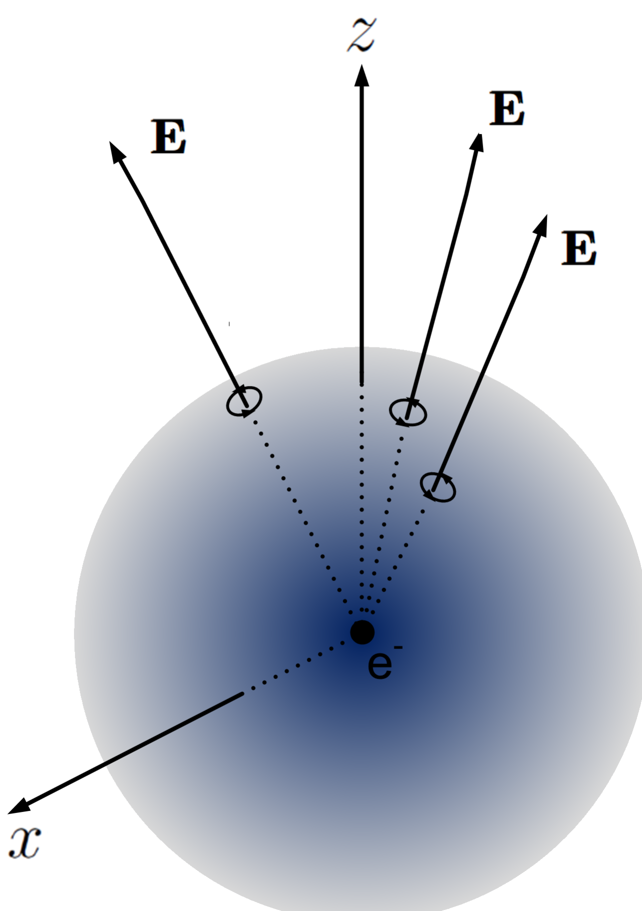
The

Maxwell Limit

April 14, 2017



> *"...the classical theory of electromagnetism is an unsatisfactory theory all by itself. There are difficulties associated with the ideas of Maxwell's theory which are not solved by and not directly associated with quantum mechanics."*
>
> R.P. Feynman, *The Feynman Lectures*, chapter 28

Abstract

Beginning with the Maxwell-Lorentz equations for a charged point particle, an application of the principle of causality will lead naturally to the vacuum gauge condition. The presence of the gauge field allows the velocity and acceleration fields of the classical particle to be formulated as independent theories determined from associated vacuum gauge potentials.

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1 Causal Theory of a Charged Particle

Define a flat spacetime using the Minkowski metric

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{1.1}$$

where the general homogeneous Lorentz transformation relating two sets of coordinates $(c\tau, \boldsymbol{\rho}) \longrightarrow (ct, \boldsymbol{r})$ is given by

$$c\tau = \gamma\,(ct - \boldsymbol{\beta}\cdot\boldsymbol{r}) \tag{1.2a}$$

$$\boldsymbol{\rho} = \boldsymbol{r} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{r})\boldsymbol{\beta} - \gamma ct\boldsymbol{\beta} \tag{1.2b}$$

A point charge moving through the spacetime is characterized by a four-current density J_e^ν and the associated field satisfies the ***Maxwell-Lorentz equations***

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J_e^\nu \tag{1.3}$$

The name is appropriate since the original Maxwell theory of the electromagnetic field was a macroscopic theory, which Lorentz first applied to a charged particle.

Unfortunately, although equation (1.3) generates a workable theory of the electron, it cannot be the correct theory for the simple reason that the particle can neither be created nor destroyed. This is inconsistent with the modern philosophy of quantum field theory which allows for the creation and annihilation of electrons and positrons out of the vacuum. As a primary example, it is a violation of the causality principle to assign a Coulomb field

$$\mathbf{E} = \frac{e}{\rho^2}\hat{\boldsymbol{\rho}} \tag{1.4}$$

to any charged particle which was created at a finite time $\vartriangle\tau$ in the past. This field extends to spatial infinity and requires the particle to have an eternal lifespan.

To eliminate this problem suppose classical electron theory is re-formulated based on an interaction like

$$\gamma + \gamma \longrightarrow e^+ + e^- \tag{1.5}$$

As illustrated by the spacetime diagram of Figure 1 if both particles are created at the origin then the present time field strength must be constrained to the region shaded in red by a causality function ϑ so that the total field will be given by

$$F^{\mu\nu}_{tot} = [F^{\mu\nu}_+ + F^{\mu\nu}_-]\cdot\vartheta \tag{1.6}$$

To simplify the problem further, forget about the anti-particle which can be re-inserted at a later time. Then write a new (more rational) version of Coulombs'

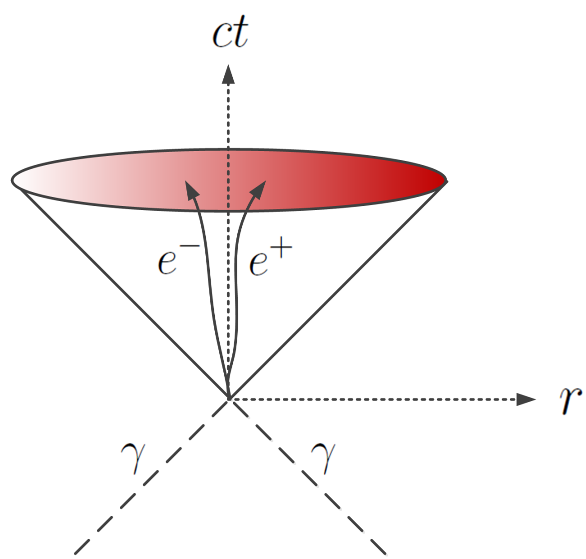

Figure 1: *Spacetime diagram for pair production. Approximating the interaction as pointlike, a causal theory constrains the field of both charged particles to the finite reach of the light cone. In a frame S at some time ct, this is the area shaded in red.*

law for an electron at the origin created at time $c\tau = 0$ as

$$\boxed{\mathbf{E}(\boldsymbol{\rho}, \tau) = \frac{e}{\rho^2}\hat{\boldsymbol{\rho}} \cdot \vartheta(c\tau - \rho)} \tag{1.7}$$

In this equation the radial step function takes a value of 1 when $c\tau \geq \rho$ and vanishes otherwise.

Beginning with (1.7) the essential goal will be to re-develop classical electron theory for strict enforcement of the causality principle. Any modifications required by the causal theory may then be compared to the original Maxwell-Lorentz theory.

1.1 Causality Step Function

Although the definition of the causality function in equation (1.7) is somewhat vague it is not difficult to show Lorentz invariance. On purely physical grounds its argument is nothing more than a variation of the relativistic interval, which is form invariant and vanishes for a lightlike interval. A more formal proof follows by first defining the argument by $g \equiv c\tau - \rho$ and then transforming to a moving frame. Using the transformation in (1.2) gives

$$g(ct, \boldsymbol{r}) = \gamma ct - \gamma \boldsymbol{\beta} \cdot \boldsymbol{r} - \rho \tag{1.8}$$

Now Taylor expand in a small variable ϵ for values of ct which are only slightly larger than r. Writing $\boldsymbol{\beta}\cdot\boldsymbol{r} = \beta r\cos\phi_r$, the function and its first derivative at $\epsilon = 0$ are:

$$g(ct,\boldsymbol{r})\Big|_{ct=r} = 0 \tag{1.9a}$$

$$\frac{\partial g(ct,\boldsymbol{r})}{\partial ct}\Big|_{ct=r} = \frac{(1-\beta^2)^{1/2}}{1-\beta\cos\phi_r} \equiv g' \tag{1.9b}$$

The proof now follows by inspection of g' in (1.9b). This term is a classic Doppler shift but of importance is the observation that it's value is larger than zero for any $\boldsymbol{\beta}$ and ϕ_r which are chosen. As ϵ tends to zero the expansion

$$g(\epsilon) \approx g'\epsilon + O(\epsilon^2) \tag{1.10}$$

therefore vanishes smoothly over the entire sphere so that one may write $\vartheta = \vartheta(ct - r)$ in the moving frame[1].

