Vacuum Dilatation Functions

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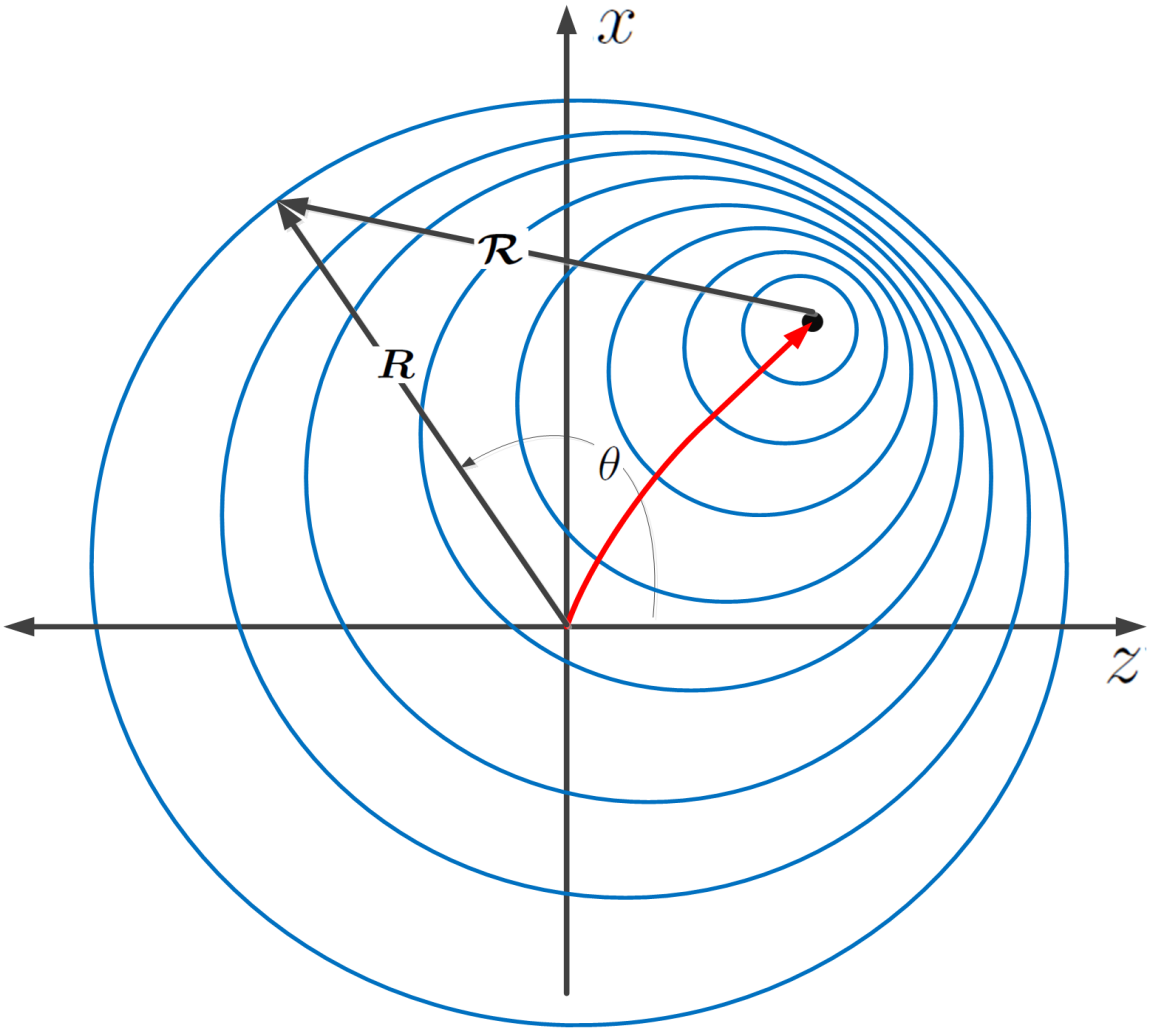

"The concepts of simple charged particles and the electromagnetic field are in some way inconsistent."

R.P. Feynman, *The Feynman Lectures*, chapter 28

Abstract

The charge density of the classical electron can be de-coupled from the vacuum Lagrangian giving way to an R-space Lagrangian for a set of vacuum dilatation functions. Dilatation functions are purely geometrical objects describing a theoretical orifice in the vacuum—they operate in three dimensions and radiate volume only. Dilatation functions may also be analyzed in terms of Fourier modes leading to a Fourier mode Lagrangian along with an associated Fourier mode stres tensor.

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1 Dilatation Functions

The theory of the vacuum gauge electron arises from the necessity to impose the causality principle on its electromagnetic field. These fields may be written

$$\mathbf{E} = \mathbf{E}_{M} \cdot \vartheta \tag{1.1a}$$
$$\mathbf{B} = \mathbf{B}_{M} \cdot \vartheta \tag{1.1b}$$

where the subscript M indicates the well known Maxwell fields and the causality sphere—labeled ϑ—is a function of the retarded time and may also include a phase

$$\vartheta = \vartheta[\gamma^{-1}(ct_r + r_e)] \tag{1.2}$$

Causal fields place the description of an inertial electron into the scientific field of signal theory—even if no measurement if possible. The mathematics appropriate for this description is generated by the covariant ***vacuum gauge*** condition

$$\boxed{\partial_\nu A^\nu_v \equiv \sqrt{\mathrm{E}^2 - \mathrm{B}^2}} \tag{1.3}$$

leading to a de-coupled theory of velocity and acceleration potentials which obey the equations

$$\Box^2 A^\nu_v - \partial^\nu \partial_\mu A^\mu_v = \frac{4\pi}{c} J^{*\nu}_e \qquad\qquad \Box^2 A^\nu_a - \partial^\nu \partial_\mu A^\mu_a = \frac{4\pi}{c} J^\nu_a \tag{1.4}$$

The implications of a de-coupled theory are far reaching since it allows for the development of a theory of moving velocity fields completely independent of the presence of acceleration fields.

The velocity potentials are a covariant null vector with scalar and vector components written separately as

$$A_v = \frac{eR}{\rho^2} \cdot \vartheta \qquad\qquad \boldsymbol{A}_v = \frac{e\boldsymbol{R}}{\rho^2} \cdot \vartheta \tag{1.5}$$

In the Maxwell limit they become continuous functions which may be interpreted in terms of a surface charge density $\sigma_e(\theta, \phi)$ and a set of functions $[\mathrm{u}, \mathbf{u}]$

$$A(\boldsymbol{r}, t) = 4\pi \cdot \sigma_e(\theta, \phi)\, \mathrm{u}(R) \qquad\qquad \boldsymbol{A}(\boldsymbol{r}, t) = 4\pi \cdot \sigma_e(\theta, \phi)\, \mathbf{u}(R) \tag{1.6}$$

Explicit forms of the fields are

$$\sigma_e(\theta, \phi) \equiv \frac{\sigma_e}{\gamma^2(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^2} \qquad\qquad \mathbf{u}(R) \equiv \frac{{r_e}^2}{R}\, \hat{\boldsymbol{n}} \tag{1.7}$$

The angular coordinates (θ, ϕ) are measuered from the retarded position of the charge and it should be observed that the three coordinates (R, θ, ϕ) are all implicit functions of the present time coordinates $(\boldsymbol{r}, t)$. It is suitable to refer to $\mathbf{u}(R)$ as a ***vector***

dilatation and its magnitude u(R) as a ***scalar dilatation*** . Together, they characterize a microscopic spherical aperture in what would otherwise be a continuous vacuum. In what follows, the essential and forthcoming stategy will be the implementation of the theory of vacuum dilatation to propagate the velocity fields of the electron in the form of spherical vacuum waves.

De-coupling of the charge density from the dilatation functions is not limited to vacuum gauge potentials only. With the definition of an R-space gradient operator it is possible to develop a Lagrangian formulation for the dilatation functions which can be used to demonstrate particle stability without the presence of the charge density. The three-space theory can then be linked to the covariant total energy tensor of the electron using an appropriate set of unit vectors derivable from the theory.

1.1 Mathematics of R-Space

In R-space, the components of the vector $\boldsymbol{R}$ from the retarded position of the electron to the present time field point are written in terms of a spherical polar coordinate system

$$R_x = R\sin\theta\cos\phi \qquad R_y = R\sin\theta\sin\phi \qquad R_z = R\cos\theta \tag{1.8}$$

The R-space gradient operator in this coordinate system is

$$\boldsymbol{\nabla}_R \equiv \hat{\boldsymbol{n}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi} \tag{1.9}$$

Now consider the gradient of a function $\boldsymbol{\mathfrak{D}}$ which depends on present time coordinates $(ct, \boldsymbol{r})$ implicitly through $\boldsymbol{R}$. The bold type suggests that $\boldsymbol{\mathfrak{D}}$ is a vector function but it may also be a scalar—or perhaps a tensor of higher rank. Present time differential operators applied to $\boldsymbol{\mathfrak{D}}$ can be written in terms of $\boldsymbol{\nabla}_R$ using the chain rule

$$\boldsymbol{\nabla}\boldsymbol{\mathfrak{D}}(\boldsymbol{R}) = \boldsymbol{\nabla}\boldsymbol{R}\cdot\boldsymbol{\nabla}_R\boldsymbol{\mathfrak{D}} = \left(\mathbf{1} + \frac{\gamma\boldsymbol{R}\boldsymbol{\beta}}{\rho}\right)\cdot\boldsymbol{\nabla}_R\boldsymbol{\mathfrak{D}} \tag{1.10a}$$

$$\frac{\partial\boldsymbol{\mathfrak{D}}(\boldsymbol{R})}{\partial ct} = \frac{\partial\boldsymbol{R}}{\partial ct}\cdot\boldsymbol{\nabla}_R\boldsymbol{\mathfrak{D}} = -\frac{\gamma}{\rho}R\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_R\boldsymbol{\mathfrak{D}} \tag{1.10b}$$

Components of the covariant operator ∂_ν are then

$$\frac{\partial}{\partial ct} = -\frac{\gamma}{\rho}R\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_R \qquad\qquad \boldsymbol{\nabla} = \boldsymbol{\nabla}_R + \frac{\gamma}{\rho}\boldsymbol{R}(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_R) \tag{1.11}$$

A useful bi-product of equations (1.10) is the first order equation

$$\boxed{\boldsymbol{\nabla}\boldsymbol{\mathfrak{D}} + \hat{\boldsymbol{n}}\frac{\partial\boldsymbol{\mathfrak{D}}}{\partial ct} = \boldsymbol{\nabla}_R\boldsymbol{\mathfrak{D}}} \tag{1.12}$$

Among other things, this relation can be used to generate a set of spacelike unit four-vectors appropriate for the vacuum theory.

