Multipole Fields in the Vacuum Gauge

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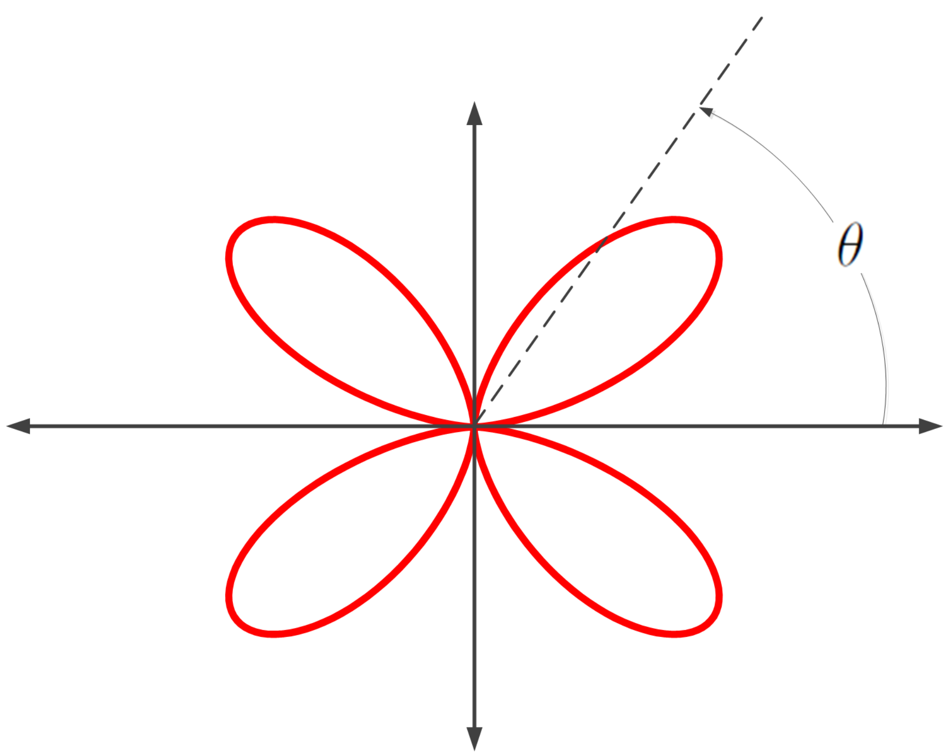

"Whatever you call them—rubber bands, or Poincaré stresses, or something else—there have to be other forces in nature to make a consistent theory of this kind."

R.P. Feynman, *The Feynman Lectures*, chapter 28

Abstract

The vacuum gauge requires velocity and acceleration fields of the electron to be considered as independent phenomena—and this remains valid for a description of multipole fields. For the velocity theory, all electric and magnetic multipoles can be written as linear combinations of vector multipole potentials. For the acceleration fields the proportionality between the fields and potentials is equally beneficial allowing for simple calculations associated with multipole radiation.

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1 Multipole Velocity Fields

In the vacuum gauge both scalar and vector potentials require Taylor expansions for small displacements from the origin. The presence of the vector potential is not only useful, but also necessary for an accurate description of multipole electric and magnetic velocity fluxes.

1.1 Multipole Expansion in the Vacuum Gauge

A charged particle is displaced from the origin of a coordinate system by an amount $\boldsymbol{r}_o$ having arbitrary rectangular coordinates

$$\boldsymbol{r}_o = r_o(\sin\theta' \cos\phi' \hat{\boldsymbol{x}} + \sin\theta' \sin\phi' \hat{\boldsymbol{y}} + \cos\theta' \hat{\boldsymbol{z}}) \tag{1.1}$$

A graphic showing the coordinate system along with vectors $\boldsymbol{r}$ and $\boldsymbol{r}_o$ is shown in figure 1. The angle γ between the vectors is determined by the trigonometric identity

$$\cos\gamma = \sin\theta' \sin\theta \cos(\phi - \phi') + \cos\theta' \cos\theta \tag{1.2}$$

Now assume $r_o \ll r$ and define the small quantity

$$\begin{aligned} \boldsymbol{\varepsilon} &\equiv \frac{r_o}{r}\left[\cos\gamma\,\hat{\boldsymbol{r}} + \frac{\partial}{\partial\theta}\cos\gamma\,\hat{\boldsymbol{\theta}} + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\cos\gamma\,\hat{\boldsymbol{\phi}}\right] \\ &= \frac{r_o}{r}\left[\cos\gamma\,\hat{\boldsymbol{r}} + r\boldsymbol{\nabla}\cos\gamma\right] \end{aligned} \tag{1.3}$$

The well known scalar potential for this particle can be written as a sum over Legendre polynomials in $\cos\gamma$ and is given by

$$A(r,\theta,\phi) = e\sum_{l=0}^{\infty}\frac{r_o^l}{r^{l+1}}\mathcal{P}_l(\cos\gamma) \tag{1.4}$$

However, as already discussed, the expansion of the potentials is not yet complete and must include the vector potential also. The vector potential proves to be a more difficult expansion since the general displacement includes two instances of the small variable $\boldsymbol{\varepsilon}$:

$$\boldsymbol{A} = \frac{e}{r}\left[\frac{\hat{\boldsymbol{r}} - \boldsymbol{\varepsilon}}{1 - 2\hat{\boldsymbol{r}}\cdot\boldsymbol{\varepsilon} + \varepsilon^2}\right] \tag{1.5}$$

A power series in $\boldsymbol{A}$ is

$$\begin{aligned} \boldsymbol{A} &= \sum_{n=0}^{\infty}\boldsymbol{A}_n \\ &= \sum_{n=0}^{\infty}\frac{e r_o^n}{r^{n+1}}\left[\cos n\gamma\,\hat{\boldsymbol{r}} - \frac{r}{n}\boldsymbol{\nabla}\cos n\gamma\right] \end{aligned} \tag{1.6}$$