A more formal definition of the causality step results by first considering the location $\boldsymbol{w}(t_r)$ of a constant velocity electron at retarded time t_r:

$$\boldsymbol{w}(t_r) = \boldsymbol{\beta} ct_r \tag{1.11}$$

An associated four-vector is $w^\nu = (ct_r, \boldsymbol{w})$ and the null vector R^ν from the retarded position to the present time point $r^\nu = (ct, \boldsymbol{r})$ is given by

$$R^\nu = r^\nu - w^\nu \tag{1.12}$$

With this information the causality step takes the form $\vartheta(ct_r/\gamma)$. The factor of γ is brought about by the Lorentz transformation and can be easily removed using well known properties of ϑ-functions. However, this may not be a good idea depending on the problem. As a compromise both possibilities will be included in the simple definition[2]

$$\vartheta(ct_r) = \vartheta(ct_r/\gamma) \equiv \begin{cases} 1 & ct_r \geq 0 \\ 0 & ct_r < 0 \end{cases} \tag{1.13}$$

Its four-gradient is

$$\begin{aligned} \partial^\nu\vartheta(ct_r) &= \frac{\partial\vartheta(ct_r)}{\partial ct_r}\cdot\partial^\nu ct_r \\ &= \delta(ct_r)\cdot\frac{\gamma R^\nu}{\rho} = \delta(ct_r/\gamma)\cdot\frac{R^\nu}{\rho} \end{aligned} \tag{1.14}$$

[1] It is also possible to proceed with the analysis here by considering g as a function of r instead of ct. In this case one would calculate $g'(r) = dg/dr$ and evaluate the derivative at $r = ct$. The end result is the same.

[2] In the name of simplicity, and to circumvent energy considerations, it is appropriate not to consider transformations of the step into the frequency domain at this time.

and it should be emphasized here that $\partial^\nu\vartheta$ is not functionally tied to the instantaneous acceleration of the particle—either when it was created, or at any other time. A simple operation to re-capture ϑ in terms of Lorentz scalars $(c\tau, \rho)$ is to form the scalar

$$\vartheta(c\tau - \rho) \equiv \int_0^{c\tau} \partial^\nu\vartheta\,\beta_\nu c\,d\tau' = \int_0^{c\tau} \delta(c\tau' - \rho)c\,d\tau' \tag{1.15}$$

This definition is completely equivalent to (1.13).

1.2 Theory of Electric Charge

With a reasonable definition of the causality step, the rest frame electric field vector can now be promoted to a covariant status by writing the field strength tensor

$$F^{\mu\nu} = F^{\mu\nu}_{M} \cdot \vartheta \tag{1.16}$$

Taking the divergence brings about an added complexity to the conventional notion of the particles' charge and current density since

$$\partial_\mu F^{\mu\nu} \equiv \frac{4\pi}{c} J^{*\nu}_e \tag{1.17}$$

where

$$\boxed{J^{*\nu}_e \equiv J^\nu_e \cdot \vartheta + J^\nu_N} \qquad \boxed{J^\nu_N \equiv \frac{c}{4\pi} F^{\mu\nu}_M \cdot \partial_\mu\vartheta} \tag{1.18}$$

and where J^ν_N may be referred to as the ***null current***—having a four-space norm of zero.

One possibility for understanding properties of $J^{*\nu}_e$ is through a divergence calculation. This is a continuity equation for electric charge which might not be expected to hold for a single particle since the charge has been created from nothing. Curiously enough, this is a false assumption:

$$\partial_\nu J^{*\nu}_e = \partial_\nu J^\nu_e \cdot \vartheta + J^\nu_e \cdot \partial_\nu\vartheta + \partial_\nu J^\nu_N \tag{1.19}$$

The first term on the right is obviously zero. The last term requires a calculation of $F^{\mu\nu}_M \cdot \partial_\nu\partial_\mu\vartheta = 0$ and what remains cancels with the second term so that $\partial_\nu J^{*\nu}_e = 0$.

The physics of this calculation can be understood from the spacetime diagram in figure 2. An application of Gauss' law over the volume of the light cone is

$$\int \partial_\nu J^{*\nu}_e d^4x = \int J^{*\nu}_e \beta_\nu d^3\sigma + \int J^{*\nu}_e R_\nu d^2\omega \tag{1.20}$$

but the left side of this equation is zero. Moreover, since the null current is orthogonal to the light cone—and since the light cone does not include the position of the charge except at its vertex—then the second integral on the right is also zero. The remaining integral over the spacelike plane is

$$\int J_e^{*\nu}\beta_\nu d^3\sigma = \int (J_e^\nu \cdot \vartheta + J_{\scriptscriptstyle N}^\nu)\beta_\nu d^3\sigma = 0 \tag{1.21}$$

For all times $ct > 0$ the integral over $J_e^\nu \cdot \vartheta$ is just ec so

$$\int J_{\scriptscriptstyle N}^\nu \beta_\nu d^3\sigma = -ec \tag{1.22}$$

This result is most easily determined in the rest frame where $\beta_\nu \, ds = (d^3r, \mathbf{0})$. The total causal charge density therefore consists of a charge with a finite lifetime moving at velocity β^ν, and an oppositely charged radial delta-current which propagates into the cosmos at the speed of light. Since the net charge is always zero, continuity of the four-current is preserved. Note the essential function of the null current here, which serves as a mask to hide the charge from any observer separated from the origin by a spacelike interval.

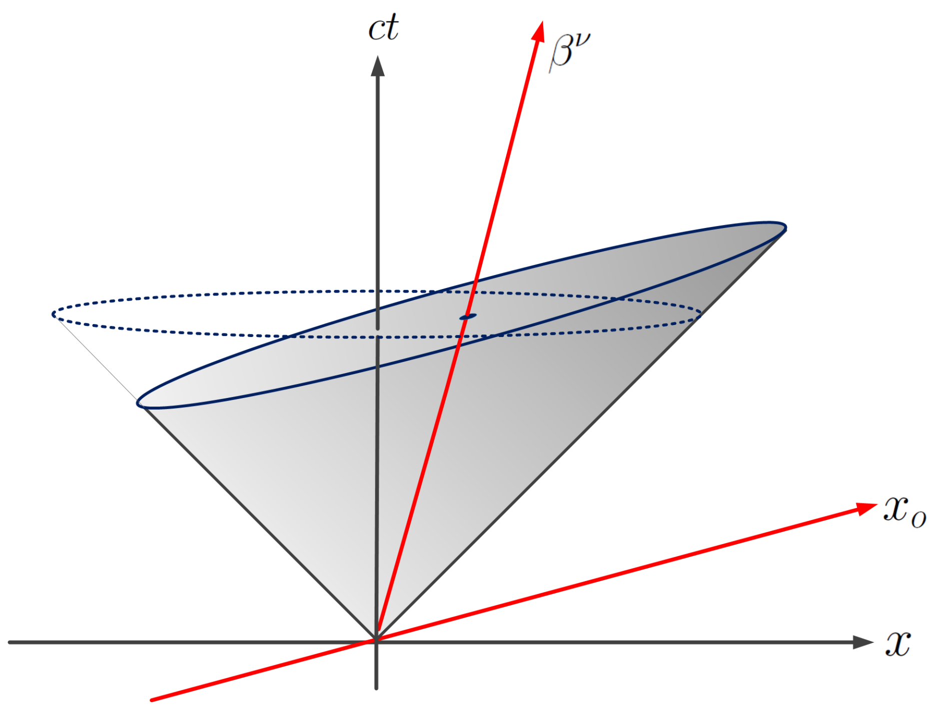

Figure 2: *Point charge density and propagating null current indicated in blue relative to a moving frame. The null current is ill-defined in this frame because a moving observer will see the equal time null current shown by the dashed line.*

More generally, one would also like to show that the null current seen by a moving observer has a net four-velocity β^ν like the particle itself. This can be accomplished beginning with an integral $J^{\nu}_{_N}$ over all space

$$I^{\nu}_{_N} = \int J^{\nu}_{_N} d^4x = -ec\beta^{\nu} \tag{1.23}$$

Details of this integral have been relegated to the appendix.

As a means of comparison between the causal and conventional theories, now define the ***Maxwell limit*** by

$$\lim_{ct\to\infty} J^{*\nu}_{e} = J^{\nu}_{e} \tag{1.24}$$

As this 'irrational' limit is approached, all possible observers in the universe find themselves within the causal reach of the particle. The null current evaporates and what remains is the conventional Maxwell-Lorentz theory.