Application to the Vector Potential: A pair of arbitrary second rank tensors $\boldsymbol{X}$ and $\boldsymbol{Y}$ can be combined with the R-space dot product, and double dot product contractions

$$(\boldsymbol{X}\cdot\boldsymbol{Y})_{ij} \equiv X_{ik}Y_{kj} \qquad \text{and} \qquad \boldsymbol{X}:\boldsymbol{Y} \equiv X_{ij}\,Y_{ji} \tag{1.13}$$

For second rank tensors composed of two vectors, associative properties are

$$\boldsymbol{AB}\cdot\boldsymbol{CD} = \boldsymbol{AD}(\boldsymbol{B}\cdot\boldsymbol{C}) \qquad \boldsymbol{AB}\cdot\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B}\cdot\boldsymbol{C}) \tag{1.14}$$

Armed with this knowledge equation (1.12) can be applied to the vacuum gauge vector velocity potential

$$\boldsymbol{\nabla}\boldsymbol{A} + \hat{\boldsymbol{n}}\frac{\partial \boldsymbol{A}}{\partial ct} = \boldsymbol{\nabla}_{\!R}\boldsymbol{A} \tag{1.15}$$

This is a relation among second rank tensors but it can be transformed into a scalar relation by defining the identity operator (idem factor) from the gradient $\mathbf{1} \equiv \boldsymbol{\nabla}_{\!R}\boldsymbol{R}$. Contracting each term through a double dot product with the identity shows that

$$\mathbf{1}:\boldsymbol{\nabla}\boldsymbol{A} = \boldsymbol{\nabla}\cdot\boldsymbol{A} \tag{1.16a}$$

$$\mathbf{1}:\boldsymbol{\nabla}_{\!R}\boldsymbol{A} = \boldsymbol{\nabla}_{\!R}\cdot\boldsymbol{A} \tag{1.16b}$$

$$\mathbf{1}:\hat{\boldsymbol{n}}\frac{\partial \boldsymbol{A}}{\partial t} = \frac{\partial A}{\partial t} \tag{1.16c}$$

Now evaluate

$$\boldsymbol{\nabla}_{\!R}\cdot\boldsymbol{A} = \frac{e}{\rho^2} \tag{1.17}$$

which re-derives the vacuum gauge condition.

Electron Charge Density: Another application of (1.12) is to the electron charge density:

$$\boldsymbol{\nabla}\sigma_e + \hat{\boldsymbol{n}}\frac{\partial \sigma_e}{\partial ct} = \boldsymbol{\nabla}_{\!R}\sigma_e \tag{1.18a}$$

This derives equations

$$\boldsymbol{\nabla}_{\!R}\sigma_e = \frac{2\gamma}{\rho}\sigma_e\left[\boldsymbol{\beta} - (\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}\right] \tag{1.18b}$$

$$\frac{1}{c}\frac{\partial \sigma_e}{\partial t} = -\frac{2\gamma^2 R}{\rho^2}\sigma_e\left[\beta^2 - (\boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})^2\right] \tag{1.18c}$$

Since the charge density is not a Lorentz scalar then $\partial^{\nu}\sigma_e$ is not expected to be a four-vector.

Spacelike Unit Vectors: One more important use of (1.12) is for the three scalar quantities (ρ, θ, ϕ) defined by

$$\rho = \gamma(R - \boldsymbol{R}\cdot\boldsymbol{\beta}) \qquad \theta = \arctan\left[\frac{(R_x^2 + R_y^2)^{1/2}}{R_z}\right] \qquad \phi = \arctan\left[\frac{R_y}{R_x}\right] \tag{1.19}$$

Gradients and time derivatives for each of these scalars are available in Table 2.2. Of

	ρ	θ	ϕ
1.	$\boldsymbol{\nabla}_R\rho = \gamma(\hat{\boldsymbol{n}} - \boldsymbol{\beta})$	$\boldsymbol{\nabla}_R\theta = \dfrac{\hat{\boldsymbol{\theta}}}{R}$	$\boldsymbol{\nabla}_R\phi = \dfrac{\hat{\boldsymbol{\phi}}}{R\sin\theta}$
2.	$\boldsymbol{\nabla}\rho = \dfrac{\boldsymbol{R}}{\rho} - \gamma\boldsymbol{\beta}$	$\boldsymbol{\nabla}\theta = \dfrac{\gamma}{\rho}\left(\hat{\boldsymbol{\theta}} + \boldsymbol{\beta}\times\hat{\boldsymbol{\phi}}\right)$	$\boldsymbol{\nabla}\phi = \dfrac{\gamma}{\rho}\left(\hat{\boldsymbol{\phi}} - \boldsymbol{\beta}\times\hat{\boldsymbol{\theta}}\right)$
3.	$\dfrac{\partial\rho}{\partial ct} = \gamma - \dfrac{R}{\rho}$	$\dfrac{\partial\theta}{\partial ct} = -\dfrac{\gamma}{\rho}\boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}}$	$\dfrac{\partial\phi}{\partial ct} = -\dfrac{\gamma}{\rho}\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}}$
4.	$\mathcal{U}^\nu = -\partial^\nu\rho$	$\theta^\nu = -R\partial^\nu\theta$	$\phi^\nu = -R\sin\theta\partial^\nu\phi$

Table 1: Gradients and time derivatives applied to the set of scalar fields (ρ, θ, ϕ) using equation (1.12). For all three fields, the time derivatives on line 3 can be determined by dotting the gradients on line 2 with $-\boldsymbol{\beta}$.

the three scalars, only ρ is a Lorentz scalar. This means that four-gradients $\partial^\nu\theta$ and $\partial^\nu\phi$ will not be four-vectors. As the last row in the table shows however, covariance of the angular coordinates can be restored upon multiplication by R and $R\sin\theta$, respectively. Explicit representations of both four-vectors are

$$\theta^\nu = \frac{1}{(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})}\begin{bmatrix}\boldsymbol{\beta}\cdot\hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\theta}} + \boldsymbol{\beta}\times\hat{\boldsymbol{\phi}}\end{bmatrix} \qquad \phi^\nu = \frac{1}{(1 - \boldsymbol{\beta}\cdot\hat{\boldsymbol{n}})}\begin{bmatrix}\boldsymbol{\beta}\cdot\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\phi}} - \boldsymbol{\beta}\times\hat{\boldsymbol{\theta}}\end{bmatrix} \tag{1.20}$$

When combined with the vector $\mathcal{U}^\nu$, they form a viable set of spacelike unit vectors for the Minkowski spacetime. Moreover, the set can also be combined with the timelike

vector β^ν to form the complete set

$$\beta^\nu\beta_\nu = 1 \qquad \mathcal{U}^\nu\mathcal{U}_\nu = -1 \qquad \theta^\nu\theta_\nu = -1 \qquad \phi^\nu\phi_\nu = -1 \tag{1.21}$$

Proof of completeness will require verification of six different orthogonality relations.

1.2 Lagrangian Formulation of the R-space Vacuum

The free field portion of the electron vacuum Lagrangian is constructed from vacuum gauge potentials and takes the form

$$\mathcal{L} = -\frac{1}{8\pi}\left[\partial^\mu A^\nu \partial_\mu A_\nu - (\partial_\nu A^\nu)^2\right] \tag{1.22}$$

Excluding the location of the charge, this Lagrangian correctly predicts the Maxwell-Lorentz equations for the velocity fields. Inserting (1.6) into $\mathcal{L}$ and showing that $\partial^\mu \mathrm{u}^\nu \cdot \mathrm{u}_\nu = 0$ results in the de-coupled Lagrangian

$$\mathcal{L} = b \cdot L/D^4 \tag{1.23}$$

where

$$L = -\frac{1}{2}\left[\partial^\mu \mathrm{u}^\nu \partial_\mu \mathrm{u}_\nu - (\partial_\nu \mathrm{u}^\nu)^2\right] \qquad D = \gamma(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta}) \tag{1.24}$$

The constant term $b = 4\pi\sigma_e^2$ has a value of 4.65×10^{30} N and may be referred to as a vacuum bulk modulus, while the dilatation Lagrangian can be expanded as

$$L = -\frac{1}{2}\left[\left(\frac{\partial \mathrm{u}}{\partial t}\right)^2 - \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} - \boldsymbol{\nabla}\mathrm{u} \cdot \boldsymbol{\nabla}\mathrm{u} + \boldsymbol{\nabla}\mathbf{u} * \boldsymbol{\nabla}\mathbf{u} - \left(\frac{\partial \mathrm{u}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{u}\right)^2\right] \tag{1.25}$$

Unfortunately, this Lagrangian is still not useful because all derivatives are relative to present time and present position coordinates $(\boldsymbol{r}, t)$. However, a transformation to R-space differentials can be implemented using the four relations

$$a.\quad \boldsymbol{\nabla}_R \mathrm{u} = \boldsymbol{\nabla}\mathrm{u} + \frac{1}{c}\frac{\partial \mathrm{u}}{\partial t}\hat{\boldsymbol{n}} \qquad b.\quad \boldsymbol{\nabla}_R \mathbf{u} = \boldsymbol{\nabla}\mathbf{u} + \hat{\boldsymbol{n}}\frac{1}{c}\frac{\partial \mathbf{u}}{\partial t}$$

$$c.\quad \boldsymbol{\nabla}_R \cdot \mathbf{u} = \boldsymbol{\nabla} \cdot \mathbf{u} + \frac{1}{c}\frac{\partial \mathrm{u}}{\partial t} \qquad d.\quad \frac{\partial \mathbf{u}}{\partial t} \cdot \hat{\boldsymbol{n}} = \frac{\partial \mathrm{u}}{\partial t}$$

Inserting these into equation (1.25) will elicit a large set of cancellations among time derivatives. The resulting Lagrangian will then be given exclusively by R-Space derivatives

$$L = \frac{1}{2}\left[\boldsymbol{\nabla}_R \mathrm{u} \cdot \boldsymbol{\nabla}_R \mathrm{u} - \boldsymbol{\nabla}_R \mathbf{u} : \boldsymbol{\nabla}_R \mathbf{u} + (\boldsymbol{\nabla}_R \cdot \mathbf{u})^2\right] \tag{1.26}$$

There are several important features of this Lagrangian. It has a value

$$L = -r_e^4/2R^4 \tag{1.27}$$

which can be inserted into (1.23) to show that the value of $\mathcal{L}$ is unchanged by the transformation. Next, it will be appropriate to include a point source interaction term

$$\epsilon = 4\pi r_e^2 \delta^3(\boldsymbol{R}) \tag{1.28}$$

leading to scalar and vector field Lagrangians

$$L^{(s)} \equiv \frac{1}{2}\boldsymbol{\nabla}_R \mathrm{u} \cdot \boldsymbol{\nabla}_R \mathrm{u} - \epsilon \mathrm{u} \qquad L^{(v)} \equiv -\frac{1}{2}[\boldsymbol{\nabla}_R \mathbf{u} : \boldsymbol{\nabla}_R \mathbf{u} - (\boldsymbol{\nabla}_R \cdot \mathbf{u})^2] \tag{1.29}$$

Both Lagrangians are unitless and parameterized only by the electron radius r_e, while producing equations of motion given by

$$\boldsymbol{\nabla}_R^2 \mathrm{u} = -\epsilon \qquad \boldsymbol{\nabla}_R \times \boldsymbol{\nabla}_R \times \mathbf{u} = 0 \tag{1.30}$$

