VG-4
Vacuum Gauge
Electron
in the
Spherical Basis

Coordinate Transformation

One of the more striking features of vacuum gauge electrodynamics is the ability to write the entire theory in terms of a spherically based 4-coordinate system. In an arbitrary frame, this transformation is

$$\boxed{x^\nu = c\tau\beta^\nu + \rho\,\mathcal{U}^\nu}$$

Generally, one has $\mathcal{U}^\nu = \mathcal{U}^\nu(\theta,\phi)$ so the coordinate transformation is

$$x^\nu = x^\nu(c\tau,\rho,\theta,\phi)$$

The time and space coordinates are scalar invariants while the two angles are measured with respect to the retarded position of the charge.

Basis Vectors

A set of mutually orthogonal basis vectors is

$$\vec{\mathrm{e}}_\tau = \frac{\partial x_\nu}{\partial c\tau} = \beta_\nu \qquad \vec{\mathrm{e}}_\rho = \frac{\partial x_\nu}{\partial \rho} = \mathcal{U}_\nu$$

$$\vec{\mathrm{e}}_\theta = \frac{\partial x_\nu}{\partial \theta} = R\,\theta_\nu \qquad \vec{\mathrm{e}}_\phi = \frac{\partial x_\nu}{\partial \phi} = R\sin\theta\,\phi_\nu$$

Basis vectors on the top row are four-vectors while the two angular basis vectors are proportional to four-vectors. A set of scale factors is

$$h_\tau = 1 \qquad h_\rho = 1 \qquad h_\theta = R \qquad h_\phi = R\sin\theta$$

Unit Vectors

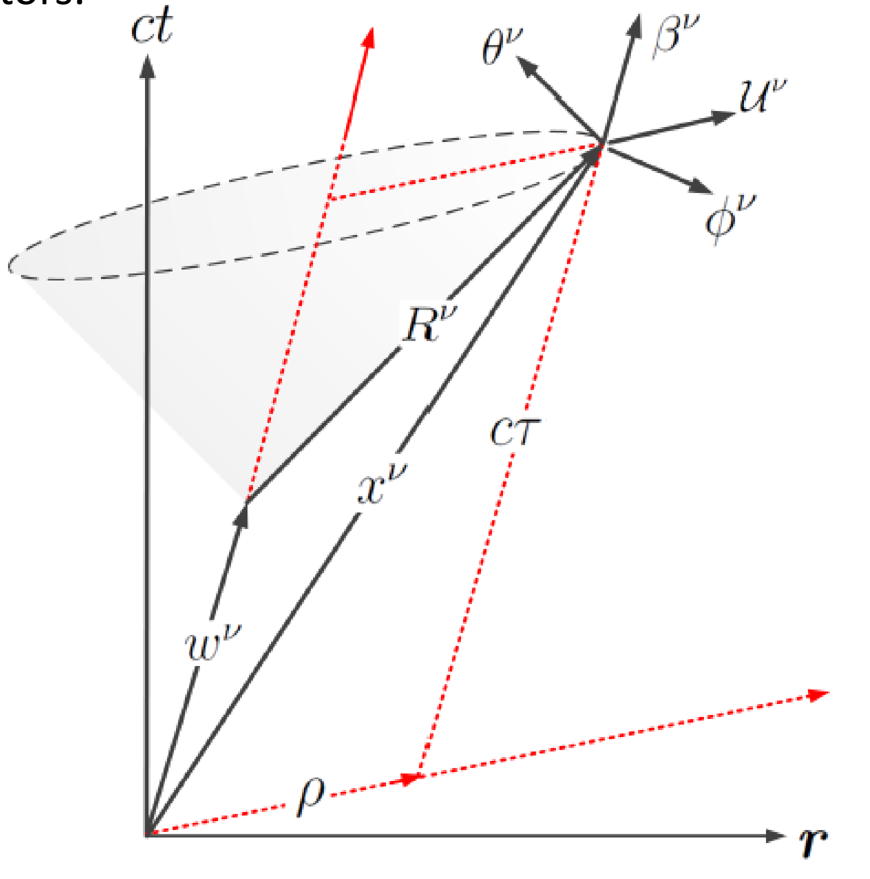
As with 3D curvi-linear coordinate systems, it is much easier to work with unit vectors in the 4-space theory instead of basis vectors.