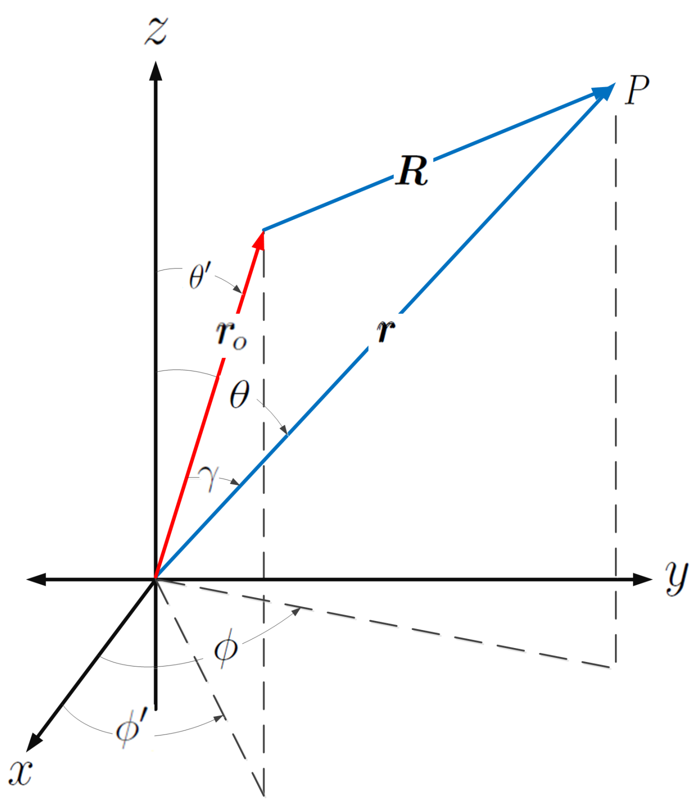

Figure 1: *Plot showing variables appropriate for a classical electron displaced from the origin of coordinates.*

For a stationary charge there is no need to consider magnetic fields at all. This means each multipole vector potential must obey

$$\boldsymbol{\nabla} \times \boldsymbol{A}_n = 0 \tag{1.7}$$

as can be easily verified.

The importance of the vector potential arises when considering expansions of the electric field vector. Since the field is moving an appropriate electric flux vector must be written

$$\boldsymbol{\pi}_{E} = \sigma_e \mathbf{E} \tag{1.8}$$

It is not difficult to show that a given electric flux multipole is determined by admixture of potentials according to the formula

$$\boldsymbol{\pi}_{Em} = \frac{\sigma_e}{r} \sum_{n=0}^{m} \mathcal{P}_{m-n}(\cos\gamma)\varepsilon^{m-n}\, \boldsymbol{A}_n \tag{1.9}$$

Now observe that a sum of equation (1.9) over all multipoles can be written

$$\boxed{\boldsymbol{\pi}_E = \frac{1}{\mathrm{a}_e} \sum_{m,n=0}^{\infty} A_m \boldsymbol{A}_n} \tag{1.10}$$

where a_e is the minimum classical surface element given by $\mathrm{a}_e = 4\pi r_e^2$. Written this way the total electric flux is purely a function of vacuum gauge potentials and each multipole l can be read from the formula by including all terms such that $l = m + n$. For the record, the three lowest order contributions are

$$\boldsymbol{\pi}_{E0} = \frac{1}{\mathrm{a}_e} A_0 \boldsymbol{A}_0 \tag{1.11a}$$

$$\boldsymbol{\pi}_{E1} = \frac{1}{\mathrm{a}_e} \left[A_1 \boldsymbol{A}_0 + A_0 \boldsymbol{A}_1 \right] \tag{1.11b}$$

$$\boldsymbol{\pi}_{E2} = \frac{1}{\mathrm{a}_e} \left[A_2 \boldsymbol{A}_0 + A_1 \boldsymbol{A}_1 + A_0 \boldsymbol{A}_2 \right] \tag{1.11c}$$

This completes the multipole expansion of the electric flux vector, but it is also useful to write each multipole field in terms of $\cos\gamma$. This can be accomplished using the relations

$$(l+1)\mathcal{P}_l(\cos\gamma) = \sum_{n=0}^{l} \mathcal{P}_{l-n}(\cos\gamma) \cos n\gamma \tag{1.12}$$

$$\boldsymbol{\nabla} \mathcal{P}_l(\cos\gamma) = \sum_{n=1}^{l} \frac{1}{n} \mathcal{P}_{l-n}(\cos\gamma) \boldsymbol{\nabla} \cos n\gamma \tag{1.13}$$

which leads to a sum over all multipoles

$$\boldsymbol{\pi}_E = \sigma_e e \sum_{l=0}^{\infty} \frac{r_o^l}{r^{l+2}} \left[(l+1)\mathcal{P}_l(\cos\gamma)\hat{\boldsymbol{r}} - r \boldsymbol{\nabla} \mathcal{P}_l(\cos\gamma) \right] \tag{1.14}$$

Of course this equation can also be derived directly from equation (1.4) from a gradient operation.

1.2 Multipole Expansion along the Z-Axis:

A nifty simplification occurs in the expansion of the vector potential when considering displacement from the origin along the z-axis. Equation (1.1) simplifies to $\boldsymbol{r}_o = z_o \hat{\boldsymbol{z}}$ while the value of $\boldsymbol{\varepsilon}$ in spherical-polar coordinates reads

$$\boldsymbol{\varepsilon} = \frac{z_o}{r} (\cos\theta\, \hat{\boldsymbol{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}) \tag{1.15}$$

In the cartesian coordinate system the multipole potentials are determined from

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{n=1}^{\infty} \frac{e z_o^{n-1}}{r^n} \left[\sin n\theta \cos\phi,\, \sin n\theta \sin\phi,\, \cos n\theta\right] \tag{1.16}$$

but an expansion in curvi-linear coordinates gives pure multi-angle contributions along each of two components:

$$\boldsymbol{A}(r,\theta) = \frac{e}{r} \sum_{n=0}^{\infty} \left(\frac{z_o}{r}\right)^n \left[\cos n\theta\, \hat{\boldsymbol{r}} + \sin n\theta\, \hat{\boldsymbol{\theta}}\right] \tag{1.17}$$