Properties of $\boldsymbol{\Delta}$: The Lagrangian in (1.26) can be crafted into a more elaborate theory beginning with the definition of a strain tensor $\boldsymbol{\nabla}_R \mathbf{u} \equiv \boldsymbol{\eta}$ with a scalar contraction

$$\eta = \boldsymbol{\nabla}_R \cdot \mathbf{u} = \frac{r_e^2}{R^2} \tag{1.31}$$

An associated stress tensor can be defined by

$$\boxed{\boldsymbol{\Delta} \equiv \tfrac{1}{2}\left(-\boldsymbol{\eta} + \mathbf{1}\eta\right)} \tag{1.32}$$

having a determinant of zero and obeying the divergence relation $\boldsymbol{\nabla}_R \cdot \boldsymbol{\Delta} = \mathbf{0}$. The fact that $\boldsymbol{\Delta}$ is purely symmetric indicates that the stress it induces on the vacuum is not to be associated with any local rotations. Principal axes can be determined by solving the eigenvalue problem with solutions

$$\boldsymbol{\Delta} \cdot \hat{\boldsymbol{n}} = \eta \hat{\boldsymbol{n}} \qquad \boldsymbol{\Delta} \cdot \hat{\boldsymbol{\theta}} = 0 \qquad \boldsymbol{\Delta} \cdot \hat{\boldsymbol{\phi}} = 0 \tag{1.33}$$

Viewing the principal values as coefficients of dilatation along the principal axes establishes that the vacuum can only support a radial stress. This can also be inferred by dividing $\boldsymbol{\Delta}$ into distortional and spherical components and performing integrations over the particle radius. Using $\mathrm{Tr}\boldsymbol{\Delta} = \eta$, results are:

$$\boldsymbol{\Delta}_{dis} = \boldsymbol{\Delta} - \tfrac{1}{3}\mathbf{1}\mathrm{Tr}\boldsymbol{\Delta} \qquad \longrightarrow \qquad \int_{r_e} \boldsymbol{\Delta}_{dis} d^3R = \mathbf{0} \tag{1.34a}$$

$$\boldsymbol{\Delta}_{sph} = \tfrac{1}{3}\mathbf{1}\mathrm{Tr}\boldsymbol{\Delta} \qquad \longrightarrow \qquad \int_{r_e} \boldsymbol{\Delta}_{sph} d^3R = \mathbf{1}\mathcal{V} \tag{1.34b}$$

where the spherical volume is given by $\mathcal{V} = 4\pi r_e^3/3$.

The zero divergence of $\mathbf{\Delta}$ also implies a Gauss integral law. Moreover, the three-dimensional nature of R-space allows for the inclusion of other three-space vectors like electromagnetic velocity fluxes $\boldsymbol{\pi}_{\rm E}$ and $\boldsymbol{\pi}_{\rm B}$ to formulate integrals with $\mathbf{\Delta}$. If $d\boldsymbol{s}$ is the R-space surface element of the particle then two important integrals are:

$$\int_{r_e} \mathbf{\Delta} \cdot d\boldsymbol{s} = \mathbf{0} \qquad\qquad \int_{r_e} (\mathbf{\Delta} \cdot \boldsymbol{\pi}_{\rm E}) \cdot d\boldsymbol{s} = \dot{\varrho} \tag{1.35}$$

The second integral is the total scalar momentum radiated by the particle per unit time. On the other hand, $\mathbf{\Delta} \cdot \boldsymbol{\pi}_{\rm B} = \mathbf{0}$ eliminates the need for a third integral.

Total Stress Tensor: Now suppose a scalar strain variable is defined by $\boldsymbol{\xi} \equiv \boldsymbol{\nabla}_R \mathrm{u}$. In terms of the new variables the interacting Lagrangian density becomes

$$\boxed{L = \tfrac{1}{2}\boldsymbol{\xi} \cdot \boldsymbol{\xi} + \mathbf{\Delta} : \boldsymbol{\eta} - \epsilon \mathrm{u}} \tag{1.36}$$

Each variable in this Lagrangian will contribute to the total stress tensor with independent portions given by

$$\mathbf{T}^{(s)} = \frac{\partial L^{(s)}}{\partial \boldsymbol{\xi}} \boldsymbol{\xi} - \mathbf{1} L^{(s)} = \boldsymbol{\xi}\,\boldsymbol{\xi} - \tfrac{1}{2}\mathbf{1}(\boldsymbol{\xi} \cdot \boldsymbol{\xi}) = -\tfrac{1}{2}\eta\,\boldsymbol{\eta} \tag{1.37a}$$

$$\mathbf{T}^{(v)} = \frac{\partial L^{(v)}}{\partial \boldsymbol{\eta}} \cdot \boldsymbol{\eta} - \mathbf{1} L^{(v)} = 2\mathbf{\Delta} \cdot \boldsymbol{\eta} - \mathbf{1}(\mathbf{\Delta} : \boldsymbol{\eta}) = \eta\,\boldsymbol{\eta} \tag{1.37b}$$

Divergence operations can also be applied yielding a source term for $\mathbf{T}^{(s)}$:

$$\boldsymbol{\nabla}_R \cdot \mathbf{T}^{(s)} = (\boldsymbol{\nabla}_R \cdot \boldsymbol{\xi})\boldsymbol{\xi} + \boldsymbol{\xi} \cdot \boldsymbol{\nabla}_R \boldsymbol{\xi} - \tfrac{1}{2}\boldsymbol{\nabla}_R(\boldsymbol{\xi} \cdot \boldsymbol{\xi}) = -\epsilon\,\boldsymbol{\xi} \tag{1.38a}$$

$$\boldsymbol{\nabla}_R \cdot \mathbf{T}^{(v)} = 2(\boldsymbol{\nabla}_R \cdot \mathbf{\Delta}) \cdot \boldsymbol{\eta} + 2\mathbf{\Delta} \cdot (\boldsymbol{\nabla}_R \cdot \boldsymbol{\eta}) - \boldsymbol{\nabla}_R(\mathbf{\Delta} : \boldsymbol{\eta}) = \mathbf{0} \tag{1.38b}$$

Both $\mathbf{T}^{(s)}$ and $\mathbf{T}^{(v)}$ will need to be added together but first it will be imperative to impose the causality principle on the dilatation functions so that $\mathrm{u}^\nu \to \mathrm{u}^\nu \cdot \vartheta$. The immediate consequence will be the propagation of the field which requires the addition of a linear component to the Lagrangian. This term can be generated in equation (1.36) from a displacement of $\mathbf{\Delta}$ proportional to the identity:

$$\mathbf{\Delta} \longrightarrow \mathbf{\Delta} - \tfrac{1}{2}\mathbf{1} \tag{1.39}$$

Unfortunately, unlike the four-space theory, the resulting Lagrangian will not vanish at the radius of the electron. The reason for this is most likely due to the absence of time-components in the theory but the modified total stress tensor is

$$\boxed{\mathbf{T} = \tfrac{1}{2}\eta\,\boldsymbol{\eta} + \mathbf{\Delta}} \tag{1.40}$$

This new $\mathbf{T}$ still exhibits a zero divergence and the stability of the particle can be addressed by writing

$$\mathbf{T} = \tfrac{1}{2}\left[(\eta - 1)\boldsymbol{\eta} + \mathbf{1}\eta\right] \tag{1.41}$$

Evaluating at $\eta = 1$ removes the quadratic stress and leaves only the metric

$$\mathbf{T} = \tfrac{1}{2}\mathbf{1} \tag{1.42}$$

It is useful to integrate both the quadratic and linear components of (1.40). It is also instructive to introduce the casuality function in equation (1.2) for both integrals. The total volume displaced by the quadratic stress in the limit of large times is

$$\boldsymbol{\mathcal{T}} = \int_{r_e}^{\infty}\int_{\Omega} \tfrac{1}{2}\eta\boldsymbol{\eta}\cdot\vartheta d^3R = \tfrac{1}{2}\mathbf{1}\mathcal{V}\cdot\vartheta(\tau) \tag{1.43}$$

where we assume that $t = \gamma\tau$. In the time interval $c\tau$ an amount of volume

$$\boldsymbol{\mathcal{S}} = \int_{r_e}^{ct+r_e}\int_{\Omega} \boldsymbol{\Delta}\cdot\vartheta d^3R = \mathbf{1}\,\mathcal{V}\frac{c\tau}{r_e} \tag{1.44}$$

is also propagated through the dilatation. Of interest here is a measure of the total volume radiated through the particle radius in time τ given by

$$\mathbf{1} : \boldsymbol{\mathcal{S}} = 4\pi r_e^2 c\tau \tag{1.45}$$

The volume radiated per unit proper time is then given by

$$\frac{dV}{d\tau} = 7.48\times 10^{-21}\,\mathrm{m}^3/\mathrm{s} \tag{1.46}$$

For some perspective, this result can be multiplied by Avogadro's number giving a volume flux of about $4500\,\mathrm{m}^3/\mathrm{s}$.

Re-construction of the Covariant Theory: While the 3-space theory only includes spacelike components, it is still possible to re-construct the covariant theory. First it will be necessary to determine a Hamiltonian h for the volume theory. If a causality operator is included in equation (1.43) then

$$h(\tau) = \mathbf{1} : \boldsymbol{\mathcal{T}}\cdot\vartheta(\tau) - \frac{1}{2}\mathbf{1} : \boldsymbol{\mathcal{S}} \tag{1.47}$$

To re-capture the invariant Hamiltonian of the 4-space theory it is only necessary to multiply this equation by the bulk modulus $b = 4\pi\sigma_e^2$ leading to

$$\mathcal{H}(\tau) = mc^2\cdot\vartheta(\tau) - \tfrac{1}{2}\dot{\rho}c\tau \tag{1.48}$$

While this is an impressive calculation the entire second rank energy tensor follows beginning with the 4×4 construction

$$\mathcal{V}^{\mu\nu} = \begin{bmatrix} h & 0 \\ 0 & \boldsymbol{\mathcal{S}} \end{bmatrix} \tag{1.49}$$

Including a set of independent coordinate vectors this tensor may also be written

$$\mathcal{V}^{\mu\nu} = h(\tau)\hat{\boldsymbol{\tau}}\,\hat{\boldsymbol{\tau}} + \mathcal{V}\frac{c\tau}{r_e}\left[\hat{\mathbf{n}}\,\hat{\mathbf{n}} + \hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\,\hat{\boldsymbol{\phi}}\right] \tag{1.50}$$

Covariance is then established by replacing the coordinate vectors with the complete set introduced in equation (1.21) and multiplying through by the bulk modulus

$$\mathcal{E}^{\mu\nu} = \left(mc^2 - \tfrac{1}{2}\dot{\varrho}c\tau\right)\beta^\mu\beta^\nu + \tfrac{1}{3}\dot{\varrho}c\tau\left(\mathcal{U}^\mu\mathcal{U}^\nu + \theta^\mu\theta^\nu + \phi^\mu\phi^\nu\right) \tag{1.51}$$

The volume theory may also be used to construct the tensor $\mathcal{E}^{\mu\nu}_{rad}$. Here it is necessary to begin with equation (1.42). The covariant tensor has no energy component but it will be necessary to add an off-diagonal time-space component given by[1]

$$\mathbf{T}' = \tfrac{1}{2}\hat{\boldsymbol{\tau}}\,\hat{\mathbf{n}} \tag{1.52}$$

As before covariance is established by writing

$$\mathbf{T} + \mathbf{T}' = \frac{1}{2}\left[\hat{\boldsymbol{\tau}}\,\hat{\mathbf{n}} + \hat{\mathbf{n}}\,\hat{\mathbf{n}} + \hat{\boldsymbol{\theta}}\,\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\,\hat{\boldsymbol{\phi}}\right] \tag{1.53}$$

Now multiply by the bulk modulus, and replace coordinate vectors with their covariant counterparts so that

$$\mathcal{E}^{\mu\nu}_{rad} = \frac{e^2}{8\pi r_e^4}\left[\beta^\mu\mathcal{U}^\nu + \mathcal{U}^\mu\mathcal{U}^\nu + \theta^\mu\theta^\nu + \phi^\mu\phi^\nu\right] \tag{1.54}$$

2 Fourier Analysis of Dilatation Functions

Fourier modes of scalar and vector dilatation functions are derived from the vacuum gauge velocity potentials. Linear second order differential equations solved by these functions are also derived from a Fourier analysis of the vacuum Lagrangian. Differential equations are then solved using the method of Green functions re-producing initial results.

