = (V^\alpha\beta_\alpha)\beta^\nu - (V^\alpha\mathcal{U}_\alpha)\mathcal{U}^\nu - (V^\alpha\theta_\alpha)\theta^\nu - (V^\alpha\phi_\alpha)\phi^\nu \tag{1.11}$$

The four-gradient operator in the spherical basis applied to an arbitrary scalar field ψ also determines a vector which can be written

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial c\tau}\beta^{\nu} - \frac{\partial\psi}{\partial\rho}\mathcal{U}^{\nu} - \frac{1}{R}\frac{\partial\psi}{\partial\theta}\theta^{\nu} - \frac{1}{R\sin\theta}\frac{\partial\psi}{\partial\phi}\phi^{\nu} \tag{1.12}$$

Although proper use of this operator can be deceptive, it is worthwhile to apply it most simply to the retarded time

$$\vec{\nabla}ct_r = \vec{\nabla}\gamma(c\tau - \rho) = \frac{\gamma R^{\nu}}{\rho} \tag{1.13}$$

One way to generate second rank tensors is through the covariant derivative of a vector field. Two different forms of the covariant derivative may be written

$$\vec{\nabla}\vec{V} = V_{\nu\,;\,\mu}\,\vec{\omega}^{\mu}\cdot\vec{\omega}^{\nu} = \left[V_{\nu,\,\mu} - V_{\lambda}\Gamma^{\lambda}_{\nu\mu}\right]\vec{\omega}^{\mu\nu} \tag{1.14a}$$

$$\vec{\nabla}\vec{V} = V^{\nu}_{\ ;\,\mu}\,\vec{\omega}^{\mu}\cdot\vec{e}_{\nu} = \left[V^{\nu}_{\ ,\,\mu} - V^{\lambda}\Gamma^{\nu}_{\lambda\mu}\right]\vec{\omega}^{\mu}\cdot\vec{e}_{\nu} \tag{1.14b}$$

Application of this equation and the four-gradient operator are best demonstrated by a number of examples.

Covariant Derivative of the Coordinate Transformation: The coordinate transformation in (1.5) can be written formally as

$$\vec{x} = c\tau\,\vec{\omega}^{\tau} + \rho\,\vec{\omega}^{\rho} \longrightarrow (c\tau,\,-\rho,\,0,\,0) \tag{1.15}$$

The covariant derivative is the metric tensor defined by $\widehat{\mathrm{g}} \equiv \vec{\nabla}\vec{x}$. Only the diagonal components of this tensor are non-zero while the two terms associated with angles θ and ϕ are derived from Christoffel symbols

$$x_{\theta\,;\,\theta} = -x_{\rho}\,\Gamma^{\rho}_{\theta\theta} \tag{1.16a}$$

$$x_{\phi\,;\,\phi} = -x_{\rho}\Gamma^{\rho}_{\phi\phi} \tag{1.16b}$$

The final result is simply

$$\widehat{\mathrm{g}} = \vec{\omega}^{\tau\tau} - \vec{\omega}^{\rho\rho} - R^2\,\vec{\omega}^{\theta\theta} - R^2\sin^2\theta\,\vec{\omega}^{\phi\phi} \tag{1.17}$$

where $R = R(\rho,\theta,\phi)$. A matrix representation of this tensor is

$$\mathrm{g}_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2\sin^2\theta \end{bmatrix} \tag{1.18}$$

It is not difficult to show that the spacetime described by this metric is flat implying that all components of the Riemann curvature tensor are zero. However, curvature is not required to show that differentials of all metric components satisfy

$$\mathrm{g}_{\mu\nu;\,\alpha} = 0 \tag{1.19}$$

Covariant Derivative of the Four-Velocity: While β^ν is a unit coordinate vector in the spherically-based coordinate system, it also has a covariant derivative which can be addressed using the chain rule. Generally speaking β^ν is a function of the retarded time, but in the spherically-based system it must be written $\beta^\nu = \beta^\nu(c\tau - \rho)$ so that a proper application of the chain rule is

$$\vec{\nabla}\vec{\beta} = \frac{\partial\vec{\beta}}{\partial c\tau}\,\vec{\nabla}c\tau + \frac{\partial\vec{\beta}}{\partial\rho}\,\vec{\nabla}\rho \tag{1.20}$$

Now

$$\vec{a} = \frac{\partial\vec{\beta}}{\partial c\tau} = -\frac{\partial\vec{\beta}}{\partial\rho} \tag{1.21}$$

so that

$$\vec{\nabla}\vec{\beta} = \vec{a}(\vec{e}_\tau + \vec{e}_\rho) \tag{1.22}$$

In the spherically-based system a_ν has an expansion in terms of spacelike unit vectors so that the covariant derivative may be written

$$\beta_{\nu\,;\,\mu} = -(\beta_\mu + \mathcal{U}_\mu)\cdot(a_u\mathcal{U}_\nu + a_\theta\theta_\nu + a_\phi\phi_\nu) \tag{1.23}$$

In terms of a matrix of numbers along unit vectors, the final result may also be written

$$\beta_{\nu\,;\,\mu} = -\begin{bmatrix} 0 & a_u & a_\theta & a_\phi \\ 0 & a_u & a_\theta & a_\phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.24}$$

Gradient Operation on ρ: Blindly choosing equation (1.12) to calculate the quantity $\vec{\nabla}\rho$ will produce the vector $\mathcal{U}^\nu$. Unfortunately, this will not yield any extra terms associated with particle accelerations. For this define the order pairs

$$\vec{R} = R^\nu\vec{e}_\nu \longrightarrow (\rho,\rho,0,0) \tag{1.25}$$

$$\vec{\beta} = \beta_\nu\,\vec{\omega}^\nu \longrightarrow (1,0,0,0) \tag{1.26}$$

The appropriate gradient operation is then $\vec{\nabla}(\vec{R}\cdot\vec{\beta})$. In terms of sums over indices this may be written

$$\rho_{\,;\,\nu} = R^\lambda_{\;;\,\nu}\cdot\beta_\lambda + R^\lambda\cdot\beta_{\lambda\,;\,\nu} \tag{1.27}$$

The first term here requires a calculation of the covariant derivative of R^λ:

$$R^\lambda_{\;;\,\nu} = -\mathcal{U}^\lambda\mathcal{U}_\nu - \theta^\lambda\theta_\nu - \phi^\lambda\phi_\nu - \beta^\lambda\mathcal{U}_\nu \tag{1.28}$$

Coupling this with the result of the previous example gives the correct result

$$\rho_{\,;\,\nu} = -\mathcal{U}_\nu + a_uR_\nu \tag{1.29}$$

2 Integration

2.1 Volume and Surface Elements:

Scale factors for the spherical system can be read directly from (1.18):

$$h_\tau = 1 \qquad h_\rho = 1 \qquad h_\theta = R \qquad h_\phi = R\sin\theta \tag{2.1}$$

Defining the distorted solid angle element by

$$d\Omega' \equiv \frac{d\Omega}{\gamma^2(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})^2} \tag{2.2}$$

The scale factors may be collected together to construct the four-volume element.