Each component of the vector potential is also a Fourier series in the polar angle. Fourier coefficients along each direction are identical but follow from separate integrals,

$$\varepsilon^n = \frac{2}{\pi} \int_0^{\pi} g_\varepsilon(\theta) \cos n\theta \, d\theta = \frac{2}{\pi} \int_0^{\pi} h_\varepsilon(\theta) \sin n\theta \, d\theta \tag{1.18}$$

where the two functions $g_\varepsilon(\theta)$ and $h_\varepsilon(\theta)$ are given by

$$g_\varepsilon(\theta) = \frac{1 - \varepsilon \cos\theta}{1 - 2\varepsilon\cos\theta + \varepsilon^2} \tag{1.19}$$

$$h_\varepsilon(\theta) = \frac{\varepsilon \sin\theta}{1 - 2\varepsilon\cos\theta + \varepsilon^2} \tag{1.20}$$

Based on the construction of equation (1.17), it may also be reasonable to consider individual multipole potentials in term of the complex scalar potential $\widetilde{A}$ defined by

$$\widetilde{A} \equiv \frac{e}{r} \sum_{n=0}^{\infty} \left(\frac{z_o}{r} e^{i\theta}\right)^n \tag{1.21}$$

Components of the vector potential can then be written

$$A_r = \mathrm{Re}\widetilde{A} = \frac{e}{r}\, g_\varepsilon(\theta) \qquad\qquad A_\theta = \mathrm{Im}\widetilde{A} = \frac{e}{r}\, h_\varepsilon(\theta) \tag{1.22}$$

The divergence of $\boldsymbol{A}$ is also important. Since the scalar potential is not a function of time then the measure of vacuum dilatation is given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{e}{r^2} \sum_{n=0}^{\infty} \left[\frac{z_o}{r}\right]^n \left[\frac{\sin(n+1)\theta}{\sin\theta}\right] \tag{1.23}$$

According to this formula the total dilatation receives contributions from multipoles of all orders. The dilatation from each multipole is a function of angle and can be either positive or negative. However, the total dilation is positive and this can be shown by replacing $\sin(n+1)\theta$ with complex exponentials and summing the series to determine

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{e}{r^2(1 - 2\varepsilon\cos\theta + \varepsilon^2)} \tag{1.24}$$

Multipole Fields of an Oscillating Charge: Now suppose that the static problem is replaced with a particle undergoing oscillations at angular frequency ω. The position of the particle as a function of time on the z-axis is given by

$$\boldsymbol{z}(t_r) = z_o e^{-i\omega t_r}\hat{\boldsymbol{z}} \tag{1.25}$$

where t_r is the retarded time. In general

$$ct_r = ct - \|\boldsymbol{r} - \boldsymbol{z}(t_r)\| \tag{1.26}$$

but if $\|\boldsymbol{z}(t_r)\| \ll r$ for all t_r then (1.26) can be replaced with

$$ct_r = ct - r \tag{1.27}$$

and this adds time dependence to the electric field vector so that (1.10) generalizes to

$$\boldsymbol{\pi}_E = \frac{1}{\mathrm{a}_e}\sum_{m,n=0}^{\infty} A_m \boldsymbol{A}_n e^{i(kr-\omega t)} \tag{1.28}$$

Meanwhile, a changing electric multipole field of order ℓ generates a magnetic flux multipole of order $\ell+1$ calculated from the expression

$$\boldsymbol{\pi}_{B\ell+1} = \boldsymbol{\beta} \times \boldsymbol{\pi}_{E\ell} \tag{1.29}$$

where $\boldsymbol{\beta} = -i\omega\boldsymbol{z}/c$. By design, the lowest order magnetic pole is $\boldsymbol{\pi}_{B1}$ leaving only the radiating electric monopole $\boldsymbol{\pi}_{E0}$. For the dipole fields it is convenient to define the electric dipole moment $\boldsymbol{p} = ez_o\hat{\boldsymbol{z}}$. The associated velocity fields can then be written in coordinate–free form as

$$\boldsymbol{\pi}_{E1} = \sigma_e \left[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}\right]\frac{e^{i(kr-\omega t)}}{r^3} \tag{1.30a}$$

$$\boldsymbol{\pi}_{B1} = ik\sigma_e(\hat{\boldsymbol{r}} \times \boldsymbol{p})\frac{e^{i(kr-\omega t)}}{r^2} \tag{1.30b}$$

A similar procedure can also be applied to determine the quadrupole fields. In this case it is convenient to calculate the traceless quadrupole tensor determined from

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\boldsymbol{r})d^3r \tag{1.31}$$

The charge density can be written $\rho = e\delta(x)\delta(y)\delta(z-z_o)$ producing the tensor

$$\widehat{\boldsymbol{Q}} = ez_o^2 \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{1.32}$$

Quadrupole electric and magnetic flux vectors are then

$$\boldsymbol{\pi}_{E2} = \sigma_e \left[\frac{3}{2}(\hat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - (\hat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}}\right] \frac{e^{i(kr-\omega t)}}{r^4} \tag{1.33a}$$

$$\boldsymbol{\pi}_{B2} = ik\sigma_e \left[\hat{\boldsymbol{r}} \times \widehat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}}\right] \frac{e^{i(kr-\omega t)}}{r^3} \tag{1.33b}$$

As a check it can be shown that the correct order in r satisfies

$$\boldsymbol{\nabla} \times \boldsymbol{\pi}_{Bn} = -ink\,\boldsymbol{\pi}_{En} \tag{1.34}$$

for both dipole and quadrupole fields.