2.1 Fourier Mode Lagrangian

Dilatation Functions from Vacuum Waves: The theory presented thus far treats dilatation functions as essentially continuous fields. However, causality requires a spatial limit to vacuum dilatation governed by the radial step function described in equation (1.2). This function can be precisely defined by

$$\vartheta(t_r + \tau_e) \equiv \begin{cases} \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\sin[\omega(t - R/c + \tau_e)]}{\omega}\,d\omega & t + \tau_e \geq R/c \\ \\ 0 & t + \tau_e < R/c \end{cases} \tag{2.1}$$

[1]The off-diagonal element is not to be associated with any quadratic stresses and is rightfully determined only from the vacuum tensor.

In the causal region the function ϑ propagates travelling waves over all frequencies from the retarded position. These waves have an amplitude falling off like $1/\omega$ and give a value of exactly 1 when summed over all frequencies. An important operation is the four-gradient $\partial^\nu\vartheta$. Operating inside the integrand will produce two δ-functions but only one is kept since the function is defined to be zero in the elsewhere region:

$$\partial^\nu\vartheta = \frac{R^\nu}{\rho}\cdot\delta(t_r+\tau_e) \tag{2.2}$$

One may also use a complex exponential which may be easier to work with

$$\vartheta(ct_r) = \mathrm{Im}\left[\frac{2}{\pi}\int_0^\infty \tilde{\phi}_\omega \frac{e^{i\omega(t-R/c)}}{\omega}\,d\omega\right] \qquad R\le ct+r_e \tag{2.3}$$

This seems to be a useful definition since the radius appears as a phase factor $\tilde{\phi}_\omega = e^{i\omega\tau_e}$ and the time and space variables become separable.

With the inclusion of causality, the vector potential in equation (1.5) can be written

$$\boldsymbol{A}(\boldsymbol{R},t) = 4\pi\cdot\sigma_e(\theta,\phi)\cdot\mathbf{u}(R)\cdot\vartheta \tag{2.4}$$

The charge density bears no direct connection to the causality sphere but the dilatation can be coupled with equation (2.1) leading to the transformation equation

$$\mathbf{u}(R,t) = \mathrm{Im}\left[\frac{2}{\pi}\int_0^\infty \mathbf{u}_\omega(R)e^{i\omega t}\,d\omega\right] \tag{2.5a}$$

$$\mathbf{u}_\omega(R) = \tilde{a}_\omega\frac{e^{-i\omega R/c}}{\omega R}\hat{\boldsymbol{n}} \quad \text{where} \quad \tilde{a}_\omega \equiv r_e^2 e^{i\omega\tau_e} \tag{2.5b}$$

The form of (2.5) shows that $\mathbf{u}(R,t)$ radiates longitudinal travelling spherical waves at all frequencies from the instantaneous retarded position of the particle. Such waves can be referred to as electromagnetic pressure waves or ***vacuum waves***. Two Fourier components have been plotted in figure 1 for reference—appearing as lowest order spherical Bessel functions. It is important to evaluate an individual mode at the electron radius

$$\mathbf{u}_\omega(R,t)\Big|_{r_e} = \frac{1}{\omega}\left[r_e\sin\omega t\,\hat{\boldsymbol{n}}\right] \tag{2.6}$$

According to this equation longitudinal modes at frequency ω are generated by oscillations of the electron radius vector. The factor ω^{-1} is a measure of the mode intensity at the given frequency. A Fourier component of the vector potential evaluated at the radius can be constructed as

$$\boldsymbol{A}_\omega(R,t)\Big|_{r_e} = 4\pi\sigma_e(\theta,\phi)\cdot\frac{1}{\omega}\left[r_e\sin\omega t\,\hat{\boldsymbol{n}}\right] \tag{2.7}$$

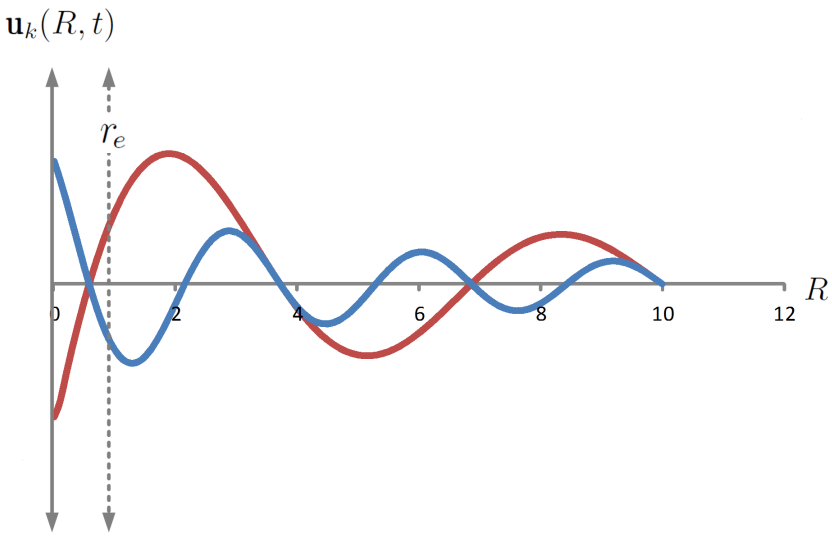

Figure 1: *Two arbitrary Fourier modes of the vacuum dilatation function at time* $ct = 10$ *propagated from the retarded position.*

This indicates the role of the charge density as the directional component of the signal amplitude. If necessary, the entire Fourier analysis can be extended to the scalar component of either the dilatation or the vector potential by simply dropping the polarization vector.

Equations of Motion: In section 1.2 the electron charge density was effectively removed from the vacuum Lagrangian leading to equation (1.25). Beginning from this Lagrangian the next logical step will be to write the dilatation as a sum over Fourier amplitudes. Since all terms in the Lagrangian are quadratic in field quantities it will be necessary to insert the transformation in equation (2.5) twice using different integration variables. Instead of frequencies, it will also be convenient to use waves number k and p resulting in the R-space Fourier mode Lagrangian

$$\begin{aligned} L_{kp} = \frac{1}{2}[&\boldsymbol{\nabla}_R \mathrm{u}_k \cdot \boldsymbol{\nabla}_R \mathrm{u}_p - \boldsymbol{\nabla}_R \mathbf{u}_k : \boldsymbol{\nabla}_R \mathbf{u}_p + (\boldsymbol{\nabla}_R \cdot \mathbf{u}_k)(\boldsymbol{\nabla}_R \cdot \mathbf{u}_p) \\ &+ i(k+p)\frac{\mathrm{u}_k \mathrm{u}_p}{R} + kp\,\mathrm{u}_k \mathrm{u}_p] \cdot e^{i(k+p)ct} \end{aligned} \tag{2.8}$$

having a functional value

$$L_{kp} = -\frac{1}{2R^2}\mathrm{u}_k \mathrm{u}_p \cdot e^{i(k+p)ct} \tag{2.9}$$

This can be immediately re-integrated to show consistency with equation (1.27). Now require $p \to -k$ allowing the Lagrangian to assume a real number while introducing

complex conjugates

$$\mathrm{u}_{-k} = -\mathrm{u}_k^* \qquad \mathbf{u}_{-k} = -\mathbf{u}_k^* \tag{2.10}$$

If complex conjugates are viewed as independent field quantities then it will be necessary to insert delta-function interaction terms for both. Separating into scalar and vector field components as before gives

$$L_k^{(s)} \equiv \frac{1}{2}\left[\boldsymbol{\nabla}_R \mathrm{u}_k \cdot \boldsymbol{\nabla}_R \mathrm{u}_k^* - k^2 \mathrm{u}_k \mathrm{u}_k^*\right] - \epsilon_k \mathrm{u}_k^* - \epsilon_k \mathrm{u}_k \tag{2.11a}$$

$$L_k^{(v)} \equiv -\frac{1}{2}[\boldsymbol{\nabla}_R \mathbf{u}_k : \boldsymbol{\nabla}_R \mathbf{u}_k^* - \boldsymbol{\nabla}_R \cdot \mathbf{u}_k \cdot \boldsymbol{\nabla}_R \cdot \mathbf{u}_k^*] \tag{2.11b}$$

Both Lagrangians are purely real scalars and a minus sign has been intentionally removed as a result of its initial appearance in (2.10). Functional values now include factors of k^2 so—at least for now—they do not have units of energy.

$$L_k^{(s)} = \frac{1}{2R^2}\mathrm{u}_k \mathrm{u}_k^* \qquad\qquad L_k^{(v)} = -\frac{1}{R^2}\mathrm{u}_k \mathrm{u}_k^* \tag{2.12}$$

Helmholtz equations resulting from the Euler-Lagrange equations are

$$\boxed{(\boldsymbol{\nabla}_R^2 + k^2)\,\mathrm{u}_k = -\epsilon_k} \qquad\qquad \boxed{(\boldsymbol{\nabla}_R^2 + k^2)\,\mathbf{u}_k = -\mathbf{J}_k} \tag{2.13}$$

with scalar and vector source densities given by

$$\epsilon_k(\boldsymbol{R}) = \frac{4\pi\tilde{a}_k}{k}\,\delta^3(\boldsymbol{R}) \qquad\qquad \mathbf{J}_k(R) = \frac{2\mathbf{u}_k}{R^2} \tag{2.14}$$

Both second order equations are unitless and couple to unitless source currents.

2.2 Fourier Analysis from First Order Equations

Differential equations (2.13) can also be derived without appealing to the Fourier mode Lagrangian. Instead, sets of coupled first order equations can be determined from either the four-space vacuum tensor or the field strength tensor. The first order analysis also has the benefit of generating additional differential equations not available from the Fourier mode Lagrangian.