2 Vacuum Gauge Velocity Potentials

An immediate objective is the development of a potential theory for a charged particle based on a causal field strength tensor. Solutions for rest frame potentials are derived from second order differential equations using the method of Green functions. The generalization to an arbitrary moving frame is straight-forward culminating in a causal calculation of the field strength tensor.

2.1 Differential Equations for Velocity Potentials

In the conventional theory, a potential formulation is initially arrived at by appealing to the homogeneous Maxwell-Lorentz equations. To proceed in this manner for the causal theory, it will be useful to verify that the fields

$$\mathbf{E} = \mathbf{E}_{_M} \cdot \vartheta \quad \text{and} \quad \mathbf{B} = \mathbf{B}_{_M} \cdot \vartheta \tag{2.1}$$

are still solutions to homogeneous equations. This is easy to do with assistance from the relation

$$\frac{1}{c}\frac{\partial \vartheta}{\partial t}\hat{\boldsymbol{n}} = -\boldsymbol{\nabla}\vartheta \tag{2.2}$$

where $\hat{\boldsymbol{n}}$ as the unit vector from the initial (retarded) position of the charge. Now suppose causal fields can be derived from causal potentials $A^{\nu} = A^{\nu}_{_M} \cdot \vartheta$ by the familiar covariant relation

$$F^{\mu\nu}_{_M} \cdot \vartheta = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{2.3}$$

Inserted into (1.17) will then produce causal source equations

$$\Box^2 A^{\nu} - \partial^{\nu}\partial_{\mu} A^{\mu} = \frac{4\pi}{c} J^{*\nu}_{e} \tag{2.4}$$

Unfortunately, the notion of gauge freedom to solve this set of equations loses its appeal here since an additional level of complexity has been added to the four-current density. In other words, solutions A^ν cannot be obtained by choosing an arbitrary gauge condition. This is also evident in equation (2.3) where the causality principle places constraints on the functional form of the potentials. As a specific example, suppose the Liénard–Wiechert potentials are amended to include the causality principle:

$$A_e^\nu \longrightarrow A_e^\nu \cdot \vartheta \tag{2.5}$$

Not only do these potentials fail to produce the correct field strength tensor, they also fall short of satisfying the Lorentz gauge condition—instead, yielding spurious delta functions in both cases.

A more sensible gauge condition for the causal theory is the covariant formula

$$\partial_\nu A_v^\nu = \frac{e}{\rho^2} \cdot \vartheta \tag{2.6}$$

where the subscript v appended to the potentials indicates that accelerations of the particle are not being considered. Inserting into (2.4) gives

$$\Box^2 A_v^\nu - \partial^\nu \left[\frac{e}{\rho^2} \cdot \vartheta\right] = \frac{4\pi}{c} J_e^{*\nu} \tag{2.7}$$

The value of the gauge condition is immediately recognized since the term

$$\frac{e}{\rho^2}\, \partial^\nu \vartheta = -J_{\scriptscriptstyle N}^\nu \tag{2.8}$$

forces an elimination of the null current from the theory. What remains is a source equation for A_v^ν which can be written

$$\Box^2 A_v^\nu = \frac{4\pi}{c} \left[J_e^\nu + J_\ell^\nu\right] \cdot \vartheta \tag{2.9}$$

and where J_ℓ^ν is the ***velocity current*** defined by writing the spacelike vector

$$\mathcal{U}^\nu = \frac{R^\nu}{\rho} - \beta^\nu \tag{2.10}$$

so that

$$\boxed{J_\ell^\nu \equiv \frac{ec}{2\pi} \frac{\mathcal{U}^\nu}{\rho^3}} \tag{2.11}$$

The solution to (2.9) is facilitated by inserting

$$A_v^\nu = A_e^\nu + A_\ell^\nu \tag{2.12}$$

and producing the de-coupled pair

$$\Box^2 A_e^\nu = \frac{4\pi}{c} J_e^\nu \cdot \vartheta \qquad\qquad \Box^2 A_\ell^\nu = \frac{4\pi}{c} J_\ell^\nu \cdot \vartheta \tag{2.13}$$

Both of these equations can be solved by conveniently returning to the rest frame. It will then be a simple matter to apply a Lorentz transformation to moving frame coordinates.

2.2 Green Function Solution for Rest Frame Potentials

Timelike Potentials: For the first equation in (2.13), $\boldsymbol{J}_e = 0$ in the rest frame implies a solution of the form $A_e^\nu = (V, \mathbf{0})$. The goal is therefore to solve the equation

$$\Box^2 V = 4\pi\rho_e \cdot \vartheta \tag{2.14}$$

The retarded Green function for the operator $\Box^2$ will satisfy

$$\Box^2 G_{ret}(\boldsymbol{r}, t; \boldsymbol{r}', t') = \delta^3(\boldsymbol{r} - \boldsymbol{r}')\delta(ct - ct') \tag{2.15}$$

which has the solution

$$G_{ret}(\boldsymbol{r}, t; \boldsymbol{r}', t') = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}\delta(ct' - ct + |\boldsymbol{r} - \boldsymbol{r}'|) \tag{2.16}$$

The scalar potential is then

$$V(\boldsymbol{r}, t) = \iint \frac{\rho_e(\boldsymbol{r}', t')}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}\delta(ct' - ct + |\boldsymbol{r} - \boldsymbol{r}'|)d^3r'\,dt' \tag{2.17}$$

Now suppose at $t = 0$ a point charge is created at the origin. The appropriate charge density is

$$\rho_e(\boldsymbol{r}', t') = e\,\delta^3(\boldsymbol{r}')\,\vartheta(ct') \tag{2.18}$$

Inserting this into (2.17) and integrating over ct' gives

$$V(\boldsymbol{r}, t) = \int \frac{e\delta^3(\boldsymbol{r}')\vartheta(ct - |\boldsymbol{r} - \boldsymbol{r}'|)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}d^3r' \tag{2.19}$$

Integrating over the spatial coordinates yields the simple result

$$V(r, t) = \frac{e}{r} \cdot \vartheta(ct - r) \tag{2.20}$$

This is the conventional Lorentz gauge solution constrained by the causality sphere.

Spacelike Gauge Potentials: The source equation for A_ℓ^μ in equation (2.13) is a vector equation in the rest frame taking the form

$$\Box^2 \boldsymbol{A}_\ell = \frac{4\pi}{c}\mathbf{J}_\ell \cdot \vartheta \tag{2.21}$$

where the spacelike current density can be written

$$\mathbf{J}_\ell \equiv \frac{ce}{2\pi r^3}\hat{\boldsymbol{n}} \tag{2.22}$$

Unfortunately, the Green function in (2.16) does not offer a simple solution here. Instead, it is sufficient to demand a causal solution of the form

$$\boldsymbol{A}_\ell(r, t) = \boldsymbol{A}_\ell(r) \cdot \vartheta(ct - r) \tag{2.23}$$

This eliminates the time dependence and reduces the problem of determining $\boldsymbol{A}_\ell(r,t)$ to solving the two equations

$$\boldsymbol{\nabla}^2 \boldsymbol{A}_\ell = -\frac{4\pi}{c}\mathbf{J}_\ell \qquad\qquad \boldsymbol{\nabla}(\boldsymbol{r}\cdot\boldsymbol{A}_\ell) = 0 \tag{2.24}$$

Saving the second of these equations for last, consider the Green function for the operator $\boldsymbol{\nabla}^2$ which satisfies

$$\boldsymbol{\nabla}^2 G_{ret}(\boldsymbol{r},\boldsymbol{r}') = -\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}') \tag{2.25}$$

and has the solution

$$G_{ret}(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{2.26}$$

Then

$$\boldsymbol{A}_\ell(r) = \frac{4\pi}{c}\int G_{ret}(\boldsymbol{r},\boldsymbol{r}')\mathbf{J}_\ell(r')d^3r \tag{2.27}$$

Now write the expansion

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\,Y_{lm}(\theta,\phi)\,Y^*_{lm}(\theta',\phi') \tag{2.28}$$

Then

$$\boldsymbol{A}_\ell(r) = \sum_{l,m}\frac{2e}{2l+1}\int\frac{r_<^l}{r_>^{l+1}}\,Y_{lm}(\theta,\phi)\,Y^*_{lm}(\theta',\phi')\,\frac{\hat{\boldsymbol{r}}}{r'^3}\,d^3r' \tag{2.29}$$

