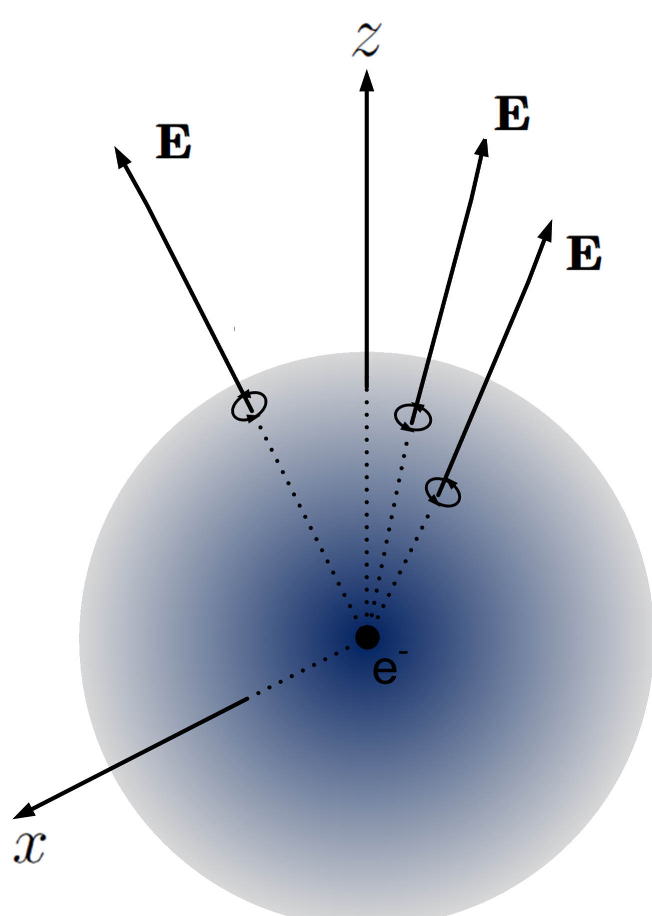
The
Maxwell Limit

July 8, 2017



Abstract

Beginning with the Maxwell-Lorentz equations for a charged point particle, an application of the principle of causality will lead naturally to the vacuum gauge condition. The presence of the gauge field allows the velocity and acceleration fields of the classical particle to be formulated as independent theories determined from associated vacuum gauge potentials.

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1 Causal Theory of a Charged Particle

Define a flat spacetime using the Minkowski metric

$$\mathrm{g}^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{1.1}$$

where the general homogeneous Lorentz transformation relating two sets of coordinates $(c\tau, \boldsymbol{\rho}) \longrightarrow (ct, \boldsymbol{r})$ is given by

$$c\tau = \gamma\,(ct - \boldsymbol{\beta}\cdot\boldsymbol{r}) \tag{1.2a}$$

$$\boldsymbol{\rho} = \boldsymbol{r} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta}\cdot\boldsymbol{r})\boldsymbol{\beta} - \gamma ct\boldsymbol{\beta} \tag{1.2b}$$

A point electron moving through the spacetime is characterized by a four-current density J_e^ν and the associated fields satisfy the ***Maxwell-Lorentz equations***

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J_e^\nu \tag{1.3}$$

The name is appropriate since the original Maxwell theory of the electromagnetic field was a macroscopic theory, which Lorentz first applied to a charged particle.

Unfortunately, although equation (1.3) generates a workable theory of the electron, it cannot be the correct theory for the simple reason that the particle can neither be created nor destroyed. This is inconsistent with the modern philosophy of quantum field theory which allows for the creation and annihilation of electrons and positrons out of the vacuum. As a primary example, it is a violation of the causality principle to assign a Coulomb field to any charged particle which was created at a finite time $\Delta\tau$ in the past. This field extends to spatial infinity and requires the particle to have an eternal lifespan.