First order equations in the continuous limit: The four-space vacuum tensor will generate sets of coupled first-order equations in the quantities (A, $\boldsymbol{A}$) by writing

$$\partial^\mu A^\nu - \mathrm{g}^{\mu\nu}\partial_\alpha A^\alpha = \frac{2R^\mu A^\nu}{\rho^2} - \frac{2\beta^\mu A^\nu}{\rho} - \frac{A^\mu \beta^\nu}{\rho} \tag{2.15}$$

The first column of Table 2.2 shows these equations explicitly. In each equation the potentials can be written in terms of the charge density and the dilatation as

	4-Space	R-Space
1.	$-\boldsymbol{\nabla}\cdot\boldsymbol{A}=\dfrac{2RA}{\rho^2}-\dfrac{3\gamma A}{\rho}$	$\boldsymbol{\nabla}_R\cdot\mathbf{u}=\dfrac{\mathrm{u}}{R}$
2.	$\dfrac{\partial\boldsymbol{A}}{\partial ct}=\dfrac{2R\boldsymbol{A}}{\rho^2}-\dfrac{2\gamma\boldsymbol{A}}{\rho}-\dfrac{\gamma A\boldsymbol{\beta}}{\rho}$	$\dfrac{\partial\mathbf{u}}{\partial ct}=\dfrac{2\gamma\mathbf{u}}{\rho}-\dfrac{2\mathbf{u}}{R}-\dfrac{\gamma\mathrm{u}\boldsymbol{\beta}}{\rho}$
3.	$-\boldsymbol{\nabla}A=\dfrac{2A\boldsymbol{R}}{\rho^2}-\dfrac{2\gamma A\boldsymbol{\beta}}{\rho}-\dfrac{\gamma\boldsymbol{A}}{\rho}$	$\boldsymbol{\nabla}_R\mathrm{u}=-\dfrac{\mathbf{u}}{R}$
4.	$-\boldsymbol{\nabla}\boldsymbol{A}+\mathbf{1}\partial_\nu A^\nu=\dfrac{2\boldsymbol{RA}}{\rho^2}-\dfrac{2\gamma\boldsymbol{\beta A}}{\rho}-\dfrac{\gamma\boldsymbol{A\beta}}{\rho}$	$\boldsymbol{\Delta}=\frac{1}{2}\left(-\boldsymbol{\nabla}_R\mathbf{u}+\mathbf{1}\boldsymbol{\nabla}_R\cdot\mathbf{u}\right)$

Table 2: First order equations satisfied by the velocity potentials in 4-space compared with their R-space counterparts

in equation (1.6). Using the product rule, and introducing R-space derivatives, will then render equations for the dilatation functions shown in the second column of the table. While the time derivatives of A and u are absent, they can be derived by dotting either equation on line 2 with the unit vector $\hat{\boldsymbol{n}}$. Of course all four equations in the second column can also be derived by simply taking derivatives of the functions themselves, but it is important to show how they arise naturally from the covariant theory. First order equations for the charge density can also be generated from the table by removing the differential operations on the dilatation functions instead. It is not difficult to extract all of equations (1.18) by considering various combinations of the 4-space equations.

Fourier modes of First Order equations: The previous analysis becomes more complicated when considering Fourier modes of the potentials. However the overall procedure remains the same. In this case a set of equations similar to the second

column of table 2.2 are

$$\boldsymbol{\nabla}_{R} \cdot \mathbf{u}_k = \left(\frac{1}{R} - ik \right) \mathrm{u}_k \tag{2.16a}$$

$$\frac{\partial \mathbf{u}_k}{\partial t} = \left(\frac{\gamma R}{\rho} - 1 \right) \left(\frac{2}{R} + ik \right) \mathbf{u}_k - \frac{\gamma}{\rho} \mathrm{u}_k \boldsymbol{\beta} \tag{2.16b}$$

$$\boldsymbol{\nabla}_{R} \mathrm{u}_k = - \left(\frac{1}{R} + ik \right) \mathbf{u}_k \tag{2.16c}$$

$$\boldsymbol{\nabla}_{R} \mathbf{u}_k = \frac{\mathrm{u}_k}{R} \mathbf{1} - \mathbf{u}_k \hat{\boldsymbol{n}} \left(\frac{2}{R} + ik \right) \tag{2.16d}$$

The scalar and vector components in (2.16a) and (2.16c) can then be combined to re-produce the second order equations in (2.13).

Vacuum Gauge Condition: A suitable example for generating first order equations uses the vacuum gauge condition written

$$\partial_\nu A^\nu = \frac{e}{\rho^2} \cdot \vartheta \tag{2.17}$$

Inserting equation (1.5) along with the Fourier transformations of (2.3) and (2.5) shows that individual Fourier components are subject to the condition

$$\partial_\nu \left[\sigma_e(\theta, \phi) \mathrm{u}_k^\nu(R) e^{ikct} \right] = \frac{\sigma_e(\theta, \phi) \mathrm{u}_k(R)}{R} e^{ikct} \tag{2.18}$$

Using $\partial_\nu \sigma_e \, \mathrm{u}_k^\nu = 0$ will then generate two first order equations

$$\boldsymbol{\nabla}_{R} \cdot \mathbf{u}_k = \left(\frac{1}{R} - ik \right) \mathrm{u}_k \qquad\qquad \frac{1}{c} \frac{\partial \sigma_e}{\partial t} + \boldsymbol{\nabla} \sigma_e \cdot \hat{\boldsymbol{n}} = 0 \tag{2.19}$$

Field Strength Tensor: The velocity portion of the field strength tensor is another source of first order equations when written

$$\partial^\mu A^\nu - \partial^\nu A^\mu = \frac{1}{\rho} \left[A^\mu, \, \beta^\nu \right] \tag{2.20}$$

Equating individual components implies

$$-\boldsymbol{\nabla} A - \frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial t} = \frac{\gamma}{\rho} \left(\boldsymbol{A} - A \boldsymbol{\beta} \right) \qquad\qquad \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{\gamma}{\rho} (\boldsymbol{\beta} \times \boldsymbol{A}) \tag{2.21}$$

Analysis of the magnetic equation leads to the results

$$\boldsymbol{\nabla}_R \times \mathbf{u}_k = 0 \qquad\qquad \frac{1}{2\pi}\mathbf{B}_k = \boldsymbol{\nabla}_R\sigma_e \times \mathbf{u}_k \tag{2.22}$$

where $\mathbf{B}_k$ is a Fourier component of the magnetic field vector.

Applying the separation procedure to the electric equation proves to be more complicated since there are more terms to contend with. Similar to the magnetic result, two equations emerge—one for the dilatation and one for the charge density

$$\boldsymbol{\nabla}_R \mathrm{u}_k = -\left(\frac{1}{R} + ik\right)\mathbf{u}_k \qquad\qquad \boldsymbol{\nabla}\sigma_e + \hat{\boldsymbol{n}}\frac{\partial\sigma_e}{\partial ct} = \boldsymbol{\nabla}_R\sigma_e \tag{2.23}$$

and the second equation is just (1.18a).

Velocity Fields from Acceleration Fields: So far, no mention has been made of particle accelerations which also determine first order equations for the velocity potentials since $A^\nu_a = -\chi A^\nu$ and

$$\partial^\mu A^\nu_a - \partial^\nu A^\mu_a = [A^\mu, a^\nu_\perp] \tag{2.24}$$

However, it can be shown that the acceleration fields provide no new information about the dilation functions. First write

$$\partial^\mu A^\nu_a - \partial^\nu A^\mu_a = A^\mu_v a^\nu - A^\nu_v a^\mu + A^\mu_v a^\lambda \mathcal{U}_\lambda \mathcal{U}^\nu - A^\nu_v a^\lambda \mathcal{U}_\lambda \mathcal{U}^\mu \tag{2.25}$$

Now insert velocity potentials on the left side of this equation and apply the chain rule

$$\partial^\mu A^\nu_a - \partial^\nu A^\mu_a = \chi\left(\partial^\nu A^\mu_v - \partial^\mu A^\nu_v\right) - \partial^\mu\chi A^\nu_v + \partial^\nu\chi A^\mu_v \tag{2.26}$$

The quantity χ is related to the acceleration four-vector by $\chi = a^\lambda R_\lambda$ and easily differentiated to produce

$$\partial^\mu\chi = a^\mu + \dot{a}^\lambda \mathcal{U}_\lambda R^\mu \tag{2.27}$$

Inserting this equation above and noting that velocity potentials point along R^μ shows that

$$\chi\left(\partial^\nu A^\mu_v - \partial^\mu A^\nu_v\right) = A^\mu_v a^\lambda \mathcal{U}_\lambda \mathcal{U}^\nu - A^\nu_v a^\lambda \mathcal{U}_\lambda \mathcal{U}^\mu = \frac{\chi}{\rho}\left[A^\mu, \mathcal{U}^\nu\right] \tag{2.28}$$

In the final step, all references to accelerations can be removed rendering

$$eF^{\mu\nu}_v = [A^\mu_\ell, A^\nu_e] \tag{2.29}$$

This is an impressive calculation showing the intimate relation between the two field stength tensors even though they have been formally addressed as independent theories. More importantly though is the realization that particle accelerations provide no new information about vacuum dilatation which has properties solely determined by the velocity theory.

3 Theory of Vacuum Radiation

An established theory of scalar and vector dilatation are a complete set of tools required for the re-formulation of the classical electron theory in terms of a moving velocity field. The total field may be constructed from a sum over Fourier components or it can be quantized to scatter vacuum quanta from the particle radius.

3.1 Stress Tensor

The Fourier mode Lagrangian derived in equation (2.8) has units of length-squared due to the presence of wave numbers k and p appearing in the denominator. It is useful to write it as

$$L_{kp} = \frac{1}{2}\left[\boldsymbol{\xi}_k \cdot \boldsymbol{\xi}_p + \boldsymbol{\Delta}_k : \boldsymbol{\eta}_p + \boldsymbol{\Delta}_p : \boldsymbol{\eta}_k + \mathbf{u}_k \cdot \mathbf{u}_p \, f_{kp}(R)\right] \cdot e^{i(k+p)ct} \tag{3.1}$$

A Fourier mode stress tensor follows by treating field quantities in k and p as independent variables leading to

$$\mathbf{T}_{kp} = \left[\boldsymbol{\xi}_k \, \boldsymbol{\xi}_p + \boldsymbol{\Delta}_k \cdot \boldsymbol{\eta}_p + \boldsymbol{\Delta}_p \cdot \boldsymbol{\eta}_k - \mathbf{1} L_{kp}\right] \cdot e^{i(k+p)ct} \tag{3.2}$$

Two integrations over wave numbers will then return $\mathbf{T}_{kp}$ to equation (1.37). Another interesting possibility is to add equations (2.11):

$$L_k = \frac{1}{2}\left(\boldsymbol{\nabla}_R \mathrm{u}_k \cdot \boldsymbol{\nabla}_R \mathrm{u}_k^* - k^2 \mathrm{u}_k \mathrm{u}_k^*\right) + \boldsymbol{\Delta}_k : \boldsymbol{\eta}_k^* \tag{3.3}$$

Individual components of the stress tensors are

$$\mathbf{T}_k^{(s)} = \boldsymbol{\nabla}_R \mathrm{u}_k \, \boldsymbol{\nabla}_R \mathrm{u}_k^* - \mathbf{1} L_k^{(s)} = -\frac{1}{2k^2}\eta\boldsymbol{\eta} + r_e^2 \boldsymbol{\Delta} \tag{3.4a}$$

$$\mathbf{T}_k^{(v)} = \boldsymbol{\Delta}_k^* \cdot \boldsymbol{\eta}_k + \boldsymbol{\Delta}_k \cdot \boldsymbol{\eta}_k^* - \mathbf{1}\boldsymbol{\Delta}_k^* : \boldsymbol{\eta}_k = \frac{1}{k^2}\eta\boldsymbol{\eta} \tag{3.4b}$$

and automatically include the propagation term $\boldsymbol{\Delta}$. Restoration of the unitless theory seems appropriate here by dividing out a factor of r_e^2 giving a total stress tensor

$$\mathbf{T}_k = \frac{1}{2k^2 r_e^2}\eta\boldsymbol{\eta} + \boldsymbol{\Delta} \tag{3.5}$$

To recapture the continuous field result let $kr_e \to 1$, even though there seems to be no good reason for this to work since wave numbers cover the entire spectrum $[0, \infty]$.