1.3 Ring of Charge

The theory of the vector potential is not limited to oscillating charges only. As an example, the formalism developed here can also be applied in a many particle theory. Begin with an electron located in the xy-plane with position vector

$$\boldsymbol{s} = r_o(\cos\phi'\,\hat{\boldsymbol{x}} + \sin\phi'\,\hat{\boldsymbol{y}}) \tag{1.35}$$

In a spherical coordinate system this vector is

$$\boldsymbol{s} = r_o\left[\sin\theta\cos(\phi-\phi')\,\hat{\boldsymbol{r}} + \cos\theta\cos(\phi-\phi')\,\hat{\boldsymbol{\theta}} - \sin(\phi-\phi')\,\hat{\boldsymbol{\phi}}\right] \tag{1.36}$$

For $\theta' = \pi/2$, the quantity $\cos\gamma$ simplifies to

$$\cos\gamma = \sin\theta\cos(\phi-\phi') \tag{1.37}$$

and the lowest order contributions to the vector potential from equation (1.6) are:

$$\boldsymbol{A}_o = \frac{e}{r}\hat{\boldsymbol{r}} \tag{1.38a}$$

$$\boldsymbol{A}_1 = \frac{er_o}{r^2}\left[\sin\theta\cos(\phi-\phi')\,\hat{\boldsymbol{r}} - \cos\theta\cos(\phi-\phi')\,\hat{\boldsymbol{\theta}} + \sin(\phi-\phi')\,\hat{\boldsymbol{\phi}}\right] \tag{1.38b}$$

The electric dipole moment about the origin is $\boldsymbol{p} = e\boldsymbol{s}$ with monopole and dipole electric flux given by

$$\boldsymbol{\pi}_{E0} = \frac{\sigma_e e}{r^2}\hat{\boldsymbol{r}} \tag{1.39a}$$

$$\boldsymbol{\pi}_{E1} = \frac{\sigma_e}{r^3}\left[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \boldsymbol{p}\right] \tag{1.39b}$$

Re-iterating the well known interpretation, the fields of the charge at coordinates (r_o, ϕ') in the xy-plane are the same as a monopole at the origin along with an associated dipole field.

Now suppose this problem is extended to the case where there are a large number of charges N at radius r_o evenly distributed about the origin. The position of each charge may be a random variable but for the sake of simplicity it will be assumed that the n^{th} particle is located at angular coordinate $\phi' = 2\pi n/N$. A graphic is available in figure 2. The potentials in (1.38) must now be summed over all coordinates. With

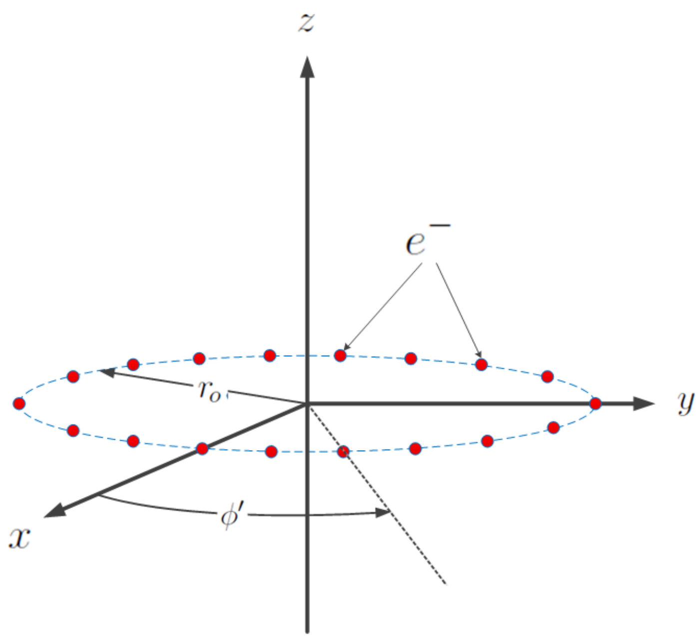

Figure 2: *Electrons distributed in a circle of radius r_o in the xy-plane.*

$$\phi_n \equiv \phi - \frac{2\pi n}{N} \tag{1.40}$$

The potentials A_1 and $\boldsymbol{A}_1$ vanish along with $\boldsymbol{\pi}_{E1}$ since

$$\sum_{n=1}^{N} \cos\phi_n = \sum_{n=1}^{N} \sin\phi_n = 0 \tag{1.41}$$

Note however that the quadrupole field is non-vanishing. Here one calculates

$$\frac{1}{\mathrm{a}_e}\sum_{n=0}^{N} A_2 \boldsymbol{A}_0 = -\frac{Ne^2r_o^2}{2\mathrm{a}_e r^4}\left[\frac{1}{2}(3\cos^2\theta - 1)\,\hat{\boldsymbol{r}}\right] \tag{1.42a}$$

$$\frac{1}{\mathrm{a}_e}\sum_{n=0}^{N} A_1 \boldsymbol{A}_1 = +\frac{Ne^2r_o^2}{2\mathrm{a}_e r^4}\left[\sin^2\theta\,\hat{\boldsymbol{r}} - \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\right] \tag{1.42b}$$

$$\frac{1}{\mathrm{a}_e}\sum_{n=0}^{N} A_0 \boldsymbol{A}_2 = -\frac{Ne^2r_o^2}{2\mathrm{a}_e r^4}\left[2\cos^2\theta\,\hat{\boldsymbol{r}} + 2\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\right] \tag{1.42c}$$

Including the monopole field and defining the total charge $Q = Ne$, the electric flux vector for the ring of charge can be written

$$\boldsymbol{\pi}_E = \frac{e}{\mathrm{a}_e}\left\{\frac{Q}{r^2}\hat{\boldsymbol{r}} - \frac{3Qr_o^2}{2r^4}\left[\frac{1}{2}(3\cos^2\theta - 1)\,\hat{\boldsymbol{r}} + \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}}\right]\right\} \tag{1.43}$$

Of course this result is also available by taking the gradient of the scalar potential

$$V(r,\theta) = \frac{Q}{r}\left[1 - \frac{r_o^2}{2r^2}\mathcal{P}_2(\cos\theta)\right] \tag{1.44}$$

and multiplying by σ_e.