$$d^4\mathcal{V} = \rho^2 d\rho\, d\Omega'\, cd\tau \tag{2.3}$$

Spacelike and timelike 3-surface elements having the same Doppler factor as $d^4\mathcal{V}$ are also determined using $d\Omega'$ and appropriate combinations of scale factors:

$$d\sigma_s^\nu = [\rho^2 d\Omega' d\rho]\,\beta^\nu \tag{2.4a}$$
$$d\sigma_\tau^\nu = [\rho^2 d\Omega'\, cd\tau]\mathcal{U}^\nu \tag{2.4b}$$

The schematic in Figure 2 shows clearly the timelike surface characterized by the

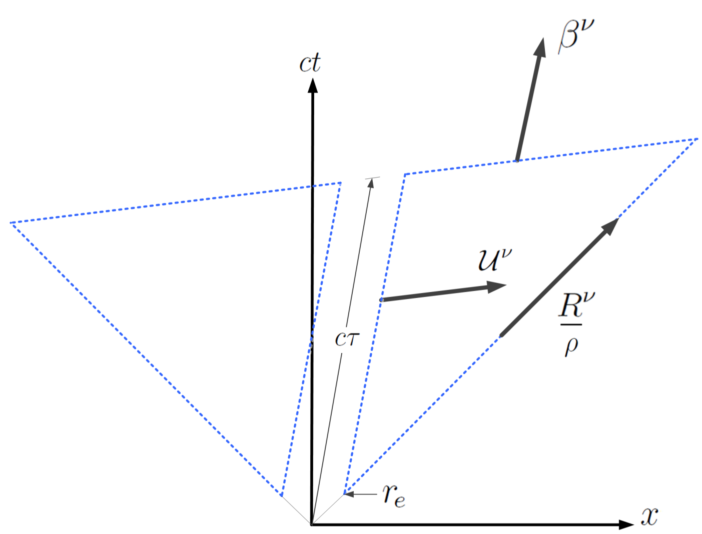

Figure 2: *Cross section of the causal light cone for the dilatated vacuum showing timelike, spacelike, and lightlike surfaces.*

vector $\mathcal{U}^\nu$ which points into the vacuum from the boundary along with the now plane surface characterized by the vector β^ν. limits on both surfaces are governed by the expansion of the light sphere.

The two hypersurfaces in (2.4) can also be combined to form the light cone 3-surface element by equating the differentials $d\rho = cd\tau$. Using the proper time element $cd\tau$—with lack of preference for either—the light cone surface element with an necessary minus sign is

$$d\sigma_{l}^{\nu} = -\frac{1}{\rho}[\rho^2 d\Omega'\, cd\tau]\, R^{\nu} \tag{2.5}$$

2.2 Volume Integrals of the Total Stress Tensor

Spacelike Integration: Four-volume integrals of $\Theta^{\mu\nu}$ and $\Lambda^{\mu\nu}$ can be performed in the spherically based system to determine a total energy tensor. For the symmetric stress tensor radial integrals extend from the radius r_e to a large time $c\tau + r_e$ which can be approximated by infinity.

$$d\mathcal{S}^{\mu\nu}_{em} = \iint \Theta^{\mu\nu} d^4\mathcal{V} = \left[mc^2\beta^\mu\beta^\nu + \tfrac{1}{3}mc^2(\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu})\right]\cdot cd\tau \tag{2.6}$$

A similar procedure for the vacuum tensor follows by first observing that only the symmetric terms make contributions to the integral. Moreover, the lower limit to the radial integration must be extended to zero so that

$$\begin{aligned} d\mathcal{S}^{\mu\nu}_{vac} = \iint \Lambda^{\mu\nu} d^4\mathcal{V} &= \left[-mc^2\beta^\mu\beta^\nu - \tfrac{1}{2}\dot{\rho}c\tau\beta^\mu\beta^\nu\right]\cdot cd\tau \\ &\quad + \left[\tfrac{2}{3}mc^2(\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu}) + \tfrac{1}{3}\dot{\rho}c\tau(\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu})\right]\cdot cd\tau \end{aligned} \tag{2.7}$$

Summing individual terms with the inclusion of a necessary integration constant leads to

$$d\mathcal{S}^{\mu\nu} = d\mathcal{S}^{\mu\nu}_{em} + d\mathcal{S}^{\mu\nu}_{vac} + \mathrm{g}^{\mu\nu}mc^2\cdot cd\tau \tag{2.8}$$

and a second rank total energy tensor follows as

$$\frac{1}{c}\frac{d\mathcal{S}^{\mu\nu}}{d\tau} = mc^2\beta^\mu\beta^\nu - \tfrac{1}{2}\dot{\rho}c\tau\beta^\mu\beta^\nu + \tfrac{1}{3}\dot{\rho}c\tau\,(\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu}) \tag{2.9a}$$

$$= \mathcal{E}^{\mu\nu}_{part} + \mathcal{E}^{\mu\nu}_{vac} + \mathcal{P}^{\mu\nu}_{vac}c \tag{2.9b}$$

Forming scalar contractions of the vacuum terms then derives the simple scalar equation

$$\boxed{\mathcal{E} = \tfrac{1}{2}\mathcal{P}c} \tag{2.10}$$

representing the fundamental relation between the energy and momentum of the vacuum.

Timelike Integration: Another important use of the volume element $d^4\mathcal{V}$ is for the construction of a timelike integration. In this case the integration over $d\rho$ and $cd\tau$ is initially witheld. For constant velocities the flux integral may be written

$$\left.\frac{d^2\mathcal{S}^{\mu\nu}_{[v]}}{d\rho}\right|_{\rho=r_e} = \int \mathcal{T}^{\mu\nu} r_e^2 d\Omega' \cdot cd\tau = \tfrac{1}{2}\dot{\varrho}c\left[\beta^\mu\beta^\nu - \mathrm{g}^{\mu\nu}\right]\cdot d\tau \tag{2.11}$$

Once again the metric tensor can be removed with the addition of an appropriate integration constant and this determines the quantity

$$\left.\frac{d^2\mathcal{S}^{\mu\nu}_{[v]}}{d\rho}\right|_{\rho=r_e} = \tfrac{1}{2}\dot{\varrho}c\,\beta^\mu\beta^\nu\cdot d\tau \tag{2.12}$$

An extension to include particle accelerations follow by writing

$$\left.\frac{d^2\mathcal{S}^{\mu\nu}_{[v]}}{d\rho}\right|_{\rho=r_e} = \int \left[\mathcal{E}^{\mu\nu}_{rad} + \mathit{\Theta}^{\mu\nu}_2 + \mathit{\Theta}^{\mu\nu}_3\right] r_e^2 d\Omega'\, cd\tau \tag{2.13}$$

In terms of vacuum gauge potentials solid angle integrals of the additional terms may be written

$$\int_{\rho=r_e} \mathit{\Theta}^{\mu\nu}_2 \rho^2 d\Omega' = \frac{1}{4\pi}\int_{\rho=r_e} \eta\left[A^\mu a^\nu_\perp + A^\nu a^\mu_\perp\right]\rho^2 d\Omega' \tag{2.14}$$

$$\int_{\rho=r_e} \mathit{\Theta}^{\mu\nu}_3 \rho^2 d\Omega' = \frac{1}{4\pi}\int_{\rho=r_e} a^\lambda_\perp a^\perp_\lambda A^\mu A^\nu \rho^2 d\Omega' \tag{2.15}$$