But the unit vector $\hat{\boldsymbol{r}} = \hat{\boldsymbol{r}}(\theta',\phi')$ has an expansion

$$\hat{\boldsymbol{r}} = \left(\frac{2\pi}{3}\right)^{\frac{1}{2}}\cdot\Big[(-Y_{11}+Y_{1-1})\hat{e}_x + i(Y_{11}+Y_{1-1})\hat{e}_y + \sqrt{2}Y_{10}\hat{e}_z\Big] \tag{2.30}$$

Integration over θ' and ϕ' can now be carried out by use of the orthogonality relation

$$\int Y_{lm}(\theta',\phi')\,Y_{l'm'}(\theta',\phi')\,d\Omega' = \delta_{ll'}\,\delta_{mm'} \tag{2.31}$$

The only term in the sum which contributes to the integral is $l=1$ leaving the radial integral

$$\boldsymbol{A}_\ell(r) = \frac{2e}{3}\hat{\boldsymbol{r}}\int_0^\infty\frac{r_<}{r_>^2}\frac{1}{r'}\,dr' = \frac{e}{r}\hat{\boldsymbol{r}} \tag{2.32}$$

As a check, it is easy to verify the second equation in (2.24). The causal solution for $\boldsymbol{A}_\ell$ is therefore

$$\boldsymbol{A}_\ell(r,t) = \frac{e}{r}\cdot\vartheta(ct-r)\hat{\boldsymbol{r}} \tag{2.33}$$

Comparing this solution with (2.20) it is prudent to drop the subscript on $\boldsymbol{A}_\ell$, and since $\|\boldsymbol{A}\| = \pm V$ re-label $V\to A$, so that the rest frame potentials are

$$A^\nu = (A,\boldsymbol{A})\cdot\vartheta \tag{2.34}$$

and satisfying the null condition $A_\nu A^\nu = 0$. To summarize, causality requires the scalar voltage to be accompanied by a vector potential—even in the rest frame.

2.3 Vacuum Gauge Condition

The most straight forward approach to determine the general causal solution for A_v^ν is to perform a Lorentz transformation on (2.34). However, it is more convenient to work in the Maxwell limit and proceed with a short discussion of the gauge condition in (2.6) which may be written as

$$\boxed{|\partial_\nu A_v^\nu| \equiv \sqrt{\mathrm{E}^2 - \mathrm{B}^2}} \tag{2.35}$$

The gauge introduced by this formula may be referred to as the ***vacuum gauge***. While its primary purpose is to facilitate enforcement of causality, its definition demonstrates an intimate link between $\partial_\nu A^\nu$—more specifically A^ν itself—and the classical theory of the electromagnetic field.

The covariant nature of the vacuum gauge gives it similar properties to the Lorentz gauge. For example, the restricted gauge transformation defined by

$$A^\nu \to A^\nu - \partial^\nu \Lambda \qquad \text{where} \qquad \Box^2 \Lambda = 0 \tag{2.36}$$

will preserve the vacuum gauge condition. This means that, like the Lorentz gauge, the vacuum gauge represents an entire class of potentials. For the special case of electromagnetic waves, the left side of (2.35) is zero and the vacuum gauge is identical to the Lorentz gauge.

When the gauges are not identical, it is possible to implement a specific gauge transformation which connects them. If A_e^ν is the Lorentz gauge potential satisfying $\partial_\nu A_e^\nu = 0$ then the vacuum gauge potential follows from

$$A^\nu = A_e^\nu + \partial^\nu \varphi \tag{2.37}$$

Applying the divergence operator to both sides of this equation and using fields for a point charge shows that the gauge field φ must satisfy

$$\Box^2 \varphi = \frac{e}{\rho^2} \tag{2.38}$$

One approach to solving this equation is by implicit differentiation. Applying the operator ∂_ν to

$$\partial^\nu \varphi = \frac{\partial \varphi}{\partial \rho} \partial^\nu \rho \tag{2.39}$$

and remembering that accelerations are not being considered results in a source equation which may be written

$$L[\,\varphi\,] = -\frac{\partial}{\partial \rho}\left[\rho^2 \frac{\partial}{\partial \rho}\right] \varphi = e \tag{2.40}$$

The operator on the left is self-adjoint and admits a general solution which includes two linearly independent solutions to the homogeneous equation:

$$\varphi = C_1\varphi_1 + C_2\varphi_2 + \varphi_p \tag{2.41}$$

For the record $\varphi_1 = C$ and $\varphi_2 = e/\rho$ but neither of these contributions are necessary for the design of the causal theory. Instead, the particular solution

$$\varphi(\boldsymbol{r}, t) = -e \ln \rho \tag{2.42}$$

is sufficient to determine the causal potentials. A solution is also available by considering a first order equation in the scalar field ψ defined through the relation

$$\varphi(\rho) = \int^{\rho} \psi(\rho')\, d\rho' \tag{2.43}$$

The associated Green function for the first order problem is determined from

$$-\frac{\partial}{\partial\rho}\left[\rho^2\, G(\rho, s)\right] = \delta(\rho - s) \tag{2.44}$$

and seems to provide a neater solution.

2.4 Velocity Potentials and Velocity Fields

A solution to the second equation in (2.13) is now easily determined from the four-gradient of φ with the inclusion of the radial step:

$$A_\ell^\nu = \frac{e}{\rho}\,\mathcal{U}^\nu \cdot \vartheta \tag{2.45}$$

This solution is also derivable by considering the continuous limit of (2.13) and writing

$$\Box^2 A_\ell^\nu = \partial^\nu \frac{e}{\rho^2} \tag{2.46}$$

But the property of gauge invariance requires $A_\ell^\nu = \partial^\nu\varphi$, and this leads to the scalar differential equation in (2.38).

Finally, the two solutions of (2.13) must be added together to produce the vacuum gauge velocity potentials for the causal particle:

$$\boxed{A_v^\nu = \frac{eR^\nu}{\rho^2} \cdot \vartheta} \tag{2.47}$$

A graphical depiction of these potentials is shown in figure 3. For constant velocity motion the effect of the vacuum gauge condition is to generate the gauge field A_ℓ^ν which rotates the Lorentz gauge (Liénard-Wiechert) potentials onto the surface of the light cone. In Minkowski space both A_e^ν and A_ℓ^ν have equal lengths except for an

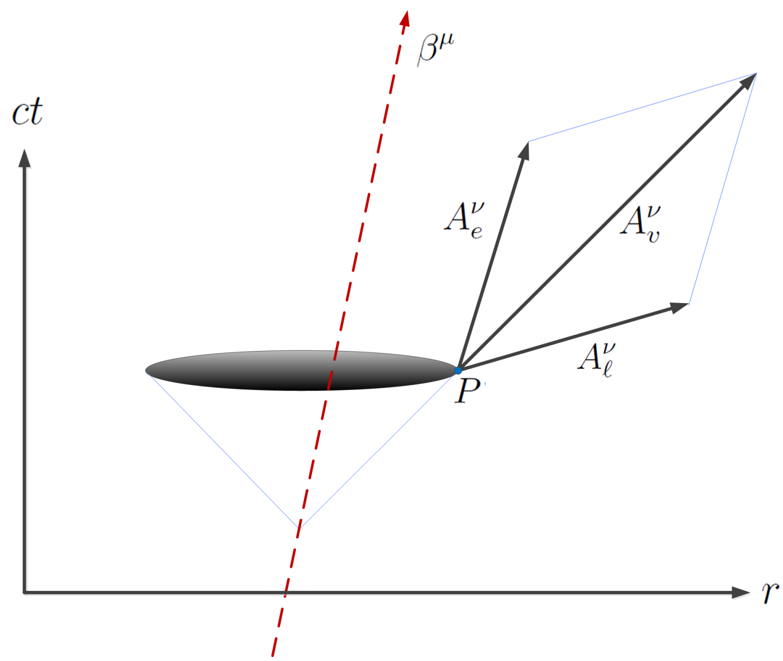

Figure 3: *Vacuum gauge velocity potentials at a point P. The light cone traces the potentials back to the world line of the charge at the retarded time.*

overall sign. They are mutually orthogonal timelike and spacelike components of the vacuum gauge velocity potential.