Magnetic Dipole Field: The previous calculation can be extended further by setting individual electrons in motion around the ring. This is easy to do by introducing a time dependence in equation (1.35) for each of the N charges:

$$\boldsymbol{s}_n(t) = r_o\left[\cos\left(\omega t + \frac{2\pi n}{N}\right)\hat{\boldsymbol{x}} + \sin\left(\omega t + \frac{2\pi n}{N}\right)\hat{\boldsymbol{y}}\right] \tag{1.45}$$

If ϕ_n is re-defined by

$$\phi_n = \phi - \omega t - \frac{2\pi n}{N} \tag{1.46}$$

then values for the position and velocity of the n^{th} electron in the spherical-polar coordinate system can be written

$$\boldsymbol{s}_n(t) = r_o\left[\sin\theta\cos\phi_n\,\hat{\boldsymbol{r}} + \cos\theta\cos\phi_n\,\hat{\boldsymbol{\theta}} - \sin\phi_n\,\hat{\boldsymbol{\phi}}\right] \tag{1.47}$$

$$\boldsymbol{\beta}_n(t) = \frac{\omega r_o}{c}\left[\sin\theta\sin\phi_n\,\hat{\boldsymbol{r}} + \cos\theta\sin\phi_n\hat{\boldsymbol{\theta}} + \cos\phi_n\hat{\boldsymbol{\phi}}\right] \tag{1.48}$$

It has already been shown that the dipole electric flux vector vanishes for the ring of charge and this holds true even if the charges are in motion. However, dipole electric flux from each charge still produces a contribution to a mangetic dipole field which can be calculated from

$$\boldsymbol{\pi}_{B1} = \sum_{n=1}^{N} \boldsymbol{\beta} \times \boldsymbol{\pi}_{E1}[n]$$

$$= \frac{1}{\mathrm{a}_e} \sum_{n=1}^{N} [A_1 \boldsymbol{\beta} \times \boldsymbol{A}_0 + A_0 \boldsymbol{\beta} \times \boldsymbol{A}_1] \tag{1.49}$$

Performing the summations leads to

$$\boldsymbol{\pi}_{B1} = \frac{1}{\mathrm{a}_e} \left[\frac{Ne^2 \omega r_o^2}{2r^3 c} \left(2\cos\theta\, \hat{\boldsymbol{r}} + \sin\theta\, \hat{\boldsymbol{\theta}} \right) \right] \tag{1.50}$$

Now define the current in the loop by $I = Ne\nu$ and the z-directed magnetic dipole moment

$$\boldsymbol{m} = I\pi r_o^2 \hat{\boldsymbol{z}} \tag{1.51}$$

In coordinate free form the dipole magnetic flux field can then be written

$$\boldsymbol{\pi}_{B1} = \frac{\sigma_e}{r^3 c} [3(\boldsymbol{m} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}} - \boldsymbol{m}] \tag{1.52}$$

It might be suggested that the results of this section qualify as a semi-classical approach to a determination of electric and magnetic flux fields since the calculations are performed by populating the ring (a crystal lattice of copper, maybe) with individual quanta of the electromagnetic field. A more complete approach might include statistics and the law of averages over a large value of N which—for a one foot long copper wire—would be $N \sim 10^{23}$. With or without statistics however, it is important to observe that no integrations or differential operations are required to determine the fields. Instead, the vacuum gauge potentials take care of everything as they represent smallest possible division of the classical electromagnetic field.

Vector Potential: As a final calculation it is important to determine an appropriate potential which will render the static dipole field of equation (1.52). This can be done using the potentials in (1.38) but it will be necessary to make careful evaluations of the retarded time. The angular coordinate of the n^{th} electron is

$$\phi' = \omega t_r + \frac{2\pi n}{N}$$

$$\sim \omega(t - r/c) - \frac{\omega r_o}{c} \sin\theta \cos(\phi - \phi') + \frac{2\pi n}{N} \tag{1.53}$$

and it will be necessary to insert this into equation (1.38b). However, if $\omega r_o/c$ is considered to be a small quantity then it will be prudent to first define the angle

$$\psi_n = \phi - \omega(t - r/c) - \frac{2\pi n}{N} \tag{1.54}$$

Expansions of the sines and cosines are

$$\cos\left[\psi_n + \frac{\omega r_o}{c}\sin\theta\cos(\phi-\phi')\right] \sim \cos\psi_n - \frac{\omega r_o}{c}\sin\theta\cos\psi_n\sin\psi_n \tag{1.55a}$$

$$\sin\left[\psi_n + \frac{\omega r_o}{c}\sin\theta\cos(\phi-\phi')\right] \sim \sin\psi_n + \frac{\omega r_o}{c}\sin\theta\cos^2\psi_n \tag{1.55b}$$

Replacing the components of $\boldsymbol{A}_1$ with these expansions and summing for all n, it becomes evident that the only non-zero contribution to $\boldsymbol{A}_1$ will come from the azimuthal component so that the total field will be

$$\begin{aligned}\boldsymbol{A}_1 &= \sum_{n=1}^{N}\frac{er_o}{r^2}\left[\sin\psi_n + \frac{\omega r_o}{c}\sin\theta\cos^2\psi_n\right]\hat{\boldsymbol{\phi}} \\ &= \frac{Ne\omega r_o^2}{2r^2c}\left[\frac{\sin\theta}{r^2}\right]\hat{\boldsymbol{\phi}}\end{aligned} \tag{1.56}$$

The complete potential will also contain contributions from the monopole potential $\boldsymbol{A}_0$. With definitions of total charge and magnetic dipole moment already given, the coordinate–free form of the vector potential through first order is

$$\boldsymbol{A} = \frac{Q}{r}\hat{\boldsymbol{r}} + \frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{r^2c} \tag{1.57}$$

The dipole term is just the lorentz gauge result characterized by $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. The divergence of the entire equation is then just the vacuum gauge condition applied to the collection of charges.