These integrals are complicated but may be addressed with the aid of angular integrals already performed in Appendix B leading to

$$\left.\frac{d^2\mathcal{S}^{\mu\nu}_{[a1]}}{d\rho}\right|_{r_e} = \frac{4}{3}mc\left[a^\mu\beta^\nu + a^\nu\beta^\mu\right]d\tau \tag{2.16a}$$

$$\left.\frac{d^2\mathcal{S}^{\mu\nu}_{[a2]}}{d\rho}\right|_{r_e} = \left[-\frac{2}{3}\frac{e^2}{c^3}a^\lambda a_\lambda\beta^\mu\beta^\nu + \frac{4}{15}\frac{e^2}{c^3}a^\lambda a_\lambda\left(\mathrm{g}^{\mu\nu} - \beta^\mu\beta^\nu\right) - \frac{2}{15}\frac{e^2}{c^3}a^\mu a^\nu\right]d\tau \tag{2.16b}$$

An easier approach towards arriving at these results begins with simple rest frame integrations followed by a Lorentz transformation. Regardless of the approach, the most general covariant symmetric total energy tensor may be written

$$\left.\frac{d^2\mathcal{S}^{\mu\nu}}{d\rho}\right|_{r_e} = \frac{d^2\mathcal{S}^{\mu\nu}_{[v]}}{d\rho} + \frac{d^2\mathcal{S}^{\mu\nu}_{[a1]}}{d\rho} + \frac{d^2\mathcal{S}^{\mu\nu}_{[a2]}}{d\rho} \tag{2.17}$$

The simplest operation on this tensor is to calculate its trace. Since both acceleration tensors are traceless this implies the scalar integral

$$\left.\frac{d\mathcal{S}}{d\rho}\right|_{r_e} = \int_{\tau_1}^{\tau_2} \frac{1}{2}\dot{\varrho} c d\tau = \mathcal{E}_{vac} \tag{2.18}$$

Also of interest is a contraction of one index with the four velocity

$$\left.\frac{d\mathcal{S}^\nu}{d\rho}\right|_{r_e} = \int_{\tau_1}^{\tau_2} \left[\frac{1}{2}\dot{\varrho} c\beta^\nu + \frac{4}{3}mca^\nu - \frac{2}{3}\frac{e^2}{c^3}a^\lambda a_\lambda \beta^\nu\right] d\tau \tag{2.19}$$

The middle term here reflects changes in the energy-momentum four-vector of the particle during accelerations and may be written

$$\int_{\tau_1}^{\tau_2} \frac{4}{3}mca^\nu d\tau = \frac{4}{3}\Delta p^\nu \tag{2.20}$$

Why this term includes a factor of 4/3 is completely unkown. The remaining two terms of the integrand constitute the total radiation rate four-vector of the particle defined by

$$P^\nu = \left[\frac{1}{2}\dot{\varrho} c - \frac{2}{3}\frac{e^2}{c^3}a^\lambda a_\lambda\right]\beta^\nu = (P_{vel} + P_{acc})\beta^\nu \tag{2.21}$$

The term P_{vel} represents radiated vacuum power while the term P_{acc} is Liénards generalization of the Larmor power formula. A second contraction with the four-velocity will then determine what appears to be a small increase in the inertia of the particle during accelerated motion.

$$\boxed{m^* = m - \frac{1}{3}\frac{e^4}{mc^8}a^\lambda a_\lambda} \tag{2.22}$$

According to this formula the mass of the electron is a relativistic invariant—even during accelerations. Although the mass appears to decrease, the additional term is actually a positive definite quantity.

The departure Δm from the inertial value can be readily evaluated for specific accelerated motions. The easiest way to do this is to write the inertia as

$$m^*c^2 = m_e c^2\left[1 + \frac{P_{acc}}{P_{in}}\right] \tag{2.23}$$

and then calculate radiative outputs. Since P_{in} is constant the largest deviations will occur when the transverse output is maximized. Bremstrahlung is a possible candidate except that the deviation only exists over collision times δt which are small. A better indicator might be the rough estimates given in the last column of Table 2 for three well known facilities. The SLAC result was estimated near the end of the linac. It is not known whether Δm could be detected by a local experiment.

Facility	Length Parameter	Energy	P_{acc}	P_{in}	Δm
SLAC	L = 2 mi.	50 GeV	976 W	1.74E10 W	5.11E-38 kg
LEP	Circ. = 27 km	91 GeV	1.55E-7 W	1.74E10 W	8.12E-48 kg
CESR	Circ. = 768 m	10 GeV	4.54E-7 W	1.74E10 W	2.38E-47 kg

Table 2: Estimates of Δm in particle accelerators.

2.3 Applications of the Divergence Theorem

In the vacuum gauge, implicit derivatives of the velocity theory do not extend to particle accelerations which are under the control of a separate theory. This requires the divergence of the total stress tensor $\mathcal{T}^{\mu\nu}$ to contain an acceleration current not known to the conventional theory

$$\partial_\mu \mathcal{T}^{\mu\nu} = -\frac{1}{c}\eta J_a^\nu \tag{2.24}$$

With the extra current density, the divergence theorem can be applied to individual parts of the total stress tensor written

$$\mathcal{T}^{\mu\nu} = \Theta_1^{\mu\nu} + \Theta_2^{\mu\nu} + \Theta_3^{\mu\nu} + \Lambda^{\mu\nu} \tag{2.25}$$

While the first and last terms on the right belong to the velocity theory, an analysis of the middle terms may be addressed by first applying the divergence theorem to the acceleration strain tensor:

Divergence Theorem applied to $\epsilon^{\mu\nu}$: Vacuum gauge electrodynamics requires an acceleration strain tensor given by

$$\epsilon^{\mu\nu} = A^\mu a_\perp^\nu \tag{2.26}$$

Acceleration strain falls off like $1/\rho$ so that values may be considered inside the electron radius. In other words, it is not neccessary to consider integrations over the timelike surface element $d\sigma_\mu^\tau$. In addition, if only contractions over the first index are considered, then no integrations are required over the lightlike surface either. The divergence theorem is therefore

$$\int \partial_\mu \epsilon^{\mu\nu} d^4\mathcal{V} = \int \epsilon^{\mu\nu} d\sigma_\mu^s \tag{2.27}$$

with integration limits for ρ and $c\tau$ given by

$$0 \leq \rho \text{ or } c\tau \leq c\tau_o \tag{2.28}$$

The divergence of the acceleration strain is simply $\eta a_\perp^\nu$. Uding this and an integration over solid angle given by

$$\int_\Omega a^\lambda \mathcal{U}_\lambda \mathcal{U}^\nu d\Omega' = -\frac{4\pi}{3} a^\nu \tag{2.29}$$

leads to the simple expression

$$\int_0^{c\tau_o}\left[\int_0^{c\tau'} a^\nu(c\tau'-\rho)d\rho\right]dc\tau' = \int_0^{c\tau_o} a^\nu(c\tau'-\rho)\rho d\rho \tag{2.30}$$

which can be easily verified.