Equation (2.47) can now be used to verify the causal field strength tensor in (1.16). To facilitate the calculation write the four potential as $A_v^\nu \to A_v^\nu \cdot \vartheta$ so that A_v^ν becomes the associated Maxwell potential. Then it must be shown that

$$F_v^{\mu\nu} = \partial^\mu (A_v^\nu \cdot \vartheta) - \partial^\nu (A_v^\mu \cdot \vartheta) = (\partial^\mu A_v^\nu - \partial^\nu A_v^\mu) \cdot \vartheta \tag{2.48}$$

The validity of this result relies on the fact that $\partial^\nu \vartheta$ is a null vector which has already been determined in equation (1.14). To complete the calculation only requires differentiating the vacuum gauge Maxwell potentials inside the causality sphere. Once again, this calculation should be performed under the assumption of non-zero particle accelerations:

$$F_v^{\mu\nu} = \frac{e}{\rho^2} \left(\partial^\mu R^\nu - \partial^\nu R^\mu\right) - \frac{2e}{\rho^3} \left(R^\nu \partial^\mu \rho - R^\mu \partial^\nu \rho\right) \tag{2.49}$$

But the covariant derivatives

$$\partial^\mu R^\nu = \mathrm{g}^{\mu\nu} - \frac{1}{\rho} R^\mu \beta^\nu \qquad\qquad \partial^\mu \rho = \beta^\mu - \frac{R^\mu}{\rho} + \frac{a^\lambda R_\lambda}{\rho} R^\mu \tag{2.50}$$

can be inserted to yield the causal tensor

$$F_v^{\mu\nu} = \frac{e}{\rho^3} \left(R^\mu \beta^\nu - R^\nu \beta^\mu\right) \cdot \vartheta \tag{2.51}$$

Once again, it is important to observe how the calculation of the field strength tensor completely rejects all terms associated with the acceleration of the particle. This property is also visible in the calculation of the vacuum gauge condition where the second equation in (2.50) will be required along with $\partial_\nu R^\nu = 3$ to determine

$$\partial_\nu A_v^\nu = \frac{e}{\rho^2} \cdot \vartheta \tag{2.52}$$

Covariant Integrals: It will be of high importance to form the covariant integral of (2.52)

$$I = \int_{\mathcal{V}} \partial_\nu A_v^\nu \, d^4x \tag{2.53}$$

where $\mathcal{V}$ represents the four-volume of the causal light cone accessible to the particle. Since the integrand and the volume element are both Lorentz scalars, this means that I is a scalar invariant. It is most easily evaluated in the proper frame where

$$I = \int_0^{ct} \left[\int_0^{ct'} \frac{e}{r^2} r^2 dr \, d\Omega \right] cdt' = 2\pi e c^2 \tau^2 \tag{2.54}$$

Since this result is proportional to the interval, in a frame moving with velocity $\boldsymbol{\beta}$ it generalizes to

$$x^\mu x_\mu = \frac{1}{2\pi e} \int_{\mathcal{V}} \partial_\nu A_v^\nu d^4x \tag{2.55}$$

where $x^\mu = (ct, \boldsymbol{\beta} ct)$ is the coordinate vector of the particle.

Another approach to evaluate I is to apply Gauss' law and integrate over the surface of the light cone shown in figure 4. First note that the volume integral can be written

$$I = \int_{\mathcal{V}} \partial_\nu A_e^\nu d^4x + \int_{\mathcal{V}} \partial_\nu A_\ell^\nu d^4x \tag{2.56}$$

The first integral is zero and this implies that the surface integrals of the Liénard-Wiechert potentials enclosing the volume $\mathcal{S}$ add to zero:

$$\int_{\mathcal{S}} A_e^\nu d_\nu s = 0 \tag{2.57}$$

More specifically, non-zero integrals over each of two hypersurfaces are

$$\int_{\mathcal{S}_1} A_e^\nu \beta_\nu \, d^3\sigma + \int_{\mathcal{S}_2} A_e^\nu R_\nu \, d^2\omega = 0 \tag{2.58}$$

As always, these integrals are most easily evaluated in the rest frame having values $\pm 2\pi e c^2 \tau^2$.

All that remains is to apply Gauss' law to the second integral in equation (2.56), but as indicating by figure 4, the gauge field only contributes along the light cone producing the result

$$I = \int_{\mathcal{S}_2} A_\ell^\nu R_\nu \, d^2\omega \tag{2.59}$$

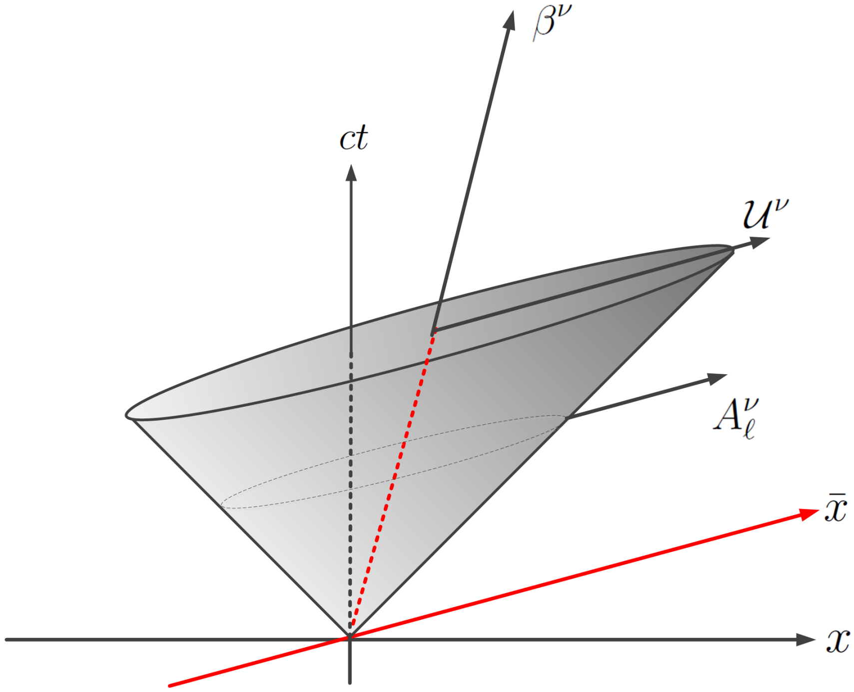

Figure 4: *The gauge field A^ν_ℓ can be integrated over the hyper-surfaces enclosing the four-volume of the light cone. There is no contribution from the integral over the hyper-ellipse.*

3 Vacuum Gauge Acceleration Potentials

Based on calculations of the previous section it is clear that a causal theory of a charged particle will require separate theories of the velocity and acceleration fields. This implies that the acceleration potentials will satisfy their own independent differential equation. Once derived, its solution and subsequent calculation of the field strength tensor $F^{\mu\nu}_a$ are straight-forward.

3.1 Differential Equation for Acceleration Potentials

Working in the Maxwell limit, the form of the vacuum gauge source equation—inclusive of accelerated motions—will be

$$\Box^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu_e \tag{3.1}$$

As a trial solution, A_v^ν can be inserted into the left side of this equation. Excluding the location of the charge, the calculation shows that

$$\Box^2 A_v^\nu = \frac{2e}{\rho^3}\mathcal{U}^\nu \tag{3.2a}$$

$$\partial^\nu \partial_\mu A_v^\mu = \frac{2e}{\rho^3}\mathcal{U}^\nu - \frac{2e}{\rho^4}(a^\lambda R_\lambda)R^\nu \tag{3.2b}$$

The trial solution therefore differs from the actual solution by a single term involving the acceleration of the particle. It is convenient to write (3.2b) as

$$\partial^\nu \partial_\mu A_v^\mu = \frac{4\pi}{c}[J_\ell^\nu + J_a^\nu] \tag{3.3}$$

which defines the *acceleration current*

$$\boxed{J_a^\nu \equiv -\frac{ec}{2\pi\rho^4}(a^\lambda R_\lambda)R^\nu} \tag{3.4}$$

Now assume that the potentials during accelerated motions can be written as

$$A^\nu = A_v^\nu + A_a^\nu \tag{3.5}$$

Inserting this into (3.1) along with the simultaneous appearance of J_a^ν shows that

$$\Box^2 A_v^\nu + \Box^2 A_a^\nu - \partial^\nu \partial_\mu A_v^\mu - \partial^\nu \partial_\mu A_a^\mu = \frac{4\pi}{c}J_e^\nu + \frac{4\pi}{c}J_a^\nu \tag{3.6}$$

Velocity and acceleration terms can now be de-coupled resulting in two independent equations which obey

$$\Box^2 A_v^\nu - \partial^\nu \partial_\mu A_v^\mu = \frac{4\pi}{c}J_e^\nu \tag{3.7a}$$

$$\Box^2 A_a^\nu - \partial^\nu \partial_\mu A_a^\mu = \frac{4\pi}{c}J_a^\nu \tag{3.7b}$$

While the first equation has already been investigated, the mathematical form of the acceleration equation implies that the acceleration fields will satisfy their own set of Maxwell-like equations

$$\partial_\mu F_a^{\mu\nu} = \frac{4\pi}{c}J_a^\nu \tag{3.8}$$

Moreover, J_a^ν is a null vector and will satisfy the continuity equation $\partial_\nu J_a^\nu = 0$ as it must since $F_a^{\mu\nu}$ is anti-symmetric. This is easily proved by noting that

$$\partial_\nu (a^\lambda R_\lambda)R^\nu = a^\lambda R_\lambda \tag{3.9}$$

A physical interpretation of the acceleration current is possible because—like the velocity potential—it lies along the light cone. It appears as a massless $1/r^2$ electrical current which radiates from the instantaneous retarded position of the charge. This means that every point in space containing a non-zero field $\mathbf{E}_a$ can be associated with a local vacuum current density at that same point.