3.2 Propagating Vacuum Fields–Continuum Limit

The unitless vacuum theory will attain a more recognizable status by adding a few well chosen constants to the dilatation function defined in equation (1.7). Using the bulk modulus b and a factor of c, two related quantities can be defined by

$$\mathsf{P}(\boldsymbol{R},t) \equiv \frac{b}{r_e}\mathrm{u}(\boldsymbol{R},t) \qquad\qquad \mathbf{V}(\boldsymbol{R},t) \equiv \frac{c}{r_e}\mathbf{u}(\boldsymbol{R},t)$$

The function $\mathsf{P}(\boldsymbol{R},t)$ is a scalar compressional field (or pressure wave) while $\mathbf{V}(\boldsymbol{R},t)$ represents a radial velocity field moving away from the source at the speed of light. Together these functions are adequate to propagate the R-space vacuum. With the inclusion of two powers of the Doppler function the energy flux is immediately available from[2]

$$\mathbf{S}_e = \frac{1}{2}|f(\theta,\phi)|^2\,\mathsf{P}\mathbf{V} \tag{3.6}$$

and this can be integrated to reproduce the electron power formula. Of special importance here is the notion that compressional wave motion does not require a background ether for its propagation. Instead, the medium defines itself by the propagation of momentum and energy from the source.

Like the dilatation functions themselves, the pressure field and the velocity field have Fourier components in the form of travelling waves:

$$\mathsf{P}_k(\boldsymbol{R},t) = br_e\,\tilde{\phi}_k\frac{e^{ik(ct-R)}}{kR} \qquad\qquad \mathbf{V}_k(\boldsymbol{R},t) = cr_e\,\tilde{\phi}_k\frac{e^{ik(ct-R)}}{kR}\hat{\boldsymbol{n}}$$

These are pressure and velocity fields per unit wave number and they simplify considerably at the vacuum boundary

$$\mathsf{P}_k(t) = \frac{b}{k}\,e^{ikct} \qquad\qquad \mathbf{V}_k(t) = \frac{c}{k}\,e^{ikct}\hat{\boldsymbol{n}}$$

First order coupled equations for both fields can be easily determining from (2.16a) and (2.16c):

$$\boldsymbol{\nabla}_R\mathsf{P}_k = -\frac{b}{c}\left[\frac{1}{R}+ik\right]\mathbf{V}_k \qquad\qquad \boldsymbol{\nabla}_R\cdot\mathbf{V}_k = \frac{c}{b}\left[\frac{1}{R}-ik\right]\mathsf{P}_k \tag{3.7}$$

These combine to produce second order de-coupled equations similar to (2.13) except the source term in the scalar equation is now an impulse force density

$$\rho_k(\boldsymbol{R}) = \frac{4\pi b\,\tilde{a}_k}{kr_e}\,\delta^3(\boldsymbol{R}) \tag{3.8}$$

Total radiated energy follows by integrating the energy flux over the radius of the particle for each wave number and then summing over all possible wave numbers:

$$E_{vac} = \frac{1}{\pi c}\int_0^\infty\left[\oint_{r_e}|f(\theta,\phi)|^2\mathsf{P}_k\mathbf{V}_k\cdot\hat{\boldsymbol{n}}R^2d\Omega\right]dk \tag{3.9}$$

[2]This equation can also be constructed from the vacuum tensor beginning with $\boldsymbol{\Delta}\cdot\hat{\boldsymbol{n}}$.

Evaluation of this integral follows by considering only imaginary parts of the complex exponential. One finds

$$E_{vac} = 4br_e^2 \int_0^\infty \frac{\sin^2 kct}{k^2} dk = 2\pi r_e^2 bct \tag{3.10}$$

A time derivative either before or after the final integration will then give the correct power formula. Note that the quantity $bc/2$ is the scalar energy flux S_e through the radius of the particle.

3.3 Quantum Theories of Vacuum Radiation

While the theory of the classical radiation field is straightforward, quantizing the longitudinal modes seems to be a more difficult problem to solve. Nevertheless, two simple quantum theories describing the emission of vacuum radiation are presented here. Both theories place limitations on the frequencies of emitted radiation and this destroys the ability to construct the causality step defined in equation (2.1) which relies on a continuous spectrum. On the other hand, quantizing the radiation field requires the observer to view the field as streams of particles which is fundamentally different than the propagation of longitudinal waves. If this is the case it might be expected that an integral like (2.1) will lose its meaning. However causality can still be enforced by simply imposing the requirement that all radiated quanta must exist within the particles finite light cone.

Equal Energy Quanta: The simplest quantum theory of the Coulomb field begins by writing the mass-energy of the electron in terms of the instantaneous radiation rate. In the rest frame only the classical formula can be written

$$m_e c^2 = \frac{1}{2}\dot{\varrho} r_e \tag{3.11}$$

where $\dot{\varrho} = 4\pi\sigma_e e$ is the total scalar momentum per second scattered by the particle radius. If this momentum is in the form of n radiated quanta per second then $\dot{\varrho} \to np_o$ where p_o is the momentum of each quantum. Next, postulate the quantum mechanical relation between the momentum of the emission and the electron radius

$$p_o = \frac{h}{r_e} \tag{3.12}$$

The required number of particles emitted per second is then

$$n = \frac{2m_e c^2}{h} \tag{3.13}$$

If emitted quanta are unobservable spin-1 bosons, this requires the spin vector of the electron to change sign at the frequency n described by (3.13). The emission of two

particles is required for one complete rotation of the spin vector implying a Dirac frequency $\nu = n/2$. Quantum theory requires rotations be described by complex exponentials. For an electron at the origin, the time dependent wave function will be

$$\psi(t) = \psi_0\, e^{-i(mc^2/\hbar)t} \tag{3.14}$$

where ψ_0 is a two component spinor. This derivation suggests that the vision of a polarized beam of electrons as a collection of spins pointing in a preferred direction is fundamentally flawed. For proponents of the vacuum gauge electron, this vision must be replaced with one of polarized randomness—detected experimentally using the laws of quantum mechanics. The energy of each particle calculates to about $\epsilon \sim 440$ MeV and the quantum mechanical energy-time uncertainty relation can be verified using the time required to cross the radius of the particle:

$$\Delta\epsilon\Delta\tau = h/2 \tag{3.15}$$

Theory of Discrete Wave Vectors: A more elaborate quantum theory can be constructed using discrete wave vectors $k = nk_o$ where n is a quantum number and k_o is yet to be determined. If

$$\Delta k = nk_o - (n-1)k_o = k_o \tag{3.16}$$

and ζ is some overall unitless constant, then pressure and velocity fields at wave vector k can be written

$$\mathsf{P}_k = \frac{br_e}{\sqrt{\zeta}}\,\frac{\sin k(c\hat{t} - R)}{kR}\,\Delta k \qquad\qquad \mathbf{V}_k = \frac{cr_e}{\sqrt{\zeta}}\,\frac{\sin k(c\hat{t} - R)}{kR}\,\Delta k\,\hat{\boldsymbol{n}}$$

The energy flux follows by summing over the index n and using orthogonality of the discrete sine functions:

$$\mathbf{S}_e = \sum_k \mathsf{P}_k\mathbf{V}_k = \frac{bcr_e^2}{\zeta R^2}\left[\sum_{n=1}^{\infty}\frac{\sin^2[nk_o(ct - R + r_e)]}{n^2}\right]\hat{\boldsymbol{n}} \tag{3.17}$$

The average energy flux summed over all possible modes is then

$$\langle\mathbf{S}_e\rangle = \frac{bcr_e^2}{2\zeta R^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2}\right]\hat{\boldsymbol{n}} \tag{3.18}$$

If the total power radiated is required to be the inertial power P_{in} then this will conveniently identify the constant ζ as the Riemann zeta-function $\zeta = \zeta[2]$. The power radiated by the n^{th} term in the sum is

$$P_n = \int_\Omega |Rf(\theta,\phi)|^2\langle\mathbf{S}_e\rangle_n\cdot\hat{\boldsymbol{n}}\,d\Omega = \frac{P_{in}}{\zeta[2]n^2} \tag{3.19}$$

Now suppose energy and momentum flux at each wave number is composed of vacuum qaunta with energy and momentum given by

$$E_n = \frac{1}{2} n\hbar c k_o \qquad\qquad \boldsymbol{p}_n = n\hbar k_o \hat{\boldsymbol{n}} \tag{3.20}$$

The total number of quanta scattered per unit time at each index n is then

$$N_n = \frac{P_n}{E_n} = \frac{2P_{in}}{\zeta[2]\hbar c k_o \, n^3} \tag{3.21}$$

The only other requirement will be to equate the total quanta radiated per second to twice the Dirac frequency.

$$2f = \sum_{n=1}^{\infty} N_n = \frac{\zeta[3]}{\zeta[2]} \cdot \frac{2P_{in}}{\hbar c k_o} \tag{3.22}$$

The ratio of zeta-functions is a constant $\alpha \sim 0.73$ and equation (3.22) solves for the fundamental wave vector

$$k_o = \frac{2\pi\alpha}{r_e} \tag{3.23}$$

The quantum mechanical energy and momentum formula's can then be re-written

$$E_n = \frac{1}{2}\frac{hc}{\lambda_n} \qquad\qquad \boldsymbol{p}_n = \frac{h}{\lambda_n}\hat{\boldsymbol{n}} \tag{3.24}$$

where the quantized wavelengths are given by

$$\lambda_n = \frac{r_e}{\alpha n} \tag{3.25}$$

This formula allows for the interpretation of the emission of vacuum particles in terms of harmonic oscillations of the particle radius. The classical radius is simply a superposition of longitudinal pulses summed over all wave vectors. Approximately 83 percent of the emissions are at the fundamental frequency having an energy of $E \sim 321$ MeV. For the 500th harmonic (overtone?) the the rate of emission is reduced by a factor of 10^8 but there are still 10^{12} particles emitted each second. The particle radius associated with each wave vector is proportional to the wavelength and given by

$$r_n = \frac{r_e}{2\pi\alpha n} \tag{3.26}$$

If we let $r_n/c \to \Delta\tau$ and $E_n \to \Delta E$ then

$$\Delta E \Delta\tau = \frac{\hbar}{2} \tag{3.27}$$

which saturates the lower bound of the uncertainty principle.

A Transformation of the Causality Sphere

A theory of an invariant particle radius follows from an inspection of the vacuum gauge velocity potentials in equations (1.7) and (2.4). The essential problem is to develop an appropriate transformation law for the causality step consistent with the theory.