It is also important to discuss the divergence theorem applied to the conjugate strain tensor $(\epsilon^{\mu\nu})^\dagger$ —equivalent to a contraction on the second index. In this case all surface integrals vanish and the volume integral reduces to

$$\int \partial_\nu \epsilon^{\mu\nu} d^4\mathcal{V} = 0 \tag{2.31}$$

This equation can also be verified using (2.29) in addition to the integral

$$\int_\Omega a^\lambda \mathcal{U}_\lambda \, d\Omega' = 0 \tag{2.32}$$

Apparently the conjugate acceleration strain tensor does not play an important role in vacuum gauge electrodynamics.

Divergence Theorem applied to $\Theta_2^{\mu\nu}$: This problem—although similar to the previous divergence calculation of acceleration strain—is much more challenging since the radius of the electron cannot be ignored. First write

$$\Theta_2^{\mu\nu} = \frac{1}{4\pi}\eta(\epsilon^{\mu\nu}+\epsilon^{\nu\mu}) \tag{2.33}$$

The divergence is then

$$\partial_\mu\Theta_2^{\mu\nu} = \frac{1}{4\pi}\left(\partial_\mu\eta\cdot\epsilon^{\mu\nu} + \eta\cdot\partial_\mu\epsilon^{\nu\mu} + \partial_\mu\eta\cdot\epsilon^{\nu\mu} + \eta\cdot\partial_\mu\epsilon^{\nu\mu}\right) = -\eta J_a^\nu \tag{2.34}$$

In this case ρ and $c\tau$ must have limits

$$r_e \le \rho \text{ or } c\tau \le c\tau_o + r_e \tag{2.35}$$

Integrations over solid angle on both sides of the equation, and noting that the light-like surface integral is zero, yields the result

$$-\int_{r_e}^{c\tau_o+r_e}\left[\int_{r_e}^{c\tau'}\frac{1}{\rho^2}a^\nu(c\tau'-\rho)d\rho\right]cd\tau'$$

$$= \int_{r_e}^{c\tau_o+r_e}\frac{1}{\rho}a^\nu(c\tau_o+r_e-\rho)d\rho - \frac{1}{r_e}\int_{r_e}^{c\tau_o+r_e} a^\nu(c\tau-r_e)cd\tau \tag{2.36}$$

The left side can be shown to be equal to the right side by changing the order of integration and then integrating by parts.

Divergence theorem applied to $\Theta_3^{\mu\nu}$: Whether defined by gauge invariance or by vacuum gauge potentials, the stress $\Theta_3^{\mu\nu}$ is associated with the production of transverse electromagnetic waves. In the vacuum gauge one writes

$$\Theta_3^{\mu\nu} = -\frac{1}{4\pi}\epsilon^{\mu\lambda}\epsilon^{\nu}{}_{\lambda} \tag{2.37}$$

so that the divergence is

$$\partial_\mu \Theta_3^{\mu\nu} = -\frac{1}{4\pi}\left(\partial_\mu \epsilon^{\mu\lambda}\cdot\epsilon^{\nu}{}_{\lambda} + \epsilon^{\mu\lambda}\cdot\partial_\mu\epsilon^{\nu}{}_{\lambda}\right) = 0 \tag{2.38}$$

Once again $\Theta_3^{\mu\nu}$ has no component along R^ν which means that the timelike and spacelike surfaces integrals must be equal to within a sign. The divergence theorem therefore reduces to

$$\int \Theta_3^{\mu\nu} d\sigma_\mu^s + \int \Theta_3^{\mu\nu} d\sigma_\mu^\tau = 0 \tag{2.39}$$

The timelike surface integral may be written

$$d\mathcal{E}^\nu_{acc} = \frac{e^2}{4\pi}\int_\Omega [a^\lambda a_\lambda + (a^\lambda \mathcal{U}_\lambda)^2][\mathcal{U}^\nu + \beta^\nu] d\Omega' c d\tau \tag{2.40}$$

but angular integrations involving $\mathcal{U}^\nu$ are zero. Moreover it can also be shown that

$$\int (a^\lambda \mathcal{U}_\lambda)^2 d\Omega' = -\frac{4\pi}{3} a^\lambda a_\lambda \tag{2.41}$$

The divergence theorem will then reduce to the equality

$$\int_{r_e}^{c\tau_o + r_e} a^\lambda(c\tau_o + r_e - \rho) a_\lambda(c\tau_o + r_e - \rho)\beta^\nu(c\tau_o + r_e - \rho) d\rho$$

$$= \int_{r_e}^{c\tau_o + r_e} a^\lambda(c\tau - r_e) a_\lambda(c\tau - r_e)\beta^\nu(c\tau - r_e) c d\tau \tag{2.42}$$

and this may be verified with a simple change of variables. On the other hand, the four-vector radiation rate for the electron may be determined by witholding the time integration in equation (2.40). Including factors of c and including an overall sign this quantity may be written

$$P^\nu_{acc} = -\frac{d\mathcal{E}^\nu_{acc}}{d\tau} = -\frac{2}{3}\frac{e^2}{c^3} a^\lambda a_\lambda \beta^\nu \tag{2.43}$$

Timelike Integration of the Stress Tensor: Using the results already obtained in the section, it is instructive to perform a timelike integral involving components of the total stress tensor. It seems prudent to leave $\Theta_2^{\mu\nu}$ out of the integral which has already been shown to satisfy its own divergence theorem. This leaves

$$\Delta\mathcal{E}^{\nu} = -\int(\Theta_1^{\mu\nu} + \Theta_3^{\mu\nu} + \Lambda^{\mu\nu})\mathcal{U}_{\nu}\rho^2 d\Omega' c d\tau \tag{2.44}$$

There is no contribution to this integral from $\Theta_1^{\mu\nu}$ leaving only radiation terms which may be written

$$\Delta\mathcal{E}^{\nu} = \int_{\tau_1}^{\tau_2}\left[\frac{1}{2}\dot{\varrho}c\beta^{\nu} - \frac{2}{3}\frac{e^2}{c^3}a^{\lambda}a_{\lambda}\beta^{\nu}\right] c d\tau \tag{2.45}$$

One can then insert the change in the energy-momentum four vector of the particle and write

$$\Delta\mathcal{E}'^{\nu} = \int_{\tau_1}^{\tau_2}(P_{in} + P_{acc})\beta^{\nu} c d\tau + c\Delta p^{\nu} \tag{2.46}$$

3 Theory of the Radiation Reaction

The theory of the radiation reaction addresses the reaction of a particle to its own fields. like all other theories in physics, its design is rooted in conservation of energy. On the other hand, causality applied to the velocity fields of the electron generates vacuum gauge theory which precludes conservation of energy from the outset. Consequently, the vacuum gauge electron will require an extensive re-assessment of the self-force problem. This will not be an injustice to the theory however which is plagued with unsuccessful attempts to design a model for a charged particle using traditional classical concepts.