$$\beta^\nu = \begin{bmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{bmatrix}$$

$$\mathcal{U}^\nu = \frac{1}{\gamma(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})}\begin{bmatrix} 1 \\ \hat{\boldsymbol{n}} \end{bmatrix} - \begin{bmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{bmatrix}$$

$$\theta^\nu = \frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})}\begin{bmatrix} \boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\theta}} + \boldsymbol{\beta}\times\hat{\boldsymbol{\phi}} \end{bmatrix}$$

$$\phi^\nu = \frac{1}{(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})}\begin{bmatrix} \boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\phi}} - \boldsymbol{\beta}\times\hat{\boldsymbol{\theta}} \end{bmatrix}$$



Interval and Metric Tensor

Normalization of the set of 4-vectors is easily verified

$$\beta^\nu \beta_\nu = 1 \qquad \mathcal{U}^\nu \mathcal{U}_\nu = -1 \qquad \theta^\nu \theta_\nu = -1 \qquad \phi^\nu \phi_\nu = -1$$

and these are accompanied by 6 orthogonality relations. The interval associated with the coordinate transformation is

$$ds^2 = c^2 d\tau^2 - d\rho^2 - \frac{\rho^2}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2}\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

The functional form here shows how angular displacements becomes distorted (or Doppler Shifted) when viewed relative to a moving frame. The metric remains flat however. In terms of the basis vectors it has the form

$$\mathrm{g}_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2\sin^2\theta \end{bmatrix}$$

Vectors and Tensors

An arbitrary vector in the spherical basis may be written

$$V^\nu = (V^\alpha \beta_\alpha)\beta^\nu - (V^\alpha \mathcal{U}_\alpha)\mathcal{U}^\nu - (V^\alpha \theta_\alpha)\theta^\nu - (V^\alpha \phi_\alpha)\phi^\nu$$

As an example, the vacuum gauge velocity potentials may be represented by

$$A_v^\nu = \left[\frac{e}{\rho}, \frac{e}{\rho}, 0, 0\right]$$

More generally, a second rank tensor will have projections along 16 curvi-linear components as shown by the graphic at right.

	β^ν	$\mathcal{U}^\nu$	θ^ν	ϕ^ν
β^μ	$\times$	$\times$	$\times$	$\times$
$\mathcal{U}^\mu$	$\times$	$\times$	$\times$	$\times$
θ^μ	$\times$	$\times$	$\times$	$\times$
ϕ^μ	$\times$	$\times$	$\times$	$\times$

Field Strength and Vacuum Tensors

Both the (velocity) field strength tensor and the vacuum tensor occupy 2D subspaces in the curvi-linear basis and are represented by

$$F_v^{\mu\nu} = \begin{bmatrix} 0 & -\eta \\ \eta & 0 \end{bmatrix} \qquad \Delta^{\mu\nu} = \begin{bmatrix} -\eta & 0 \\ \eta & 2\eta \end{bmatrix}$$

In general, the derivative in an arbitrary coordinate system includes Christoffel symbols and may be written

$$T^{\mu\nu}{}_{;\alpha} = T^{\mu\nu}{}_{,\alpha} + T^{\lambda\nu}\,\Gamma^{\mu}_{\lambda\alpha} + T^{\mu\lambda}\,\Gamma^{\nu}_{\lambda\alpha}$$

For $\alpha \to \mu$, this is a divergence operation which can be applied to the field strength and the vacuum tensor showing that

$$F^{\mu\nu}{}_{;\mu} = \frac{4\pi}{c} j_e^{*\nu} \qquad \Delta^{\mu\nu}{}_{;\mu} = \frac{4\pi}{c} j_e^{*\nu}$$