1.1 Theory of Electric Charge

To eliminate this problem suppose classical electron theory is re-formulated based on a field strength tensor given by

$$F^{\mu\nu} = F_M^{\mu\nu} \cdot \vartheta(ct_r/\gamma) \tag{1.4}$$

In this equation $F_M^{\mu\nu}$ is the conventional Maxwell-Lorentz field stength tensor, ct_r is the retarded time, and $\vartheta(ct_r/\gamma)$ is a light sphere which expands in all directions at the speed of light (see figure 1). The argument of ϑ may also be written in terms of proper frame variables $c\tau$ and ρ so that a useful definition is

$$\vartheta(c\tau - \rho) \equiv \begin{cases} 1 & c\tau > \rho \\ 0 & c\tau \leq \rho \end{cases} \tag{1.5}$$

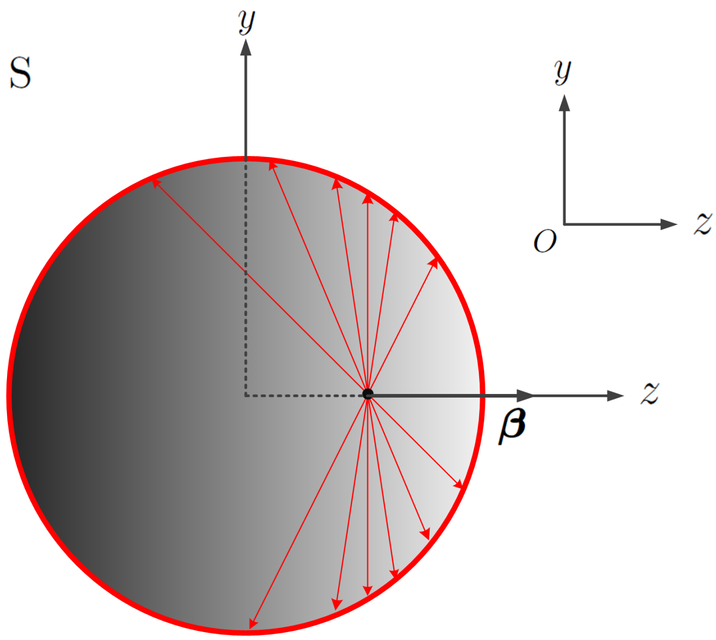

Figure 1: *The fields of a causal electron as it moves to the right. The light sphere expands about the origin of coordinates where the electron was created. The observer at O is not yet aware of the existence of the particle.*

and having a four-gradient

$$\partial^\nu \vartheta(c\tau - \rho) = \delta(c\tau - \rho) \cdot \partial^\nu (c\tau - \rho) = \delta(c\tau - \rho) \cdot \frac{R^\nu}{\rho} \tag{1.6}$$

Naturally, the inclusion of the causality step brings about an added complexity to the conventional notion of the particles' charge and current density since the divergence operation on the field strength tensor is

$$\partial_\mu F^{\mu\nu} \equiv \frac{4\pi}{c} J_e^{*\nu} \tag{1.7}$$

where

$$\boxed{J_e^{*\nu} \equiv J_e^\nu \cdot \vartheta + J_{\scriptscriptstyle N}^\nu} \qquad \boxed{J_{\scriptscriptstyle N}^\nu \equiv \frac{c}{4\pi} F_{\scriptscriptstyle M}^{\mu\nu} \cdot \partial_\mu \vartheta} \tag{1.8}$$

and where $J_{\scriptscriptstyle N}^\nu$ may be referred to as the ***null current***—having a four-space norm of zero. The properties of $J_e^{*\nu}$ can be understood through a divergence calculation. This is a continuity equation for electric charge which might not be expected to hold for a single particle since the charge has been created from nothing. Curiously enough, this is a false assumption:

$$\partial_\nu J_e^{*\nu} = \partial_\nu J_e^\nu \cdot \vartheta + J_e^\nu \cdot \partial_\nu \vartheta + \frac{c}{4\pi} \partial_\nu F_{\scriptscriptstyle M}^{\mu\nu} \cdot \partial_\mu \vartheta + \frac{c}{4\pi} F_{\scriptscriptstyle M}^{\mu\nu} \cdot \partial_\nu \partial_\mu \vartheta \tag{1.9}$$

The first term and the last term on the right are both zero. The two terms in the middle cancel out so that $\partial_\nu J_e^{*\nu} = 0$.