2 Multipole Acceleration Fields

The proportionality between fields and potentials can be extended to particle accelerations beginning with the covariant expression

$$F_a^{\mu\nu} = [A^\mu, a_\perp^\nu] \tag{2.1}$$

This gives electric and magnetic field vectors

$$\mathbf{E}_a = \boldsymbol{A}a_\perp^o - A\,\boldsymbol{a}_\perp \tag{2.2a}$$
$$\mathbf{B}_a = \boldsymbol{a}_\perp \times \boldsymbol{A} \tag{2.2b}$$

It is easy to verify the expression $\mathbf{B}_a = \hat{\boldsymbol{n}} \times \mathbf{E}_a$ by operating on the electric field vector. A general formula for radiated power also follows by making substitutions into the Poynting vector. With help from the orthogonality relation $a_\perp^\nu R_\nu = 0$, and including relevant factors of c, the equivalent formula is

$$\mathbf{S} = \frac{1}{4\pi c^3}\left[\boldsymbol{a}_\perp \cdot \boldsymbol{a}_\perp - a_\perp^o(\hat{\boldsymbol{n}} \cdot \boldsymbol{a}_\perp)\right] A\boldsymbol{A} \tag{2.3}$$

This is still the Poynting formula which can be integrated over a closed surface to determine the Liénard generalization of the Larmor power formula. This is a difficult integration since components of $a_\perp^\mu$ are complicated functions of the velocity and acceleration in the relativistic limit. In the low velocity limit however, approximations through first order in $\boldsymbol{\beta}$ are

$$a_\perp^o = \boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} - (\hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{\beta}})\hat{\boldsymbol{n}} \cdot \boldsymbol{\beta} \tag{2.4a}$$

$$\boldsymbol{a}_\perp = \dot{\boldsymbol{\beta}} - (\hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{\beta}})\hat{\boldsymbol{n}} + (\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})\hat{\boldsymbol{n}} - 2(\hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})(\hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{\beta}})\hat{\boldsymbol{n}} + (\hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{\beta}})\boldsymbol{\beta} \tag{2.4b}$$

In zeroeth order, $a_\perp^o$ vanishes and the Larmor power formula can be verified from the closed integral

$$P_{larmor} = \frac{1}{4\pi c^3}\oint \boldsymbol{a}_\perp \cdot \boldsymbol{a}_\perp A\boldsymbol{A} \cdot d\boldsymbol{s} \tag{2.5}$$

2.1 Electric Dipole Radiation

The simplest application of the new formalism is the determination of acceleration fields for an electron oscillating about the origin. Let

$$\boldsymbol{z} = z_o \cos\omega t_r \hat{\boldsymbol{z}} \tag{2.6}$$

Two time derivatives and ignoring the retardation condition in the cosine produce

$$\boldsymbol{a} = -\omega^2 z_o \cos\omega(t - r/c)\hat{\boldsymbol{z}} \tag{2.7}$$

The perpendicular acceleration is then determined to be

$$\boldsymbol{a}_\perp = \omega^2 z_o \cos\omega(t - r/c)\sin\theta\,\hat{\boldsymbol{\theta}} \tag{2.8}$$

Now insert this into (2.2) using only the lowest order potentials

$$A^\nu = \left(\frac{e}{r}, \frac{e}{r}\hat{\boldsymbol{r}}\right) \tag{2.9}$$

which leads to explicit expressions for the fields

$$\mathbf{E}_1 = -\frac{e\omega^2 z_o}{rc^2}\sin\theta\cos\omega(t - r/c)\hat{\boldsymbol{\theta}} \tag{2.10a}$$

$$\mathbf{B}_1 = -\frac{e\omega^2 z_o}{rc^2}\sin\theta\cos\omega(t - r/c)\hat{\boldsymbol{\phi}} \tag{2.10b}$$

Defining the dipole moment $\boldsymbol{p} = ez_o\hat{\boldsymbol{z}}$ and including a complex phase will then allow the fields to be written

$$\mathbf{E}_1 = k^2(\hat{\boldsymbol{r}} \times \boldsymbol{p} \times \hat{\boldsymbol{r}})\frac{e^{-i\omega(t-r/c)}}{r} \tag{2.11a}$$

$$\mathbf{B}_1 = k^2(\hat{\boldsymbol{r}} \times \boldsymbol{p})\frac{e^{-i\omega(t-r/c)}}{r} \tag{2.11b}$$

It is important to compare the derivation of the fields in (2.10) with conventional textbook derivations—for example, Griffiths *Introduction to Electrodynamics*, third edition. The vacuum gauge solution is very precise and focuses directly on the motion of the source instead of relying heavily on the potentials which must be differentiated in conventional calculations.

To complete the calculation requires a determination of the total power radiated. For this it is important to include the lowest order inertial energy flux from the velocity theory giving a total average flux

$$\langle\, \mathbf{S}\, \rangle = \frac{c}{8\pi}\left[\frac{1}{r_e^2} + k^4 z_o^2 \sin^2\theta\right] A\boldsymbol{A} \tag{2.12}$$

The total power radiated will then include the inertial power

$$P_{tot} = P_{in} + P_{dip} \tag{2.13}$$

2.2 Electric Quadrupole Radiation

Another example of the utility of equation (2.2) is that of two electrons oscillating in the z-direction near the origin of a coordinate system as shown in figure 3. The positions of the particles as a function of time are given by

$$\boldsymbol{z}_{up}(t) = +\frac{d}{2}\left[1 + \sin\omega t_r\right]\hat{\boldsymbol{z}} \tag{2.14a}$$

$$\boldsymbol{z}_{dn}(t) = -\frac{d}{2}\left[1 + \sin\omega t_r\right]\hat{\boldsymbol{z}} \tag{2.14b}$$