Replacement for the Lorentz-Dirac Equation: The equation describing the motion of a charged particle based on energy conservation is the Lorentz-Dirac equation. It is specifically designed to include the reactive effects of radiation and can be written

$$ma^{\mu} = F^{\mu}_{ext} + eF^{\mu\nu}_{em}\beta_{\nu} + \Gamma^{\mu} \tag{3.1}$$

where the Abraham four-vector is given by

$$\Gamma^{\mu} = \frac{2}{3}\frac{e^2}{c^3}\left[\dot{a}^{\mu} + \frac{1}{c}a^{\lambda}a_{\lambda}\beta^{\mu}\right] \tag{3.2}$$

Unfortunately, there are difficulties in the application of this equation. For example, the term containing $\dot{a}^{\mu}$ is problematic since the resulting equation is third order in time derivatives; this is the source of acausal motions of the particle. Another serious problem is the vanishing of the Abraham four-vector for a particle in hyperbolic motion. The implication here is that a particle under the action of a constant force neither radiates nor experiences a radiation reaction.

Based on this short review the immediate goal will be to re-fashion equation (3.1) for the radiation-based theory to eliminate the inherent difficulties and generate reasonable solutions for all possible motions of the electron. One modification already apparent from section 2 will be the need to replace m with m^*. We also propose the elimination of the Abraham four-vector altogether in favor of a radiation reaction four-force:

$$F^{\mu}_{rad} = [\gamma\boldsymbol{\beta}\cdot\boldsymbol{F}_{rad},\, \gamma\boldsymbol{F}_{rad}] \tag{3.3}$$

where the three-force $\boldsymbol{F}_{rad}$ is determined by examination of the distribution of radiation about the charge.

In the rest frame, the flow of the vacuum field about the center of radiation is symmetric and cannot exert a force. Since this is true in the rest frame it must also be true relative to any other moving frame. In contrast, the acceleration fields—which radiate preferentially in the direction of motion of the particle—will exert a force on the particle. In this case the momentum flux about the source can be derived from the Poynting vector and is proportional to the square of the acceleration field vector

$$\boldsymbol{\pi}_{acc} = \frac{1}{4\pi}|\mathbf{E}_{acc}|^2\,\hat{\boldsymbol{n}} \tag{3.4}$$

The net force exerted on the charge by the emission of transverse radiation can be calculated from

$$\boldsymbol{F}_{rad} = -\int_{R=r_e} \boldsymbol{\pi}_{acc}(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})R^2 d\Omega \tag{3.5}$$

This equation is closely related to Liénards generalization of the Larmor power formula which derives by integrating the angular power distribution

$$\frac{dP_{acc}}{d\Omega} = c\,\boldsymbol{\pi}_{acc}\cdot\hat{\boldsymbol{n}}R^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}) \tag{3.6}$$

over solid angle. A practical replacement for the reaction force $\boldsymbol{F}_{rad}$ is then

$$\boldsymbol{F}_{rad} = -\frac{1}{c}\int_{\Omega}\frac{dP_{acc}}{d\Omega}\,\hat{\boldsymbol{n}}\,d\Omega = -\frac{1}{c}P_{acc}\boldsymbol{\beta} \tag{3.7}$$

Like the power formula itself, the proof of the right side of this equation for general motions of the particle is lengthy. Once proved however a covariant formulation of the reaction force from (3.3) is

$$F^{\mu}_{rad} = -\frac{1}{c}P_{acc}\left(\gamma\beta^2,\, \gamma\boldsymbol{\beta}\right) \tag{3.8}$$

and a revised version of the Lorentz-Dirac formula is

$$\boxed{m^* a^{\mu} = F^{\mu}_{ext} + eF^{\mu\nu}_{em}\beta_{\nu} + F^{\mu}_{rad}} \tag{3.9}$$

This equation includes two separate reactions on the electron brought about by the emission of transverse radiation, and therefore only occuring when the particle accelerates: A scalar ***inertial reaction*** calculated from the total field energy within the particle radius causes a slight increase in the mass of the particle; and a vector ***self-force reaction*** resulting from the asymmetry of transverse energy flux crossing the particle radius provides a damping force opposite the direction of motion. These two quantities may be referred to collectively as the ***radiation reaction***. For the vacuum gauge electron, they both represent violations of energy conservation which can be detailed by considering specific examples.

3.1 Applications to Straight Line Motion

For straight line motion the first two terms on the right side of (3.9) have the same form and can be grouped together to represent a general external force. The entire equation can also be reduced to having a single time and space component—each of which generates the single equation

$$\gamma^3 m^* a = F_{ext} - \frac{1}{c} P_{acc} \beta \tag{3.10}$$

The force F_{ext} is generally a function of both x and t. The angular distribution of transverse power radiated by the particle is indicated in figure 3 and is a special case of equation (3.6) when the velocity and acceleration are co-linear. The simple formula

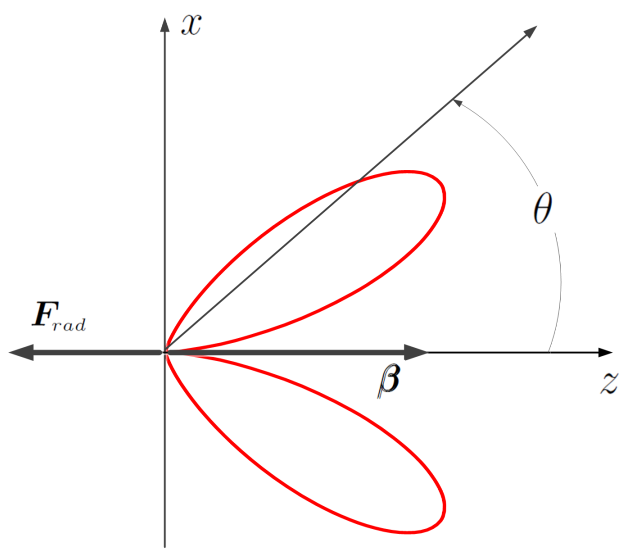

Figure 3: *Transverse power distribution for straight line motion.*

characterizing this distribution is

$$\frac{dP_{acc}}{d\Omega} = \frac{e^2 a^2}{4\pi c^3} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{3.11}$$

and can be integrated immediately over solid angle to derive[2]

$$P_{acc} = \gamma^6 m_e \tau a^2 \qquad \text{where} \qquad \tau = \frac{2}{3}\frac{e^2}{m_e c^3} \tag{3.12}$$

Functional forms of P_{acc} and m^* can now be inserted into (3.10) resulting in the cubic force equation

$$\left[\frac{3}{4}\frac{\gamma^9 \tau^2 m_e}{c^2}\right] a^3 + \left[\frac{\gamma^6 m_e \tau \beta}{c}\right] a^2 + \gamma^3 m_e a = F_{ext} \tag{3.13}$$

At this stage it is appropriate to define the unitless variables

$$y = \frac{\gamma^3 a \tau}{c} \qquad \text{and} \qquad y_o = \frac{F_{ext}\tau}{m_e c} \tag{3.14}$$

and arrive at the simplified cubic

$$\boxed{\tfrac{3}{4}y^3 + \beta y^2 + y = y_o} \tag{3.15}$$

This expression may seem alarming at first since the general cubic has three roots. However, the theory of cubic equations, summarized in the Appendix, will show that it exhibits exactly one real root. This is a very desirable feature of a radiation reaction theory since it eliminates the well known problem of runaway solutions in the conventional theory. The root is easily extracted from (3.15) by setting the force to zero leaving $y = 0$ and a quadratic equation with only complex roots.