Integral of the Four-Current Density: The physics of this calculation can be understood beginning with the integral of the four-current density

$$Q^\nu = \int J_e^{*\nu} d^4x \tag{1.10}$$

Using $d\Omega$ as a solid angle element from the particles retarded position and $\rho = R^\nu \beta_\nu$ this is actually a very simple calculation since Gauss' law can be used to write

$$dQ^\nu = \int \partial_\mu F^{\mu\nu} R^2 d\rho\, d\Omega\, cd\tau = -\oint F^{\mu\nu} \mathcal{U}_\mu R^2 d\Omega\, cd\tau \tag{1.11}$$

Then

$$\frac{dQ^\nu}{d\tau} = ec\beta^\nu \cdot \vartheta(c\tau - \rho) \tag{1.12}$$

Now suppose this result is to be obtained by explicitly evaluating the integral in (1.10). In this case the differential dQ_e^ν for the first part of $J_e^{*\nu}$ is

$$\frac{dQ_e^\nu}{d\tau} = \int J_e^\nu \cdot \vartheta R^2 d\rho\, d\Omega = ec\beta^\nu \cdot \vartheta(c\tau) \tag{1.13}$$

For the null current

$$\begin{aligned} dQ_{_N}^\nu &= \int J_{_N}^\nu R^2 d\Omega d\rho\, cd\tau \\ &= -\frac{ec}{4\pi} \int \frac{R^\nu}{\rho^3} \delta(c\tau - \rho) R^2 d\Omega d\rho\, cd\tau \end{aligned} \tag{1.14}$$

Inserting appropriate limits of integration this is

$$\frac{dQ_{_N}^\nu}{d\tau} = -\frac{ec}{4\pi} \int_\Omega \frac{\mathcal{U}^\nu + \beta^\nu}{\gamma^2(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^2} d\Omega \cdot \int_0^\rho \delta(c\tau - \rho') d\rho' \tag{1.15}$$

The solid angle integral[1] eliminates the spacelike direction $\mathcal{U}^\nu$ while the radial integral is

$$\int_0^\rho \delta(c\tau - \rho') d\rho' = \vartheta(\rho - c\tau) \tag{1.17}$$

[1] An important variation of the solid angle integral is to write

$$-\frac{ec}{4\pi} \int_\Omega \frac{[1, \hat{\boldsymbol{n}}]}{\gamma^3(1 - \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta})^3} d\Omega = -ec\beta^\nu \tag{1.16}$$

The four-velocity can then be pointed in the z-direction to give simple time and space component integrals yielding the result $-ec(\gamma, 0, 0, \gamma\beta)$.

This leaves

$$\frac{dQ^\nu_{_N}}{d\tau} = -ec\beta^\nu \cdot \vartheta(\rho - c\tau) \tag{1.18}$$

Couple this with four-current associated with the moving point charge and write the total current

$$\frac{dQ^\nu}{d\tau} = \frac{dQ^\nu_e}{d\tau} + \frac{dQ^\nu_{_N}}{d\tau}$$

$$= ec\beta^\nu \cdot [\vartheta(c\tau) - \vartheta(\rho - c\tau)] = ec\beta^\nu \cdot \vartheta(c\tau - \rho) \tag{1.19}$$

The total causal charge density therefore consists of a charge with a finite lifetime moving at velocity β^ν, and an oppositely charged radial delta-current which propagates into the cosmos at the speed of light. Since the net charge is always zero, continuity of the four-current is preserved. The essential function of the null current is therefore a mask to hide the charge from any observer separated from the origin by a spacelike interval.