This configuration of charge has no dipole moment at any time so the leading order radiation term will be the quadrupole. To solve for the acceleration fields it is useful to first write down scalar and vector potentials to leading order in the quantity d/r. Expansions will initially include the sinusoidal time dependence of each particle but this term can be averaged to zero in the expansions. Vector potentials are

$$\boldsymbol{A}_{up} = \frac{e}{r}\left[(1 + \frac{d}{2r}\cos\theta)\,\hat{\boldsymbol{r}} + \frac{d}{2r}\sin\theta\,\hat{\boldsymbol{\theta}}\right] \tag{2.15a}$$

$$\boldsymbol{A}_{dn} = \frac{e}{r}\left[(1 - \frac{d}{2r}\cos\theta)\,\hat{\boldsymbol{r}} - \frac{d}{2r}\sin\theta\,\hat{\boldsymbol{\theta}}\right] \tag{2.15b}$$

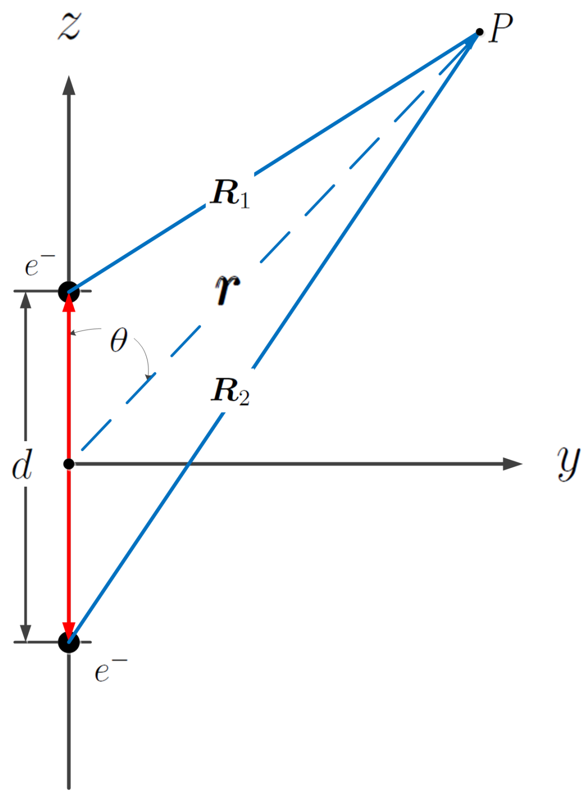

Figure 3: *Configuration of two oscillating electrons producing quadrupole radiation in the far field limit.*

The scalar potentials can either be derived from a separate expansion or they can be determined by taking the magnitude of the vector potential leading to

$$A_{up} = \frac{e}{r}\left[1 + \frac{d}{2r}\cos\theta\right] \tag{2.16}$$

$$A_{dn} = \frac{e}{r}\left[1 - \frac{d}{2r}\cos\theta\right] \tag{2.17}$$

Accelerations of the particles follow from two time derivatives

$$\boldsymbol{a}_{up} = -\frac{d\omega^2}{2}\sin\omega\left(t - \tfrac{1}{c}|\boldsymbol{r} - \boldsymbol{d}/2|\right)\hat{\boldsymbol{z}} \tag{2.18}$$

$$\boldsymbol{a}_{dn} = +\frac{d\omega^2}{2}\sin\omega\left(t - \tfrac{1}{c}|\boldsymbol{r} + \boldsymbol{d}/2|\right)\hat{\boldsymbol{z}} \tag{2.19}$$

Expanding inside the sine functions and noting that perpendicular accelerations $\boldsymbol{a}_\perp$ result by replacing the unit vector $\hat{\boldsymbol{z}}$ with the quantity $-\sin\theta\,\hat{\boldsymbol{\theta}}$ gives the approximations

$$\boldsymbol{a}_{\perp up} = +\frac{d\omega^2}{2}\left[\sin\omega(t - r/c) + \frac{\omega d}{2c}\cos\theta\cos\omega(t - r/c)\right]\sin\theta\,\hat{\boldsymbol{\theta}} \tag{2.20}$$

$$\boldsymbol{a}_{\perp dn} = -\frac{d\omega^2}{2}\left[\sin\omega(t - r/c) - \frac{\omega d}{2c}\cos\theta\cos\omega(t - r/c)\right]\sin\theta\,\hat{\boldsymbol{\theta}} \tag{2.21}$$

Again, we consider only lowest order in accelerations so that $a^o_\perp = 0$ for both particles implying a quadrupole electric field vector given by

$$\mathbf{E}_2 = -A_{up}\,\boldsymbol{a}_{\perp up} - A_{dn}\,\boldsymbol{a}_{\perp dn} \tag{2.22}$$

An explicit expression for the resulting electric field vector is then

$$\mathbf{E}_2 = \frac{ed\omega^2}{rc^2}\sin\theta\cos\theta\left[\frac{d}{r}\sin\omega(t - r/c) + \frac{\omega d}{c}\cos\omega(t - r/c)\right]\hat{\boldsymbol{\theta}} \tag{2.23}$$

In the far field limit use $r \gg c/\omega$ and the quadrupole field reduces to

$$\mathbf{E}_2 = \frac{e\omega^3 d^2}{rc^3}\sin\theta\cos\theta\cos\omega(t - r/c)\,\hat{\boldsymbol{\theta}} \tag{2.24}$$

Obviously, this result could have been obtained more simply by considering only lowest order potentials. The magnetic field vector can be determined directly from the electric field vector but it may also be determined as

$$\mathbf{B}_2 = \boldsymbol{a}_{\perp up} \times \boldsymbol{A}_{up} + \boldsymbol{a}_{\perp dn} \times \boldsymbol{A}_{dn} \tag{2.25}$$