The general procedure for solving this equation will be to first solve the cubic for y in terms of the external force F_{ext}. A general expansion of the solution valid to any order in F_{ext} will then generate an appropriate second order (probably non-linear) differential equation.

Linear Motion Under a Constant Force: An exact solution for the linear acceleration of a charged particle is available by solving equation (3.15) for y using the theory of cubic equations. While the motion $x(t)$ may have no analytic solution, it is still possible to determine the final velocity of the particle. This is simply accomplished from the observation that the solution $y = g(y_o, \beta)$ is finite for any value of β. The value of y_o is obviously constant and the acceleration of the particle can then be related to its velocity by

$$a = \frac{c}{\tau}\, g(y_o, \beta) \cdot (1-\beta^2)^{3/2} \tag{3.16}$$

[2] The time τ is defined in Jackson, chapter 17 as the *characteristic time* having a value $6.26 \times 10^{-24}\, s$ for electrons.

As the particle approaches light speed the value of the acceleration tends to zero. This means that the radiation reaction evaporates for large velocities—returning the motion to the hyperbolic curve.

Another important point is to determine when inertial and self-force reactions are comparable at high energies. From (3.15) it is evident that this will occur when $y \sim 1$ or when

$$\frac{c}{\gamma^3 a} \sim \tau \tag{3.17}$$

A 50 GeV electron at SLAC crashing into a hard target with $a \sim 10^{17} m/s^2$ comes pretty close but only for a short time.

For constant forces F_{ext} which are not too large it seems reasonable that the motion of the electron might be accurately described by neglecting the inertial reaction in equation (3.15) altogether. This leaves a much more manageable quadratic form

$$\beta y^2 + y - y_o = 0 \tag{3.18}$$

Solving for y and expanding the square root leads to the infinite series

$$a = \frac{F_{ext}}{\gamma^3 m_e} \left[\sum_{n=0}^{\infty} C_n (\beta y_o)^n \right] \tag{3.19}$$

where the C_n are numbers. In fact, with $C_0 = 1 = -C_1$ a first order approximation to the acceleration is

$$a \approx \frac{F}{\gamma^3 m_e} (1 - \beta y_o) \tag{3.20}$$

This equation indicates that a significant change to the acceleration will occur when $\beta y_o \sim 1$. In this limit the reactive power delivered to the electron by the radiation reaction force will be the same order as the inertial power delivered to the vacuum by the electron.

Simple Harmonic Oscillator: Another important application of the radiation reaction is the problem of a charged particle responding to a linear restoring force. Without a driving force an oscillating charge of initial energy E_o will undergo radiation damping until the motion stops completely. The problem is best formulated at non-relativistic speeds so that the radiated transverse power may be accurately calculated from the Larmor power formula. The equation of motion derived from (3.15) is then

$$\frac{3\tau^2}{4c^2}\ddot{x}^3 + \frac{\beta\tau}{c}\ddot{x}^2 + \ddot{x} + \omega_o^2 x = 0 \tag{3.21}$$

Probably the simplest algorithm to solve this equation for $\ddot{x}$ is to simply insert the unperturbed equation

$$\ddot{x} = -\omega_o^2 x \tag{3.22}$$

and this produces the non-linear solution

$$\ddot{x} + \omega_o^2 x + \frac{\tau}{c^2}\omega_o^4 x^2 \dot{x} - \frac{4}{3}\frac{\tau^2}{c^2}\omega_o^6 x^3 = 0 \tag{3.23}$$

According to the literature, this equation is an example of Liénard system having the general form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{3.24}$$

It has all the properties consistent with the radiation based theory since both non-linear terms vanish at $x = 0$ when the acceleration is zero, and since the self-force term is zero when $\dot{x} = 0$ indicating that no radiation is emitted along the direction of motion.

A crude approximation for the oscillator problem is to linearize (3.23) by removing the inertial reaction term all together and averaging the motion in the self-force term as

$$x^2\dot{x} \longrightarrow \tfrac{1}{2}x_o^2\dot{x} \tag{3.25}$$

This replacement will increase the rate of radiation damping somewhat but should give a reasonable estimate of the decay rate. The resulting linearized theory is

$$\ddot{x} + \frac{\epsilon}{\tau}\dot{x} + \omega_o^2 x = 0 \qquad \text{where} \qquad \epsilon = \frac{E_o}{mc^2}(\omega_o\tau)^2 \tag{3.26}$$

Inserting a trial solution $x(t) = x_o e^{-\alpha t}$ and solving the quadratic equation gives

$$\alpha = \frac{\epsilon}{2\tau} \pm i\omega_o\left[1 - \frac{\epsilon^2}{4\omega_o^2\tau^2}\right]^{1/2} \tag{3.27}$$

The decay constant and level shift of the oscillator can then be written[3]

$$\Gamma = \frac{E_o}{mc^2}\,\omega_o^2\tau \qquad\qquad \Delta\omega = -\left[\frac{E_o}{2mc^2}\right]^2\omega_o^3\tau^2 \tag{3.28}$$

Although the solution here is typical of damped oscillator equation, a problem presents itself because the power delivered to the oscillator by the self-force to slow it down is not the same as the power dispensed to the vacuum by the acceleration fields. In short, energy conservation must be violated to bring the particle to rest. The total work done by the self-force is

$$W = \tfrac{1}{2}kx_o^2 \tag{3.29}$$

and this compares to the total energy radiated by the particle

$$\Delta E_{rad} \sim P_{avg}\Delta t = \frac{P_{avg}}{\Gamma} = m_e c^2 \tag{3.30}$$

[3]Equations for decay constant and level shift can be compared with values calculated in Jackson; Second Edition, section 17.7

It could be argued at this point that a calculation such as this for an electron will not be applicable since the microscopic system will only obey the laws of quantum mechanics. One might also suggest that energy violation in the classical theory might be accommodated by a quantum mechanical theory whereby the excess of radiated energy is counterbalanced by energy absorbed from the vacuum. The path shown in figure 4 for a quantum oscillator is an example of such an effect which might be called

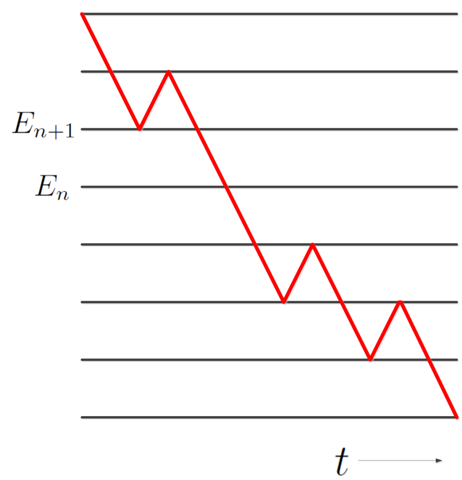

Figure 4: *Possible quantum mechanical decay curve for the vacuum gauge electron in a harmonic oscillator potential.*

spontaneous absorption. There is a non-zero matrix element for it from the electric dipole approximation

$$\langle n|ex|n'\rangle = \sqrt{\frac{\hbar}{2m_e\omega_o}}\left[\sqrt{n}\,\delta_{n,n'+1} + \sqrt{n'}\,\delta_{n,n'-1}\right] \tag{3.31}$$

One would assume that if the effect is legitimate that it is probably small.