As a means of comparison between the causal and conventional theories, now define the ***Maxwell limit*** by

$$\lim_{ct \to \infty} J^{*\nu}_e = J^\nu_e \tag{1.20}$$

As this 'irrational' limit is approached, all possible observers in the universe find themselves within the causal reach of the particle. The null current evaporates and what remains is the conventional Maxwell-Lorentz theory.

2 Vacuum Gauge Velocity Potentials

An immediate objective is the development of a potential theory for the electron based on a causal field strength tensor. For the purpose of implicit differentiation, it will be useful to replace the definition for the causality step in equation (1.5) with

$$\vartheta(ct_r/\gamma) \equiv \begin{cases} 1 & ct_r \geq 0 \\ 0 & ct_r < 0 \end{cases} \tag{2.1}$$

Its four-gradient is

$$\partial^\nu \vartheta(ct_r/\gamma) = \frac{\partial \vartheta(ct_r/\gamma)}{\partial ct_r} \cdot \partial^\nu ct_r/\gamma = \delta(ct_r/\gamma) \cdot \frac{R^\nu}{\rho} \tag{2.2}$$

and it should be emphasized here that $\partial^\nu \vartheta$ is not functionally tied to the instantaneous acceleration of the particle—either when it was created, or at any other time.

2.1 Differential Equations for Velocity Potentials

In the conventional theory, a potential formulation is initially arrived at by appealing to the homogeneous Maxwell-Lorentz equations. To proceed in this manner for the causal theory, it will be useful to verify that the velocity fields

$$\mathbf{E} = \mathbf{E}_{M} \cdot \vartheta \quad \text{and} \quad \mathbf{B} = \mathbf{B}_{M} \cdot \vartheta \tag{2.3}$$

are still solutions to homogeneous equations. This is easy to do with assistance from the relation

$$\frac{1}{c}\frac{\partial \vartheta}{\partial t}\hat{\boldsymbol{n}} = -\boldsymbol{\nabla}\vartheta \tag{2.4}$$

where $\hat{\boldsymbol{n}}$ as the unit vector from the initial (retarded) position of the charge. Now suppose causal fields can be derived from causal potentials $A^{\nu} = A^{\nu}_{M} \cdot \vartheta$ by the familiar covariant relation

$$F^{\mu\nu}_{M} \cdot \vartheta = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \tag{2.5}$$

Inserting into (1.7) will then produce causal source equations

$$\Box^2 A^{\nu} - \partial^{\nu}\partial_{\mu} A^{\mu} = \frac{4\pi}{c} J^{*\nu}_{e} \tag{2.6}$$

Unfortunately, the notion of gauge freedom to solve this set of equations loses its appeal here since the four-current density is a more complicated object than the traditional point charge density. In other words, solutions A^{ν} cannot be obtained by choosing an arbitrary gauge condition. This is also evident in equation (2.5) where the causality principle places constraints on the functional form of the potentials. As a specific example, suppose the Liénard–Wiechert potentials are amended to include the causality principle:

$$A^{\nu}_{e} \longrightarrow A^{\nu}_{e} \cdot \vartheta \tag{2.7}$$

Not only do these potentials fail to produce the correct field strength tensor, they also fall short of satisfying the Lorentz gauge condition—instead, yielding spurious delta functions in both cases.