Acceleration Strain

In terms of vacuum gauge potentials the acceleration strain tensor is

$$\epsilon^{\mu\nu} \equiv A^{\mu} a_{\perp}^{\nu}$$

Define $$\mathcal{F}_a \equiv -\frac{e}{\rho}\begin{bmatrix} a_\theta & a_\phi \\ a_\theta & a_\phi \end{bmatrix}$$ where $$a_\theta \equiv a^\nu \theta_\nu \qquad a_\phi \equiv a^\nu \phi_\nu$$

In the spherical basis the acceleration strain and its conjugate occupy separate subspaces

$$\epsilon^{\mu\nu} = \begin{bmatrix} 0 & \mathcal{F}_a \\ 0 & 0 \end{bmatrix} \qquad \epsilon^{\nu\mu} = \begin{bmatrix} 0 & 0 \\ \mathcal{F}_a^{\dagger} & 0 \end{bmatrix}$$

Now include the velocity field strengths and write the total field strength tensor in the spherical basis as

$$F^{\mu\nu} = \begin{bmatrix} \mathcal{F}_v & \mathcal{F}_a \\ -\mathcal{F}_a^{\dagger} & 0 \end{bmatrix}$$

Symmetric Stress Tensor

It has already been shown that the symmetric stress tensor can be written in terms of velocity and acceleration strains as

$$\Theta^{\mu\nu}_{vac} = \frac{1}{4\pi}\left[\tfrac{1}{2}g^{\mu\nu}\eta^2 - \eta^{\mu\lambda}\eta^{\nu}{}_{\lambda} + \eta^{\mu\lambda}\epsilon^{\nu}{}_{\lambda} - \epsilon^{\mu\lambda}\epsilon^{\nu}{}_{\lambda} + \epsilon^{\mu\lambda}\eta^{\nu}{}_{\lambda}\right]$$

A representation of this tensor in the spherical basis is quite manageable. This form of the symmetric tensor should be compared with its complicated and unwieldy structure in the Cartesian basis (see Rohrlich).

$$\Theta^{\mu\nu}_{vac} \to \frac{\eta^2}{4\pi}\begin{bmatrix} \frac{1}{2}+\rho^2(a_\theta^2+a_\phi^2) & \rho^2(a_\theta^2+a_\phi^2) & -\rho a_\theta & -\rho a_\phi \\ \rho^2(a_\theta^2+a_\phi^2) & -\frac{1}{2}+\rho^2(a_\theta^2+a_\phi^2) & -\rho a_\theta & -\rho a_\phi \\ -\rho a_\theta & -\rho a_\theta & \frac{1}{2} & 0 \\ -\rho a_\phi & -\rho a_\phi & 0 & \frac{1}{2} \end{bmatrix}$$

Integrals in the Spherical Basis

The spherical 4-coordinate system is well designed to handle integrals which arise when working in the vacuum gauge. Using

$$R = R(\rho, \theta, \phi)$$

The 4-volume element in the spherical system is

$$d^4\mathcal{V} = R^2 d\rho\, d\Omega\, cd\tau$$

Moreover, space-like and time-like hyper-surface elements for the causal theory can be written