3.2 Application to Circular Motion:

Caclulation of power radiated for motion in a circular orbit follows by orienting the instantaneous velocity vector along the z-axis. The power radiated can then be determined from equation (3.6) with the requirement that $\boldsymbol{\beta}\cdot\boldsymbol{a} = 0$. Figure 5 shows the power distribution for circular motion described by the formula

$$\frac{dP_{acc}}{d\Omega} = \frac{e^2a^2}{4\pi c^3}\left[\frac{1}{(1-\beta\cos\theta)^3} - \frac{(1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}\right] \tag{3.32}$$

Integrations over solid angle are straight forward and lead to the power formula

$$P_{acc} = \gamma^4 m_e\tau a^2 \tag{3.33}$$

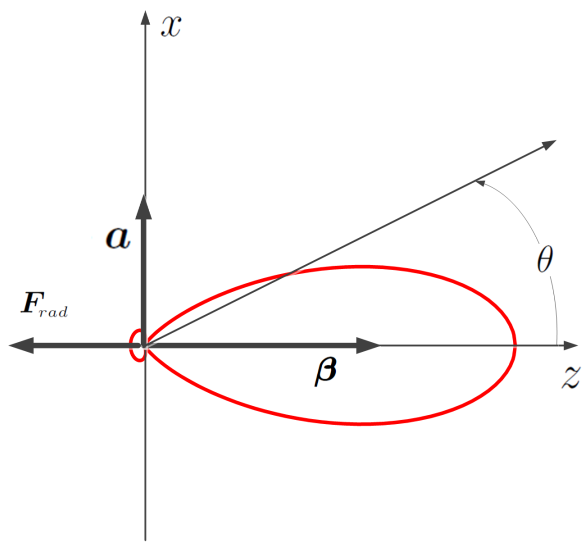

Figure 5: *Transverse power distribution for circular motion.*

Motion in a Magnetic Field: For simplicity, it will be assumed that circular motion is determined by an electron moving through a constant magnetic field directed perpendicular to the trajectory. Based on the work in section 2 assume also that $m^* \to m_e$. The governing equation for the particle orbit is then

$$m_e a^\mu = F^\mu_{ext} + eF^{\mu\nu}_{em}\beta_\nu + F^\mu_{rad} \tag{3.34}$$

In the absence of an electric field this equation de-couples. The acceleration term on the left side and the Lorentz force on the right are

$$eF^{\mu\nu}_{em}\beta_\nu = (0,\, ec\gamma\boldsymbol{\beta}\times\mathbf{B}) \qquad\qquad m_e a^\mu = (0,\, \gamma^2 m_e\boldsymbol{a}) \tag{3.35}$$

Using r as the orbit radius the equality of these four-vectors leads to the relativistic cyclotron formula

$$evB = \frac{\gamma m v^2}{r} \tag{3.36}$$

A pre-requisite for a circular orbit will then be determined by the remaining two terms

$$F^\mu_{ext} + F^\mu_{rad} = 0 \tag{3.37}$$

and this identifies the required external force as

$$\boldsymbol{F}_{ext} = \frac{\gamma^4 m_e \tau a^2}{c}\boldsymbol{\beta} \tag{3.38}$$

Once again, energy conservation is violated; the power supplied by the external force to maintain a circular orbit is less than the power radiated by the particle. This phenomenon should also be apparent for motion in a synchrotron. However, for high energy facilities like LEP and CESR, it can be asserted that the discrepancy will not be observable. A rough calculation for CESR where $\beta \sim 1$ to about three parts in a billion gives unaccounted power per electron of

$$\Delta P = P(1 - \beta^2) \sim 1.2 \times 10^{-15}\,\text{Watts} \tag{3.39}$$

This is about 10^{-9} times the power radiated by the electron.

Experiment to Measure Energy Violation: One possibility for measuring energy violation is to construct a specialized low energy cyclotron. A constant magnetic field of $B = 10\,G$ and a beam radius $r = 1.5\,m$ sets the relativistic quantities

$$\beta = 0.879 \qquad \text{and} \qquad \gamma = 2.099 \tag{3.40}$$

From equation (3.33) the power inititally radiated by an electron in a pre-determined beam current is

$$P = 2.38 \times 10^{-19}\,\text{Watts} \tag{3.41}$$

If energy is conserved, the beam will slowly spiral inward to be detected by a counter

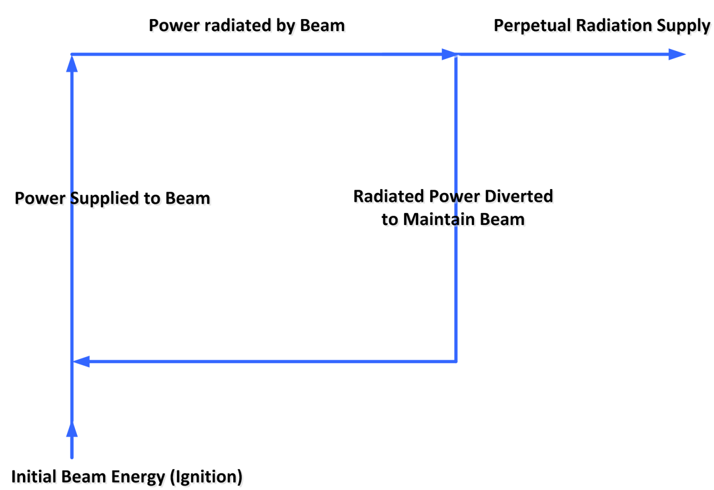

Figure 6: *Diagram illustrating the operation of a perpetual radiation cyclotron.*

placed at radius r_{in} some time ΔT_{CE} later. If the reaction force is given by equation (3.38), then the spiral inward will be much slower and detection will occur at a time ΔT_{VG} where

$$\Delta T_{VG} > \Delta T_{CE} \tag{3.42}$$

A rough calculation of the time difference is easy to calculate. The initial kinetic energy of an electron is $K = 0.56$ MeV so

$$\Delta T_{CE} \sim \frac{K}{P} \to 1,156\,s \qquad \Delta T_{VG} \sim \frac{K}{\beta^2 P} \to 1,496\,s \tag{3.43}$$

If technical difficulties can be removed this descrepancy should be easy to detect.

Perpetual Energy Production: Vaccum gauge theory allows for the possibility of building a perpetual energy machine out of cyclotron or storage ring. In an ideal situation one supposes that permanent magnets produce the necessary field required for circular motion. Furthermore, it must also be assumed that the total power radiated from the beam can be converted to useful work. Some of this work must be re-routed to maintain the beam in a circular orbit–the rest can be used as a perpetual energy source.