In the rest frame the causality sphere is

$$\vartheta = \vartheta(c\tau - r_o + r_e) \tag{A.1}$$

A graphical depiction of the this function is provided in figure 2 showing a temporal expansion over a time interval $ct_o + r_e$. Unfortunately, the presence of r_e in the

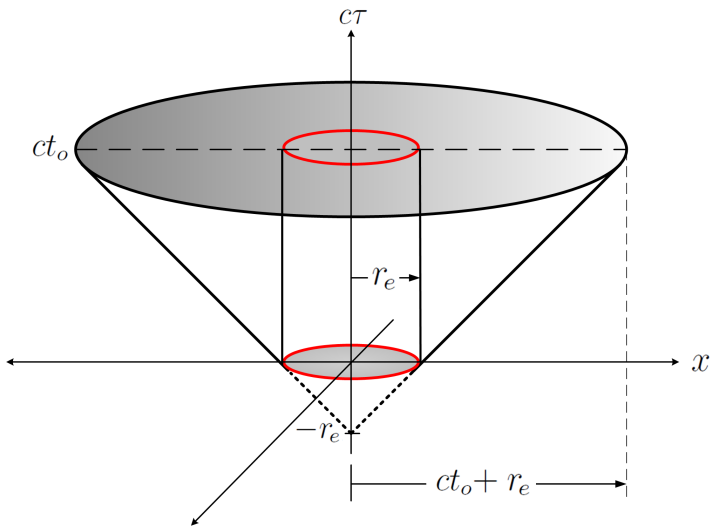

Figure 2: *Spacetime diagram indicating the expansion of the radial step* $\vartheta(c\tau - r_o + r_e)$ *in the rest frame. The inclusion of the phase* r_e *defines two regions inside and outside the particle radius.*

argument leads to difficulties when transforming to a moving frame—completely independent of any stipulated property of the radius. To understand why, insert the general homogeneous Lorentz transformation

$$c\tau = \gamma\,(ct - \boldsymbol{\beta}\cdot\boldsymbol{r}) \tag{A.2a}$$

$$\boldsymbol{r}_o = \boldsymbol{r} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{r})\boldsymbol{\beta} - \gamma ct\boldsymbol{\beta} \tag{A.2b}$$

showing that the step transforms as

$$L[\vartheta(c\tau - r_o + r_e)] = \vartheta(\gamma ct - \gamma\boldsymbol{\beta}\cdot\boldsymbol{r} - \rho + r_e) \tag{A.3}$$

The problem is to determine the collection of points which represents the boundary of the expanding step in the moving frame. Letting $\vartheta = \vartheta(Z)$, then the vanishing of Z implies the covariant condition

$$(x^\nu + r_e\beta^\nu) \cdot (x_\nu + r_e\beta_\nu) = 0 \tag{A.4}$$

This is the equation of a sphere with a radius $ct + \gamma r_e$ which expands about the point $-\gamma\boldsymbol{\beta} r_e$ so the moving frame step function is

$$L[\vartheta] = \vartheta(ct - \|\boldsymbol{r} - \gamma\boldsymbol{\beta} r_e\| + \gamma r_e) \tag{A.5}$$

In short, the homogeneous transformation does not appear to be a useful theoretical tool for keeping the mathematics simple while highlighting properties of the expanding sphere in the moving frame. Based on the previous result it is reasonable to inquire

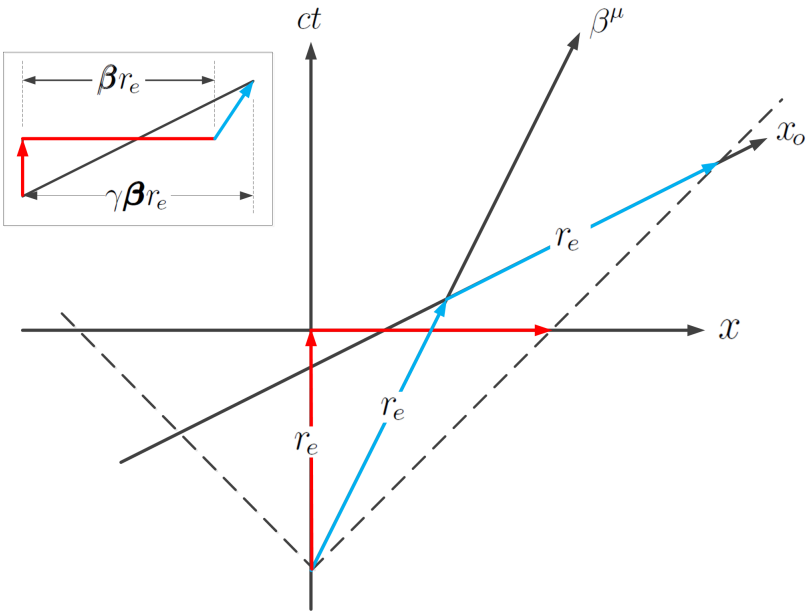

Figure 3: *Spacetime diagram based on equations (A.7). The transformation produces timelike and spacelike radius vectors (shown in red and blue) which are the same in both frames.*

how the Lorentz transformation might be tailored to shift the origin of coordinates so that the causal sphere still expands from the spatial origin in the moving frame. This can be accomplished by defining proper frame and moving frame time coordinates by

$$c\hat{\tau} \equiv c\tau + r_e \qquad\qquad c\hat{t} \equiv ct + r_e \tag{A.6}$$

Replacing the time coordinates in (A.2) with the hatted coordinates renders inhomogeneous equations

$$c\tau = \gamma\left(ct - \boldsymbol{\beta}\cdot\boldsymbol{r}\right) + \alpha_o \tag{A.7a}$$

$$\boldsymbol{r}_o = \boldsymbol{r} + \frac{\gamma-1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{r})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}ct + \boldsymbol{\alpha} \tag{A.7b}$$

where the shift in coordinates is given by

$$\alpha_\mu \equiv [\gamma - 1, -\gamma\boldsymbol{\beta}]r_e \tag{A.8}$$

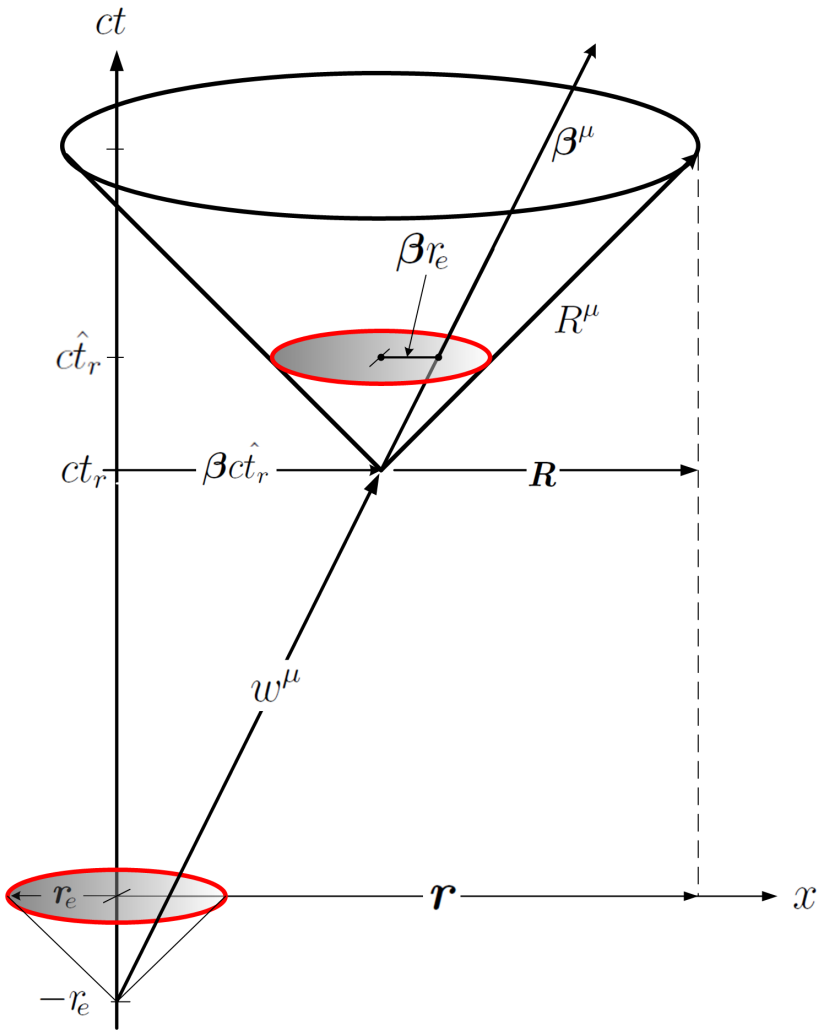

Figure 4: *World line for a particle with radius r_e moving at constant velocity.*

The two coordinate systems related by (A.7) are illustrated in figure 3. The transformation looks something like a Poincaré transformation except that the coordinate

shifts are a function of the velocity of the particle. For small non-relativistic transformations α_μ can be approximated by a purely spatial shift, proportional to the velocity of the particle. For transformations where γ is large one can write $\alpha_\mu \approx r_e\beta_\mu$. Even for large γ though, the spatial and temporal shifts are still small based on the value of the classical radius: For example $\gamma \sim 10^9$ produces a spatial shift on the order of 10^{-6} meters.

An immediate test of equations (A.7) is to use them to show that

$$L[\vartheta(c\tau - r_o + r_e)] = \vartheta(ct - r + r_e) \tag{A.9}$$

However, it is also important to consider the transformation in terms of the retarded time ct_r and its associated vector $\boldsymbol{R}$. In the moving frame the position of the electron as a function of time is given by

$$\boldsymbol{w}(ct_r) = \boldsymbol{\beta}(ct_r + r_e) \tag{A.10}$$

and the retardation condition is

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{w}(ct_r) \qquad\qquad R \equiv ct - ct_r \tag{A.11}$$

This substitution derives the following relations

$$c\tau + r_e = \rho + \frac{1}{\gamma}(ct_r + r_e)$$
$$\boldsymbol{r}_o = \boldsymbol{R} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{R})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}R$$

Extraneous terms involving the radius drop out of the second equation which is important because it preserves the definition of the covariant scalar $\rho \equiv R^\nu\beta_\nu$. In terms of the new coordinates the causality sphere can be written

$$\vartheta = \vartheta[\gamma^{-1}(c\hat{t}_r)] \tag{A.12}$$

This is a sensible result which vanishes for retarded times less then r_e.

The world line of the moving frame particle is illustrated in figure 4. The particle position at any retarded time ct_r is the center of radiation. This can be compared to the sphere which defines the particle radius at a later time $c\delta t_e = r_e$. At this time the particle position is shifted by an amount $\boldsymbol{\beta}r_e$ from the center of the sphere.