3.2 Acceleration Potentials and Acceleration Fields

It may be possible to solve the differential equation in (3.7b) by a brute force calculation, but it is easier to approach the problem from a different perspective. One possibility is to subtract the vacuum gauge velocity potentials from the Liénard Wiechert potentials obtaining

$$A^\nu = A^\nu_e - A^\nu_v \tag{3.10}$$

These potentials are just the gauge field A^μ_ℓ inclusive of a minus sign and satisfying the gauge condition

$$\partial_\nu A^\nu = \partial_\nu A^\nu_e - \partial_\nu A^\nu_v = -\partial_\nu A^\nu_v \tag{3.11}$$

The legitimacy of the solution can be verified by calculating the field strength tensor from

$$\partial^\mu A^\nu - \partial^\nu A^\mu = (F^{\mu\nu}_v + F^{\mu\nu}_a) - F^{\mu\nu}_v = F^{\mu\nu}_a \tag{3.12}$$

There are several problems with (3.10) however. For example, A^ν does not depend on $\dot{\boldsymbol{\beta}}$ and does not vanish for constant velocity motion, as required by vacuum gauge theory. To remedy the problem implement a gauge transformation:

$$A^\nu_a = A^\nu_e - A^\nu_v + \partial^\nu \lambda \tag{3.13}$$

Such a transformation will have no effect on the fields but the gauge condition becomes

$$\partial_\nu A^\nu_a = -\partial_\nu A^\nu_v + \Box^2 \lambda \tag{3.14}$$

The appropriate gauge field λ can now be determined by evaluating (3.13) assuming constant velocity motion. In this case it is expected that all components of A^ν_a will vanish so

$$\Box^2 \lambda = \partial_\nu A^\nu_v \tag{3.15}$$

As one might have guessed, λ and the scalar field φ of equation (2.42) are the same function. The acceleration potentials are therefore

$$A^\nu_a = \partial^\nu \varphi - A^\nu_v + A^\nu_e \tag{3.16}$$

But the four-gradient of the scalar field is

$$\partial^\nu \varphi = -\frac{e}{\rho}\partial^\nu \rho = A^\nu_v - A^\nu_e - \frac{e}{\rho^2}(a^\lambda R_\lambda)R^\nu \tag{3.17}$$

Comparing with (3.16) and (not so tactfully) appending the radial step, locks in the causal solution

$$\boxed{A^\nu_a(\boldsymbol{r}, t) = -\frac{e}{\rho^2}(a^\lambda R_\lambda)R^\nu \cdot \vartheta} \tag{3.18}$$

As with the velocity theory, the vacuum gauge is requiring radial potentials such that $A^a_\nu A^\nu_a = 0$. Note also that A^ν_a does not diminish for large R.

The ability of the causal potentials in (3.18) to render causal acceleration fields is determined by their direction along the light cone. Writing $A_a^\nu \to A_a^\nu \cdot \vartheta$, then the simple calculation verifies

$$F_a^{\mu\nu} = \partial^\mu(A_a^\nu \cdot \vartheta) - \partial^\nu(A_a^\mu \cdot \vartheta) = (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) \cdot \vartheta \tag{3.19}$$

Inside the causal region the acceleration potentials may be written

$$A_a^\nu = -(a^\lambda R_\lambda) A_v^\nu \tag{3.20}$$

which can be differentiated to produce

$$F_a^{\mu\nu} = -\partial^\mu(a^\lambda R_\lambda) A_v^\nu + \partial^\nu(a^\lambda R_\lambda) A_v^\mu - (a^\lambda R_\lambda) F_v^{\mu\nu} \tag{3.21}$$

A rigorous calculation of $\partial^\mu(a^\lambda R_\lambda)$ requires some computational stamina but leads to the simple result

$$\partial^\nu(a^\lambda R_\lambda) = a^\nu + \frac{\dot{a}^\lambda R_\lambda}{\rho} R^\nu \tag{3.22}$$

where $\dot{a}^\lambda = \partial a^\lambda / \partial \tau$ and where τ is the proper time. Now define the scalar field[3] $\xi \equiv a^\lambda R_\lambda / \rho$ and easily determine the final form of the field strength tensor:

$$F_a^{\mu\nu} = \frac{e}{\rho^2} \left[R^\mu a^\nu - R^\nu a^\mu - \xi (R^\mu \beta^\nu - R^\nu \beta^\mu) \right] \cdot \vartheta \tag{3.23}$$

3.3 Total Potentials

Combining the velocity potentials in (2.47) with the acceleration potentials in (3.18), the general vacuum gauge solution for arbitrary motions of a charged particle can be written

$$\boxed{A^\nu(\boldsymbol{r}, t) = \frac{e(1 - a^\lambda R_\lambda)}{\rho^2} R^\nu \cdot \vartheta} \tag{3.24}$$

Based on previous discussion, the mathematical structure of this formula also allows the potentials to be divided as

$$A^\nu = A_e^\nu + A_\ell^\nu + A_a^\nu \tag{3.25}$$

where

$$A_e^\nu = \frac{e}{\rho} \beta^\nu \cdot \vartheta \qquad A_\ell^\nu = \frac{e}{\rho} \mathcal{U}^\nu \cdot \vartheta \qquad A_a^\nu = -\frac{e}{\rho} \xi R^\nu \cdot \vartheta \tag{3.26}$$

Each of these potentials can be associated with its own vector current density, and each current density points in the same direction as its associated potentials. Two of the currents are conserved but one is not:

$$\partial_\nu J_e^\nu = 0 \qquad \partial_\nu J_\ell^\nu = -\frac{ec}{2\pi\rho^4} \qquad \partial_\nu J_a^\nu = 0 \tag{3.27}$$

[3] This definition of ξ is similar to Rohrlich's definition $a_u \equiv a^\nu \mathcal{U}_\nu$. The new definition was chosen to avoid confusion with the covariant four-acceleration.

Causality and the Scalar Field: The development of vacuum gauge theory begins from the assertion that the velocity field of a charged particle is a causal field. However, causality can also be introduced by constraining the scalar field φ as

$$\varphi = -e \ln \rho \cdot \vartheta \tag{3.28}$$

A first derivative for general accelerated motions picks up a delta function

$$\partial^\nu \varphi = [A_v^\nu + A_a^\nu - A_e^\nu] \cdot \vartheta - e \ln \rho \cdot \partial^\nu \vartheta \tag{3.29}$$

but the second derivative can be written

$$\partial^\mu \partial^\nu \varphi = [\partial^\mu A_v^\nu + \partial^\mu A_a^\nu - \partial^\mu A_e^\nu] \cdot \vartheta + S^{\mu\nu} \tag{3.30}$$

where $S^{\mu\nu}$ is the collection of remaining terms symmetric in the two indicies. Now form the object

$$\partial_\mu \left[\partial^\mu \partial^\nu \varphi - \partial^\nu \partial^\mu \varphi\right] = 0 \tag{3.31}$$

This set of operations can be written in terms of the potentials as

$$[\Box^2 A_v^\nu + \Box^2 A_a^\nu - \Box^2 A_e^\nu - \partial^\nu \partial_\mu A_v^\mu - \partial^\nu \partial_\mu A_a^\mu] \cdot \vartheta + \partial_\mu \vartheta (F^{\mu\nu} - F^{\mu\nu}) = 0 \tag{3.32}$$

But the velocity and acceleration terms in the brackets can be separated with the introduction of the acceleration current. Moreover, the point charge four-current J_e^ν replaces the wave operator acting the Liénard-Wiechert potentials. What remains is independent equations of motion given by

$$[\,\Box^2 A_v^\nu - \partial^\nu \partial_\mu A_v^\mu - \frac{4\pi}{c} J_e^\nu\,] \cdot \vartheta = 0 \qquad [\,\Box^2 A_a^\nu - \partial^\nu \partial_\mu A_a^\mu - \frac{4\pi}{c} J_a^\nu\,] \cdot \vartheta = 0 \tag{3.33}$$

Both are automatically constrained by casuality without the presence of the null current.