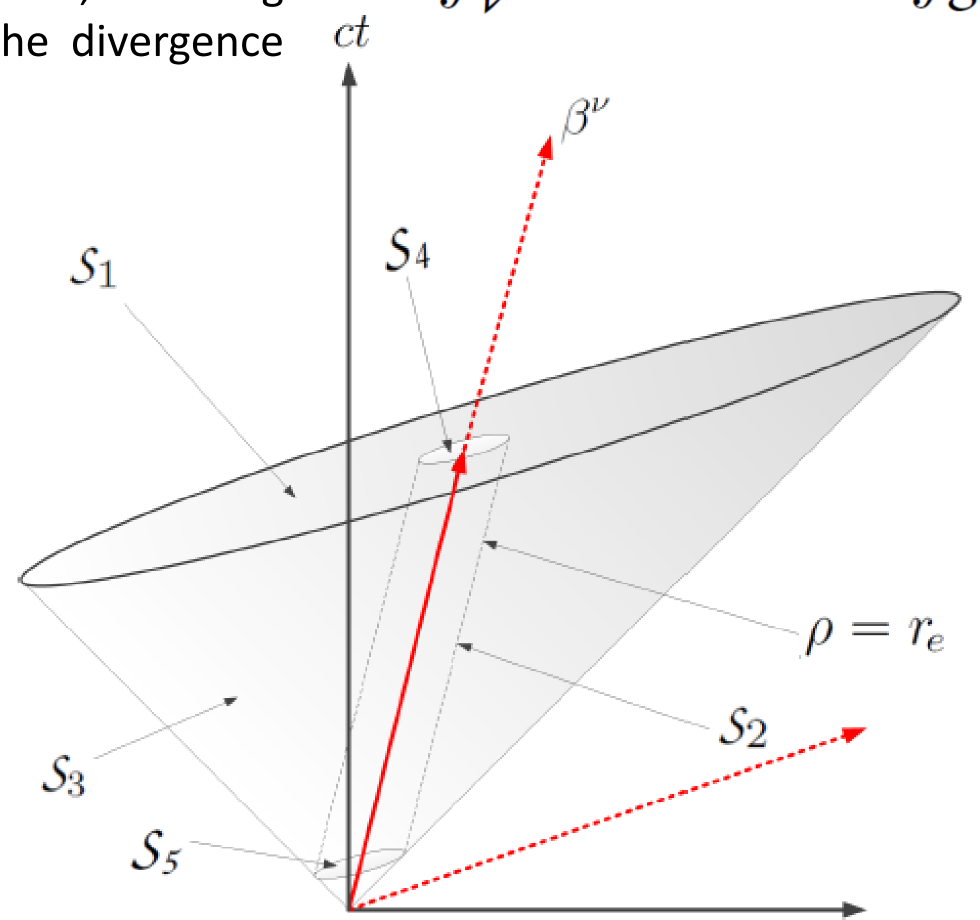
$$d\sigma_s^\mu = \left[R^2 d\rho\, d\Omega\right] \beta^\mu$$

$$d\sigma_\tau^\mu = \left[R^2 cd\tau\, d\Omega\right] \mathcal{U}^\mu$$

More on Integrals

The spherically based system of coordinates can be used to perform many integrals over the casual light cone, including applications of the divergence theorem.

$$\int_{\mathcal{V}} \partial_\nu X^\nu \, d^4\mathcal{V} = \oint_{\mathcal{S}} X^\nu d^3\mathcal{S}_\nu$$



Total Stress Tensor

The vacuum gauge and the spherically-based coordinate system also allows for an Einstein-like derivation of the total electromagnetic stress tensor using the total strain

$$\zeta^{\mu\nu} = \eta^{\mu\nu} - \epsilon^{\mu\nu}$$

Begin with the 4th rank tensor

$$\mathcal{R}^{\mu\lambda\alpha\nu} \equiv \frac{1}{4\pi}\left[\zeta^{\mu\lambda}\zeta^{\alpha\nu} - \zeta^{\lambda\mu}\zeta^{\alpha\nu} + \zeta^{\lambda\mu}\zeta^{\nu\alpha} - \zeta^{\mu\lambda}\zeta^{\nu\alpha}\right]$$

which determines a second rank tensor and its scalar contraction

$$\mathcal{R}^{\mu\nu} \equiv \mathrm{g}_{\lambda\alpha}\mathcal{R}^{\mu\lambda\alpha\nu} = \mathcal{R}^{\mu\lambda}{}_{\lambda}{}^{\nu} \qquad \mathcal{R} = \mathrm{g}_{\mu\nu}\mathcal{R}^{\mu\nu}$$

One can then show that

$$\left(\mathcal{R}^{\mu\nu} - \tfrac{1}{4}\mathrm{g}^{\mu\nu}\mathcal{R}\right)_{;\,\mu} \equiv \Theta^{\mu\nu}_{vac\,;\,\mu}$$

and this derives the total stress tensor, inclusive of the propagation term which is inserted by hand

$$\mathcal{T}^{\mu\nu} = \mathcal{R}^{\mu\nu} - \tfrac{1}{4}\mathrm{g}^{\mu\nu}\mathcal{R} + \Lambda^{\mu\nu}$$

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