B Green Function Solutions for Dilatation Functions

Source equations for the scalar and vector dilatation can be solved using the method of Green functions. The retarded Green function $G(\boldsymbol{R}, \boldsymbol{R}')$ is the same for both of equations (2.13) and is a solution to

$$\boldsymbol{\nabla}_R^2 G(\boldsymbol{R}, \boldsymbol{R}') + k^2 G(\boldsymbol{R}, \boldsymbol{R}') = -\delta^3(\boldsymbol{R} - \boldsymbol{R}') \tag{B.1}$$

Generally speaking, a Green function can always be written as a sum over independent orthogonal solutions to a Sturm-Louiville homogeneous equation. These are plane waves with a continuous index for (2.13) and requiring the Green function to take the form

$$G(\boldsymbol{R}, \boldsymbol{R}') = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{-i\boldsymbol{p}\cdot(\boldsymbol{R}-\boldsymbol{R}')}}{p^2 - k^2} d^3p \tag{B.2}$$

The integral over the primed coordinates is straight forward producing the infinite space solution

$$G(\boldsymbol{R}, \boldsymbol{R}') = \frac{e^{-ik|\boldsymbol{R}-\boldsymbol{R}'|}}{4\pi|\boldsymbol{R} - \boldsymbol{R}'|} \tag{B.3}$$

and having a useful spherical wave expansion

$$G(\boldsymbol{R}, \boldsymbol{R}') = -ik \sum_{l=0}^{\infty} \sum_{m=-l}^{l} j_l(kR_<) h_l^{(2)}(kR_>) Y_{lm}^*(\psi', \phi') Y_{lm}(\psi, \phi) \tag{B.4}$$

The forms of G in the two previous equations are actually complex conjugates of the more conventional Green function but are a convenient alteration for the vacuum gauge electron since the variable R appears with a minus sign in the causality step.

Scalar Dilation: The scalar equation is easily solved from the observation that it is—to within a constant—identical to the Green function in equation (B.1) for $\boldsymbol{R}' \to 0$. Since both u_k and ρ_k have well defined Fourier transforms it is also possible to consider a solution based on the Fourier amplitude

$$\mathrm{u}_k(p) = \sqrt{\frac{2}{\pi}} \left[\frac{\tilde{a}_k}{k(p^2 - k^2)} \right] \tag{B.5}$$

However the problem is confronted, their is little difficulty in deriving

$$\mathrm{u}_k(R) = \tilde{a}_k \frac{e^{-ikR}}{kR} \tag{B.6}$$

A theory of scalar dilatation can also be investigated using Green's theorem which combines $\mathrm{u}_k(R')$ and $G(\boldsymbol{R}, \boldsymbol{R}')$ according to

$$\int_v [\mathrm{u}_k \boldsymbol{\nabla}_R'^2 G - G\, \boldsymbol{\nabla}_R'^2 \mathrm{u}_k] dv' = \oint_s [\mathrm{u}_k \boldsymbol{\nabla}_R' G - G\, \boldsymbol{\nabla}_R' \mathrm{u}_k] \cdot d\boldsymbol{a}' \tag{B.7}$$

First, suppose that the integration volume is chosen to be infinite space. Substituting appropriate differential equations on the left side leaves two delta functions which contribute with opposite signs at $R' = 0$ and $R' = R$. The left side is therefore zero and this means the surface integral is also zero. To show this, make the replacements

$$\boldsymbol{\nabla}'_R \mathrm{u}_k \cdot \hat{\boldsymbol{n}}' = -\left(\frac{1}{R'} + ik\right) \mathrm{u}_k \qquad\qquad \boldsymbol{\nabla}'_R G \cdot \hat{\boldsymbol{n}}' = \frac{\partial G}{\partial R'} \tag{B.8}$$

and write the surface integral as

$$I_s = \lim_{R'\to\infty} \oint_s \mathrm{u}_k \left[\frac{\partial G}{\partial R'} + \left(\frac{1}{R'} + ik\right) G\right] R'^2 d\Omega \tag{B.9}$$

But u_k is a purely radial function implying that the Green function can be replaced by the leading term in the expansion (B.4) without consequence. This term is

$$G_{\ell=0}(\boldsymbol{R}, \boldsymbol{R}') = -\frac{ik}{4\pi} \begin{cases} h_0^{(2)}(kR) \cdot j_0(kR') & R' < R \\ j_0(kR) \cdot h_0^{(2)}(kR') & R' > R \end{cases} \tag{B.10}$$

Choosing the case $R' > R$ easily verifies that $I_s = 0$.

Suppose instead that the volume is chosen to be infinite space less the volume inside the vacuum boundary. Assume also that the variable R lies somewhere in the exterior region. In this case Green's theorem becomes

$$\mathrm{u}_k(R) = -\oint_s [\mathrm{u}_k \boldsymbol{\nabla}'_R G - G\, \boldsymbol{\nabla}'_R \mathrm{u}_k] \cdot d\boldsymbol{a}' \tag{B.11}$$

However, as already shown, the surface corresponding to $R' \to \infty$ makes no contribution to (B.11) so that s is to be associated exclusively with the vacuum boundary. Evaluation of the integral follows as before by replacing the Green function by its leading $\ell = 0$ term, but this time it is necessary to use the first equation in (*B*.10). The result requires the direction of the surface element to point from exterior space into the void and can be written

$$\mathrm{u}_k(R) = r_e^2 \mathrm{u}_k(r_e) \cdot \frac{e^{-ikR}}{R} \cdot \frac{e^{ikr_e}}{r_e} \qquad R' = r_e \tag{B.12}$$

In the language of scalar diffraction theory, it can be suggested here that (B.11) determines the exterior field based on an integral over an aperture where each element of the aperture is the source of a Huygens wavelet.

Vector Dilatation: Solution to the vector equation in (2.13) follows by generalizing Green's theorem to accomodate a vector field:

$$\int_v [\mathbf{u}_k \boldsymbol{\nabla}'^2_R G - G\, \boldsymbol{\nabla}'^2_R\, \mathbf{u}_k] dv' = \oint_s [\mathbf{u}_k \boldsymbol{\nabla}'_R G - G\, \boldsymbol{\nabla}'_R \mathbf{u}_k] \cdot d\boldsymbol{a}' \tag{B.13}$$

If the volume consumes all space, then the right side of this equation vanishes while the left side identifies the solution for $\mathbf{u}_k$ in the form

$$\mathbf{u}_k(R) = \int G(\boldsymbol{R}, \boldsymbol{R}')\mathbf{J}_k(R')d^3R' \tag{B.14}$$

Unfortunately, the solution for the vector problem is somewhat more complicated than its scalar counterpart. To perform the integration in (B.14) it will be necessary to use the spherical wave expansion of G given in equation (B.4). In addition to this, the current density is not a localized source which implies the full integration of the primed variables over infinite space. The spherical boundary defined by $R = R'$ divides the integral into two distinct regions and leads to the solution

$$\mathbf{u}_k(R) = -ik\sum_{l,m} h_l^{(2)}(kR)Y_{lm}(\psi,\phi)\int_0^R\int_\Omega \mathbf{J}_k(R')j_l(kR')Y_{lm}^*(\psi',\phi')R'^2\,dR'd\Omega'$$

$$-\,ik\sum_{l,m} j_l(kR)Y_{lm}^*(\psi,\phi)\int_R^\infty\int_\Omega \mathbf{J}_k(R')h_l^{(2)}(kR')Y_{lm}(\psi',\phi')R'^2\,dR'd\Omega'$$

But the integrals can be simplified considerably by performing the angular integration which eliminates all terms in the sum except for $l = 1$. For the record

$$G_{l=1}(\boldsymbol{R}, \boldsymbol{R}') = \frac{3}{4\pi}ik\cos\gamma\begin{cases} h_1^{(2)}(kR)\cdot j_1(kR') & R' < R \\ j_1(kR)\cdot h_1^{(2)}(kR') & R' > R \end{cases} \tag{B.15}$$

where $\cos\gamma$ is determined from the addition theorem of spherical harmonics

$$\cos\gamma = \sum_{m=-1}^{1} Y_{1m}^*(\theta',\phi')Y_{1m}(\theta,\phi) \tag{B.16}$$

The remaining radial integrals determine $\mathbf{u}_k(R)$ as

$$\mathbf{u}_k(R) = -ik\,\hat{\boldsymbol{n}}\left[h_1^{(2)}(kR)\int_0^R J_k(R')j_1(kR')R'^2\,dR' + j_1(kR)\int_R^\infty J_k(R')h_1^{(2)}(kR')R'^2\,dR'\right]$$

Where $J_k(R')$ is the magnitude of the vector current density which can be cast in terms of the $l = 0$ spherical Hankel function of the second kind

$$J_k(R) = -2i\frac{\tilde{a}_k}{R^2}\,h_0^{(2)}(kR) \tag{B.17}$$

Now define the unit free coordinate $x \equiv kR$, and insert the source current which leads to the expression

$$\mathbf{u}_k(x) = -2\tilde{a}_k\,\left[h_1^{(2)}(x)I_1(x) + j_1(x)I_2(x)\right]\hat{\boldsymbol{n}} \tag{B.18}$$

with integrals given by

$$I_1(x) \equiv \int_0^x h_0^{(2)}(x') j_1(x')\, dx' \qquad I_2(x) \equiv \int_x^\infty h_0^{(2)}(x') h_1^{(2)}(x')\, dx' \tag{B.19}$$

While $I_2(x)$ can be integrated by parts, $I_1(x)$ is somewhat more difficult to evaluate. One finds the spherical wave solution

$$\mathbf{u}_k(R) = \tilde{a}_k \left[\frac{e^{-ikR}}{kR}\right] \hat{\boldsymbol{n}} \tag{B.20}$$

which agrees with (2.5b).

Having verified the solution $\mathbf{u}_k$ it is important to return to equation (B.13) and apply Green's theorem over other regions. For example, suppose the selected volume is infinite space less the volume contained within the particle radius r_e. On the right side, if it is further assumed that the surface at $R' \to \infty$ does not contribute to the integral, than s represents the vacuum boundary at r_e. For a value of $\boldsymbol{R}$ exterior to the boundary, Green's theorem reads

$$\mathbf{u}_k(\boldsymbol{R}) = \int_{>r_e} G\mathbf{J}_k\, dv' - \oint_s [\mathbf{u}_k \boldsymbol{\nabla}'_R G - G\, \boldsymbol{\nabla}'_R \mathbf{u}_k] \cdot d\boldsymbol{a}' \tag{B.21}$$

but according to (B.14), this requires the volume inside the vacuum boundary to satisfy

$$\int_{<r_e} G\mathbf{J}_k\, dv' = -\oint_s [\mathbf{u}_k \boldsymbol{\nabla}'_R G - G\, \boldsymbol{\nabla}'_R \mathbf{u}_k] \cdot d\boldsymbol{a}' \tag{B.22}$$

Evaluation of the volume integral is immediate by referring to the Green function calculation of $\mathbf{u}_k$. Defining $x_e = kr_e$ implies

$$\int_{<r_e} G\mathbf{J}_k\, dv' = -2\tilde{a}_k\, h_1^{(2)}(x)\, I_1(x_e) \tag{B.23}$$

To arrive at this result through a brute force calculation of the surface integral, first calculate

$$\boldsymbol{\nabla}'_R \mathbf{u}_k \cdot \hat{\boldsymbol{n}}' = -\left[\frac{1}{R'} + ik\right] \mathbf{u}_k \qquad \boldsymbol{\nabla}'_R G \cdot \hat{\boldsymbol{n}}' = \frac{\partial G}{\partial R'} \tag{B.24}$$

At $R' = r_e$, the surface integral is therefore linear in the dilatation

$$\boldsymbol{I} = \oint_s \mathbf{u}_k \left[\left.\frac{\partial G}{\partial R'}\right|_{R'=r_e} + \left(\frac{1}{r_e} + ik\right) G\right] r_e^2 d\Omega \tag{B.25}$$

The angular integrals can now be performed with the observation that only the component of G corresponding to $l = 1$ will contribute to the integral. Inserting the $R' < R$ term in equation (B.15) will then verify the result of (B.23).