Pair Production/Annihilation: An excellent example of the use of total vacuum gauge potentials is for the description of the creation and annihilation of an $e^+\, e^-$ pair. The spacetime diagram in figure 5 shows the pair created at the origin of coordinates and annihilated at a later time $c\tau_o$. First, write the position vector of each particle as

$$\boldsymbol{w}_+ = \boldsymbol{w}_+(\tau) \tag{3.34a}$$
$$\boldsymbol{w}_- = \boldsymbol{w}_-(\tau) \tag{3.34b}$$

Since the pair created and annihilated itself, their positions should be identical at time $\tau = 0$ and $\tau = \tau_o$:

$$\boldsymbol{w}_+(0) = \boldsymbol{w}_-(0) \tag{3.35a}$$
$$\boldsymbol{w}_+(\tau_o) = \boldsymbol{w}_-(\tau_o) \tag{3.35b}$$

Add to these constraints the retardation condition, which is generally different for each particle

$$R^\nu_+ = x^\nu - w^\nu_+(t_r) \tag{3.36a}$$
$$R^\nu_- = x^\nu - w^\nu_-(t_r) \tag{3.36b}$$

In the frame S the vacuum gauge potentials $A^\nu_{tot}(ct, \boldsymbol{r})$ are the sum of vacuum gauge

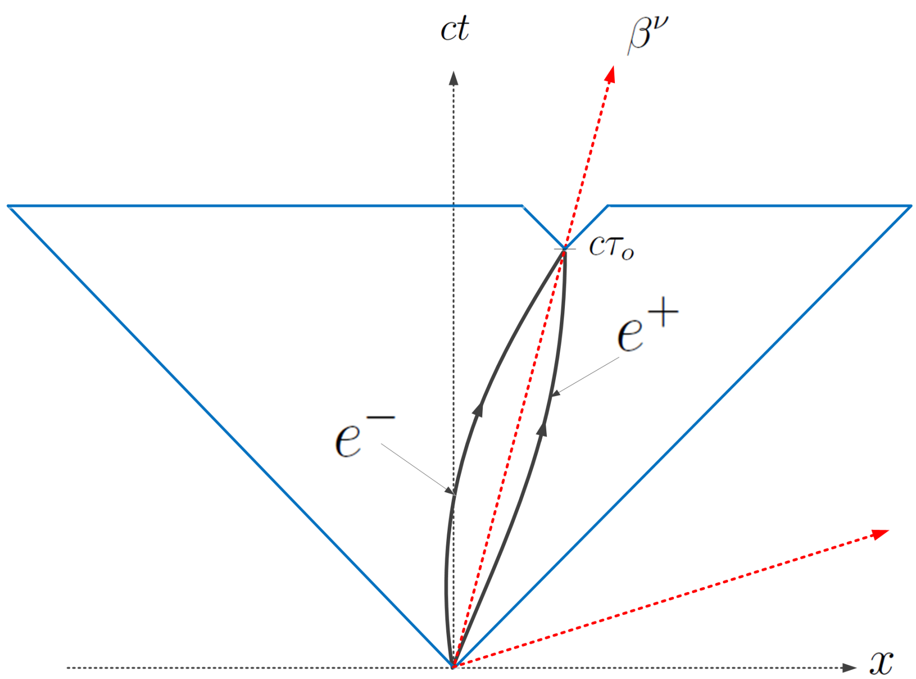

Figure 5: *Spacetime diagram exhibiting a two particle* $e^+ e^-$ *system in the vacuum gauge. The blue line marks the limit of the causal fields as viewed in the frame* S*.*

potentials associated with each particle. In addition, since both particles share the same causality requirements at both ends of their life span, the form of the potentials is

$$A^\nu_{tot}(ct, \boldsymbol{r}) = \left[A^\nu_{v+} + A^\nu_{a+} + A^\nu_{v-} + A^\nu_{a+}\right] \cdot \vartheta(ct_r) \cdot \vartheta(\gamma c\tau_o - ct_r) \tag{3.37}$$

Derivatives then determine the appropriate form of the causal field stength tensor

$$F^{\mu\nu}_{tot}(ct, \boldsymbol{r}) = [F^{\mu\nu}_+ + F^{\mu\nu}_-] \cdot \vartheta(ct_r) \cdot \vartheta(\gamma c\tau_o - ct_r) \tag{3.38}$$

The four-current density associated with this system is determined by the two point charges and also by two causality induced delta-currents. It would be delightful if the delta-currents canceled for the two equal and opposite charges, but this is not quite the case since each particle has a different velocity during creation and annihilation. For example, using $e = -|e|$ the first delta current may be written

$$J^{\nu}_{N1} = -\frac{ec}{4\pi}\frac{(1,\hat{\boldsymbol{r}})}{r^2}\left[\frac{1}{\gamma_-^3(1-\hat{\boldsymbol{r}}\cdot\boldsymbol{\beta}_-)^3} - \frac{1}{\gamma_+^3(1-\hat{\boldsymbol{r}}\cdot\boldsymbol{\beta}_+)^3}\right]\cdot\delta(ct-r) \tag{3.39}$$

where all velocities are evaluated at $ct_r = 0$.

Certainly the time $c\tau_o$ can be anything, but there exists a real possibility that a model such as this can be linked to the brief appearance of $e^+\,e^-$ pairs described by Quantum Field Theory. In support of this idea the two particle system already has two causality functions serving as classical creation and annihilation operators. If an appropriate classical Hamiltonian can be constructed for this system, it would be a tremendous accomplishment to develop a simple quantization procedure leading directly to the quantized Dirac field Hamiltonian operator.

A Integral of the Null Current

The goal here is to perform the integral of the null current defined in equation (1.18) and show that it moves with the four-velocity of the particle. A differential dQ^ν associated with the four-volume integral is

$$\begin{aligned} dQ^\nu &= \int J_N^\nu R^2 d\Omega d\rho\, cd\tau \\ &= -\frac{ec}{4\pi}\int \frac{R^\nu}{\rho^3}\delta(c\tau-\rho)R^2 d\Omega d\rho\, cd\tau \end{aligned} \tag{A.1}$$

First suppose that the time integral is witheld. Using $\rho = \gamma R(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})$ and

$$\frac{R^\nu}{\rho} = \mathcal{U}^\nu + \beta^\nu \tag{A.2}$$

allows for an all space integral over the independent variable $d\rho$ resulting in

$$\frac{dQ^\nu}{dc\tau} = -\frac{ec}{4\pi}\int_\Omega \frac{\mathcal{U}^\nu+\beta^\nu}{\gamma^2(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^2}d\Omega \cdot \int_0^\infty \delta(c\tau-\rho)d\rho \tag{A.3}$$

The solid angle integral[4] eliminates the spacelike direction $\mathcal{U}^\nu$ leading to the final result:

$$\frac{dQ^\nu}{dc\tau} = -ec\beta^\nu \tag{A.5}$$

A more interesting possibility is to relax the requirement to integrate ρ to spatial infinity. The last integral in (A.3) may then be written

$$\int_0^\rho \delta(c\tau-\rho')d\rho' = \vartheta(\rho-c\tau) \tag{A.6}$$

Couple this with four-current associated with the moving point charge and write the total current

$$I^\nu_{total} = ec\beta^\nu\cdot[\vartheta(c\tau)-\vartheta(\rho-c\tau)] \tag{A.7}$$

If $\rho > c\tau$, the value of this function is zero indicating that the observer has no knowledge of the charge. For large enough times however, the null current eventually passes the observer who then becomes aware of the four-current $I^\nu_{total} = ec\beta^\nu$.