Putting together equations (2.15) and (2.20) confirms that

$$\mathbf{B}_2 = \hat{\boldsymbol{r}} \times \mathbf{E}_2 \tag{2.26}$$

Quadrupole Tensor: It is important to write quadrupole electric and magnetic field strengths in terms of a quadrupole tensor This tensor is twice that of equation (1.31) since both electrons contribute. In terms of the quadrupole tensor and rewriting in terms of complex exponentials one finds

$$\mathbf{E}_{_2} = \frac{2ik^3}{3}\left[\hat{\boldsymbol{r}} \times (\widehat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}}) \times \hat{\boldsymbol{r}}\right]\frac{e^{i\omega(t-r/c)}}{r} \tag{2.27a}$$

$$\mathbf{B}_{_2} = \frac{2ik^3}{3}\left[\hat{\boldsymbol{r}} \times \widehat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}}\right]\frac{e^{i\omega(t-r/c)}}{r} \tag{2.27b}$$

Finally, the time averaged momentum flux can be calculated either from equation (2.3) or from the complex Poynting vector

$$\mathbf{S} = \frac{c}{8\pi}(\mathbf{E} \times \mathbf{B}^*) \tag{2.28}$$

Power radiated per unit solid angle including two instances of the inertial power will then be

$$\frac{dP}{d\Omega} = \sigma_e ec + \frac{ce^2 d^4 k^6}{36\pi}\sin^2\theta\cos^2\theta \tag{2.29}$$

2.3 Magnetic Dipole Radiation

Acceleration fields produced by an oscillating magnetic dipole can be calculated using the ring of charge in section 1.3 except that each of the N electrons in the ring must undergo oscillations about their specificed angle $2\pi n/N$. If α is a small angle in radians, then the position of each charge can be written

$$\boldsymbol{s}_n(t) = r_o \cos\left(\phi_n + \alpha \cos \omega t_r\right) \hat{\boldsymbol{x}} + r_o \sin\left(\phi_n + \alpha \cos \omega t_r\right) \hat{\boldsymbol{y}} \tag{2.30}$$

$$\approx r_o \left(\cos \phi_n - \alpha \sin \phi_n \cos \omega t_r\right) \hat{\boldsymbol{x}} + r_o \left(\sin \phi_n + \alpha \cos \phi_n \cos \omega t_r\right) \hat{\boldsymbol{y}} \tag{2.31}$$

The magnitude of this vector to lowest order in α is just r_o while velocity and acceleration vectors are

$$\boldsymbol{v}_n(t) = \alpha \omega r_o \sin \omega t_r \left[\sin \phi_n \hat{\boldsymbol{x}} - \cos \phi_n \hat{\boldsymbol{y}}\right] \tag{2.32}$$

$$\boldsymbol{a}_n(t) = \alpha \omega^2 r_o \cos \omega t_r \left[\sin \phi_n \hat{\boldsymbol{x}} - \cos \phi_n \hat{\boldsymbol{y}}\right] \tag{2.33}$$

and in the spherical-polar coordinate system the acceleration vector perpendicular to $\hat{\boldsymbol{r}}$ reads

$$\boldsymbol{a}_{\perp n}(t) = \alpha \omega^2 r_o \cos \omega t_r \left[\cos\theta \sin(\phi - \phi_n)\, \hat{\boldsymbol{\theta}} - \cos(\phi - \phi_n) \hat{\boldsymbol{\phi}}\right] \tag{2.34}$$

Since the n^{th} particle is oscillating at the retarded time we expand

$$\cos \omega[t - |\boldsymbol{r} - \boldsymbol{s}_n|/c] \approx \cos \omega(t - r/c) - \frac{\omega r_o}{c} \sin\theta \cos(\phi - \phi_n) \sin \omega(t - r/c) \tag{2.35}$$

The value of $\boldsymbol{a}_\perp$ summed over all particles is

$$\sum_{n=1}^{N} \boldsymbol{a}_{\perp n} = \frac{N \alpha \omega^3 r_o^2}{2c} \sin\theta \sin \omega(t - r/c) \hat{\boldsymbol{\phi}} \tag{2.36}$$

An extension of equations (2.2) is then

$$\mathbf{E}_a = -A \sum_{n=1}^{N} \boldsymbol{a}_{\perp n} \qquad\qquad \mathbf{B}_a = -\boldsymbol{A} \times \sum_{n=1}^{N} \boldsymbol{a}_{\perp n} \tag{2.37}$$

which renders the fields

$$\mathbf{E}_a = -\frac{N e \alpha \omega^3 r_o^2}{2c^3} \left[\frac{\sin\theta}{r}\right] \sin \omega(t - r/c)\, \hat{\boldsymbol{\phi}} \tag{2.38a}$$

$$\mathbf{B}_a = +\frac{N e \alpha \omega^3 r_o^2}{2c^3} \left[\frac{\sin\theta}{r}\right] \sin \omega(t - r/c)\, \hat{\boldsymbol{\theta}} \tag{2.38b}$$

To complete the calculation define the current and magnetic dipole moment by

$$I(t_r) = Ne\alpha\nu \sin \omega t_r \qquad\qquad \boldsymbol{m}(t_r) = I(t_r)\pi r_o^2 \hat{\boldsymbol{z}} \tag{2.39}$$

allowing the dipole fields to be written more concisely as

$$\mathbf{E}_a = -\frac{m_o\,\omega^2}{c}\left[\frac{\sin\theta}{r}\right]\sin\omega(t - r/c)\,\hat{\boldsymbol{\phi}} \tag{2.40a}$$

$$\mathbf{B}_a = +\frac{m_o\,\omega^2}{c}\left[\frac{\sin\theta}{r}\right]\sin\omega(t - r/c)\,\hat{\boldsymbol{\theta}} \tag{2.40b}$$

The average energy flux from this charge configuration follows from

$$\langle\,\mathbf{S}\,\rangle = \frac{1}{4\pi c^3}\langle\,\boldsymbol{a}_\perp \cdot \boldsymbol{a}_\perp\rangle A\boldsymbol{A} = \frac{m_o^2\omega^4}{8\pi c^5}\left[\frac{\sin^2\theta}{r^2}\right]\hat{\boldsymbol{r}} \tag{2.41}$$

Once again, the vacuum gauge reigns supreme in its ability to provide simple, straightforward calculations of complex quantities. All logical steps are purely algebraic and require no knowledge of vector calculus.