A Cubic Equations

The most general cubic equation is

$$y^3 + py^2 + qy + r = 0 \tag{A.1}$$

where p, q, and r are real and constant coefficients. It can be shown however, that any cubic equation such as this can alway's be reduced to the form

$$x^3 + ax + b = 0 \tag{A.2}$$

by making the substitution

$$y = x - p/3 \tag{A.3}$$

This substitution then implies the following relations among the constants

$$a = \frac{1}{3}(3q - p^2) \qquad \text{and} \qquad b = \frac{1}{27}(2p^3 - 9pq + 27r) \tag{A.4}$$

The exact solutions to the cubic equation are given by

$$x_1 = R + S \tag{A.5a}$$

$$x_2 = -\frac{1}{2}(R + S) + \frac{\sqrt{3}i}{2}(R - S) \tag{A.5b}$$

$$x_3 = -\frac{1}{2}(R + S) - \frac{\sqrt{3}i}{2}(R - S) \tag{A.5c}$$

Where

$$R = \left[\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{1/2} - \frac{b}{2}\right]^{1/3} \qquad S = -\left[\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{1/2} + \frac{b}{2}\right]^{1/3} \tag{A.6}$$

Now note the following properties of solutions to the cubic equation for given values of a and b:

If $\frac{b^2}{4} + \frac{a^3}{27} > 0$, there will be one real root and two complex roots.

If $\frac{b^2}{4} + \frac{a^3}{27} = 0$, there will be three real roots of which at least two are equal.

If $\frac{b^2}{4} + \frac{a^3}{27} < 0$, there will be three unequal, real roots.

Power Series Solution For $b \ll a$: A solution may be sought for a cubic equation where $b \ll a$. If this is the case, then a power series solution may be appropriate. For the case of a single real root the solution can be approximated by

$$x \approx -\frac{b}{a} + \frac{b^3}{a^4} - \frac{3b^5}{a^7} + \cdots$$

This solution can be verified by inserting it into the cubic equation and setting all higher orders to zero. suppose next that $b \ll a$ but there are three unequal real roots. In this case both R and S can be power expanded:

$$R = (\frac{a}{3})^{1/2} - \frac{1}{2a}b + \frac{1}{72(\frac{a}{3})^{5/2}}b^2 + \frac{1}{2a^4}b^3 - \frac{35}{1296}\frac{1}{(\frac{a}{3})^{11/2}}b^4 - \frac{3}{2a^7}b^5 + \cdots$$

$$S = (\frac{a}{3})^{1/2} + \frac{1}{2a}b + \frac{1}{72(\frac{a}{3})^{5/2}}b^2 - \frac{1}{2a^4}b^3 - \frac{35}{1296}\frac{1}{(\frac{a}{3})^{11/2}}b^4 + \frac{3}{2a^7}b^5 + \cdots$$

B Useful Integrals

1. $\displaystyle\int_0^\pi \frac{\sin\theta}{(1-\beta\cos\theta)^2}d\theta = 2\gamma^2$

2. $\displaystyle\int_0^\pi \frac{\sin\theta}{(1-\beta\cos\theta)^3}d\theta = 2\gamma^4$

3. $\displaystyle\int_0^\pi \frac{\cos\theta\sin\theta}{(1-\beta\cos\theta)^3}d\theta = 2\gamma^4\beta$

4. $\displaystyle\int_0^\pi \frac{\cos\theta\sin\theta}{(1-\beta\cos\theta)^4}d\theta = \tfrac{8}{3}\gamma^6\beta$

5. $\displaystyle\int_0^\pi \frac{\sin\theta}{(1-\beta\cos\theta)^4}d\theta = 2\gamma^6(1+\tfrac{1}{3}\beta^2)$

6. $\displaystyle\int_0^\pi \frac{\sin^3\theta}{(1-\beta\cos\theta)^4}d\theta = \tfrac{4}{3}\gamma^4$

7. $\displaystyle\int_0^\pi \frac{\cos^2\theta\sin\theta}{(1-\beta\cos\theta)^4}d\theta = 2\gamma^6(\tfrac{1}{3}+\beta^2)$

8. $\displaystyle\int_0^\pi \frac{\sin^3\theta}{(1-\beta\cos\theta)^5}d\theta = \tfrac{4}{3}\gamma^6$

9. $\displaystyle\int_0^\pi \frac{\sin\theta}{(1-\beta\cos\theta)^5}d\theta = 2\gamma^8(1+\beta^2)$

10. $\displaystyle\int_0^\pi \frac{\cos\theta\sin^3\theta}{(1-\beta\cos\theta)^5}d\theta = \tfrac{4}{3}\gamma^6\beta$

11. $\displaystyle\int_0^\pi \frac{\cos\theta\sin\theta}{(1-\beta\cos\theta)^5}d\theta = \tfrac{2}{3}\gamma^8\beta(5+\beta^2)$

12. $\displaystyle\int_0^\pi \frac{\cos^3\theta\sin\theta}{(1-\beta\cos\theta)^5}d\theta = 2\gamma^8\beta(1+\beta^2)$

13. $\displaystyle\int_0^\pi \frac{\cos^2\theta\sin\theta}{(1-\beta\cos\theta)^5}d\theta = \tfrac{1}{3}\gamma^8(2+10\beta^2)$

C Christoffel Symbols

Derivatives of the basis vectors define ten independent Christoffel symbols thru the relations

$$\frac{\partial\,\vec{\mathrm{e}}_\mu}{\partial q^\nu} = \Gamma^\lambda_{\mu\nu}\,\vec{\mathrm{e}}_\lambda \qquad\qquad \frac{\partial\,\vec{\omega}^{\,\mu}}{\partial q^\nu} = -\Gamma^\mu_{\lambda\nu}\,\vec{\omega}^{\,\lambda} \tag{C.1}$$

Christoffel symbols can also be determined directly from the metric tensor:

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\mathrm{g}^{\lambda\alpha}\left[\mathrm{g}_{\lambda\mu,\nu} + \mathrm{g}_{\lambda\nu,\mu} - \mathrm{g}_{\mu\nu,\lambda}\right] \tag{C.2}$$

For the coordinate transformation described by equation (1.5):

1. $\Gamma^\phi_{\rho\phi} = \dfrac{1}{\rho}$
2. $\Gamma^\theta_{\rho\theta} = \dfrac{1}{\rho}$
3. $\Gamma^\rho_{\theta\theta} = -\dfrac{\rho}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2}$
4. $\Gamma^\rho_{\phi\phi} = -\dfrac{\rho\sin^2\theta}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2}$
5. $\Gamma^\theta_{\theta\theta} = \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}$
6. $\Gamma^\phi_{\theta\theta} = -\dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}$
7. $\Gamma^\phi_{\theta\phi} = \cot\theta + \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}$
8. $\Gamma^\theta_{\theta\phi} = \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}\sin\theta$
9. $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta - \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}\sin^2\theta$
10. $\Gamma^\phi_{\phi\phi} = \dfrac{\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}}}{1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}}\sin\theta$