[4]An important variation of the solid angle integral is to write

$$-\frac{ec}{4\pi}\int_\Omega \frac{[1,\hat{\boldsymbol{n}}]}{\gamma^3(1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta})^3}d\Omega = -ec\beta^\nu \tag{A.4}$$

The four-velocity can then be pointed in the z-direction to give simple time and space component integrals yielding the result $-ec(\gamma,0,0,\gamma\beta)$.

B Lorentz Transformation of Potentials and Fields

The velocity and acceleration potentials are both four-vectors so it will be possible to perform separate Lorentz transformations on each. For the velocity potentials the transformation equations mirror those of the coordinate transformation:

$$A'_v = \gamma(A_v + \boldsymbol{\beta} \cdot \boldsymbol{A}_v) \tag{B.1a}$$

$$\boldsymbol{A}'_v = \boldsymbol{A}_v + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{A}_v)\boldsymbol{\beta} + \gamma A_v \boldsymbol{\beta} \tag{B.1b}$$

However, velocity potentials are composed of timelike and spacelike components—each of which may be transformed by itself. Component transformations from the rest frame are particularly simple.

$$A'_e = \gamma A_e \qquad \boldsymbol{A}'_e = \gamma \boldsymbol{\beta} A_e \tag{B.2a}$$

$$A'_\ell = \gamma \boldsymbol{\beta} \cdot \boldsymbol{A}_\ell \qquad \boldsymbol{A}'_\ell = \boldsymbol{A}_\ell + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{A}_\ell)\boldsymbol{\beta} \tag{B.2b}$$

The norms of both four-vectors are easily shown to be Lorentz scalars. Moreover, their norms are the same to within sign which will enforce a zero norm for the velocity potential

$$A'^{\nu}_v = A'^{\nu}_e + A'^{\nu}_\ell \tag{B.3}$$

The importance of the spacelike potentials emerge when the velocity field strength tensor is written

$$eF^{\mu\nu}_v = [A'^{\mu}_\ell,\, A'^{\nu}_e] \tag{B.4}$$

The anti-symmetric tensor loses its traditional identity as a fundamental object—being replaced by an interaction among four-potentials. The second order transformation law for $F^{\mu\nu}_v$ is then a combination of first order transformations

$$[A'^{\mu}_\ell,\, A'^{\nu}_e] = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \, [A^{\alpha}_\ell,\, A^{\beta}_e] \tag{B.5}$$

Transformation Law for Velocity Fields: As an example of the use of the gauge field, velocity electric and magnetic field vectors in a moving frame are

$$\mathbf{E}'_v = \frac{\gamma}{\rho}\,(\boldsymbol{A}'_\ell - A'_\ell \boldsymbol{\beta}) \qquad \mathbf{B}'_v = \frac{\gamma}{\rho}\,(\boldsymbol{\beta} \times \boldsymbol{A}'_\ell) \tag{B.6}$$

Now suppose the transformation of the gauge field in equation (B.2b) is inserted into (B.6). In terms of rest frame potentials one finds

$$\mathbf{E}'_v = \frac{\gamma}{\rho}\left[\boldsymbol{A}_\ell - \frac{\gamma}{\gamma + 1}(\boldsymbol{\beta} \cdot \boldsymbol{A}_\ell)\boldsymbol{\beta}\right] \tag{B.7a}$$

$$\mathbf{B}'_v = \frac{\gamma}{\rho}(\boldsymbol{\beta} \times \boldsymbol{A}_\ell) \tag{B.7b}$$

In the rest frame, $\boldsymbol{A}_\ell/\rho = \mathbf{E}_v$ which derives the transformation law for the electric and magnetic field vectors of the particle

$$\mathbf{E}'_v = \gamma\mathbf{E}_v - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{E}_v)\boldsymbol{\beta} \tag{B.8a}$$

$$\mathbf{B}'_v = \gamma\boldsymbol{\beta}\times\mathbf{E}_v \tag{B.8b}$$

Transformation of Acceleration Potentials: In the rest frame the acceleration potentials take the form

$$A^\nu_a = \boldsymbol{r}\cdot\dot{\boldsymbol{\beta}}A^\nu_v \tag{B.9}$$

A Lorentz transformation to a moving frame is easily accomplished by simply transforming the velocity potentials followed by the dot product. Unfortunately, $\dot{\boldsymbol{\beta}}$ does not transform as a four-vector. In fact, the transformation law is

$$\dot{\boldsymbol{\beta}} \to \gamma^3\dot{\boldsymbol{\beta}}'_\perp + \gamma^2\dot{\boldsymbol{\beta}}'_{||} = \frac{\gamma^4}{(\gamma+1)}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}')\boldsymbol{\beta} + \gamma^2\dot{\boldsymbol{\beta}}' \tag{B.10}$$

Combining this equation with

$$\boldsymbol{r} = \boldsymbol{R} + \frac{(\gamma-1)}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{R})\boldsymbol{\beta} - \gamma R\boldsymbol{\beta} \tag{B.11}$$

shows that the dot product transforms as

$$\boldsymbol{r}\cdot\dot{\boldsymbol{\beta}} \to -a^\lambda R_\lambda \tag{B.12}$$

where

$$a^\lambda = [\gamma^4\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}',\, \gamma^2\dot{\boldsymbol{\beta}}' + \gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}')\boldsymbol{\beta}] \tag{B.13}$$

is the four-acceleration.

Transformation of Acceleration Fields: A bracket for the acceleration fields follows by first defining the quantity

$$a^\nu_\perp \equiv a^\nu + (a^\lambda\mathcal{U}_\lambda)\mathcal{U}^\nu \tag{B.14}$$

which allows the field stength tensor to be written as

$$F^{\mu\nu}_a = [A^\mu_v,\, a^\nu_\perp] \tag{B.15}$$

Once again, the second rank tensor is determined by an anti-symmetric combination of four-vectors. In this case however, the vacuum gauge potentials cannot be traded in for the gauge field. Instead, the expansion of the bracket is

$$F^{\mu\nu}_a = [A^\mu_e,\, a^\nu_\perp] + [A^\mu_\ell,\, a^\nu_\perp] \tag{B.16}$$

and both terms are non-zero. To derive the appropriate transformation laws for the fields, first write electric and magnetic field vectors as linear functions of the velocity potentials:

$$\mathbf{E}'_a = \frac{\gamma^3}{\rho}\boldsymbol{R}\cdot\dot{\boldsymbol{\beta}}'(\boldsymbol{A}'_v - A'_v\boldsymbol{\beta}) - \gamma^2 A'_v\dot{\boldsymbol{\beta}}' \tag{B.17a}$$

$$\mathbf{B}'_a = \frac{\gamma^3}{\rho}(\boldsymbol{R}\cdot\dot{\boldsymbol{\beta}}')\boldsymbol{\beta}\times\boldsymbol{A}'_v + \gamma^2\dot{\boldsymbol{\beta}}'\times\boldsymbol{A}'_v \tag{B.17b}$$

Unfortunately, inserting transformation equations for velocity potentials will not be sufficient here as the inverse of (B.10) is also required:

$$\dot{\boldsymbol{\beta}}' \to \frac{1}{\gamma^2}\dot{\boldsymbol{\beta}} - \frac{1}{\gamma(\gamma+1)}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta} \tag{B.18}$$

The relation $\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = \gamma^3(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}')$ is also useful and if the rest frame potentials are written

$$\mathbf{E}_a = \frac{1}{\rho}(\boldsymbol{R}\cdot\dot{\boldsymbol{\beta}}')\boldsymbol{A}_v - A_v\dot{\boldsymbol{\beta}} \qquad \text{and} \qquad \mathbf{B}_a = \dot{\boldsymbol{\beta}}\times\boldsymbol{A}_v \tag{B.19}$$

then the transformation laws become

$$\mathbf{E}'_a = \gamma(\mathbf{E}_a - \boldsymbol{\beta}\times\mathbf{B}_a) - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{E}_a)\boldsymbol{\beta} \tag{B.20}$$

$$\mathbf{B}'_a = \gamma(\mathbf{B}_a + \boldsymbol{\beta}\times\mathbf{E}_a) - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B}_a)\boldsymbol{\beta} \tag{B.21}$$

This derivation is somewhat cumbersome and does not carry the same appeal as the velocity field transformation law derived in equation (B.8).