A more sensible gauge condition for the causal theory is the covariant formula

$$\partial_{\nu} A^{\nu}_{v} = \frac{e}{\rho^2} \cdot \vartheta \tag{2.8}$$

where the subscript v appended to the potentials indicates that accelerations of the particle are not being considered. Inserting into (2.6) gives

$$\Box^2 A^{\nu}_{v} - \partial^{\nu}\left[\frac{e}{\rho^2} \cdot \vartheta\right] = \frac{4\pi}{c} J^{*\nu}_{e} \tag{2.9}$$

The value of the gauge condition is immediately recognized since the term

$$\frac{e}{\rho^2}\,\partial^{\nu}\vartheta = -J^{\nu}_{N} \tag{2.10}$$

forces an elimination of the null current from the theory. What remains is a source equation for A_v^ν which can be written

$$\Box^2 A_v^\nu = \frac{4\pi}{c}\left[J_e^\nu + J_\ell^\nu\right] \cdot \vartheta \tag{2.11}$$

and where J_ℓ^ν is the ***velocity current*** defined by writing the spacelike vector

$$\mathcal{U}^\nu = \frac{R^\nu}{\rho} - \beta^\nu \tag{2.12}$$

so that

$$J_\ell^\nu \equiv \frac{ec}{2\pi}\frac{\mathcal{U}^\nu}{\rho^3} \tag{2.13}$$

The solution to (2.11) is facilitated by inserting $A_v^\nu = A_e^\nu + A_\ell^\nu$ to produce the decoupled pair

$$\Box^2 A_e^\nu = \frac{4\pi}{c} J_e^\nu \cdot \vartheta \tag{2.14a}$$

$$\Box^2 A_\ell^\nu = \frac{4\pi}{c} J_\ell^\nu \cdot \vartheta \tag{2.14b}$$

2.2 Vacuum Gauge Condition

Before solutions to equations (2.14) are determined, it is convenient to work in the Maxwell limit and proceed with a short discussion of the gauge condition in (2.8) which may be written as

$$\boxed{|\partial_\nu A_v^\nu| \equiv \sqrt{\mathrm{E}^2 - \mathrm{B}^2}} \tag{2.15}$$

The gauge introduced by this formula may be referred to as the ***vacuum gauge***. While its primary purpose is to facilitate enforcement of causality, its definition demonstrates an intimate link between $\partial_\nu A^\nu$—more specifically A^ν itself—and the classical theory of the electromagnetic field.

The covariant nature of the vacuum gauge gives it similar properties to the Lorentz gauge. For example, the restricted gauge transformation defined by

$$A^\nu \rightarrow A^\nu - \partial^\nu \Lambda \qquad \text{where} \qquad \Box^2 \Lambda = 0 \tag{2.16}$$

will preserve the vacuum gauge condition. This means that, like the Lorentz gauge, the vacuum gauge represents an entire class of potentials. For the special case of electromagnetic waves, the left side of (2.15) is zero and the vacuum gauge is identical to the Lorentz gauge.

When the gauges are not identical, it is possible to implement a specific gauge transformation which connects them. If A_e^ν are the Lorentz gauge potentials satisfying $\partial_\nu A_e^\nu = 0$ then the vacuum gauge potentials follow from

$$A^\nu = A_e^\nu + \partial^\nu \varphi \tag{2.17}$$

Applying the divergence operator to both sides of this equation and using fields for a point charge shows that the gauge field φ must satisfy

$$\Box^2 \varphi = \frac{e}{\rho^2} \tag{2.18}$$

One approach to solving this equation is by implicit differentiation. Applying the operator ∂_ν to

$$\partial^\nu \varphi = \frac{\partial \varphi}{\partial \rho} \partial^\nu \rho \tag{2.19}$$

and remembering that accelerations are not being considered results in a source equation which may be written

$$L[\,\varphi\,] = -\frac{\partial}{\partial \rho}\left[\rho^2 \frac{\partial}{\partial \rho}\right] \varphi = e \tag{2.20}$$

The operator on the left is self-adjoint and admits a general solution which includes two linearly independent solutions to the homogeneous equation:

$$\varphi = C_1 \varphi_1 + C_2 \varphi_2 + \varphi_p \tag{2.21}$$

For the record $\varphi_1 = C$ and $\varphi_2 = e/\rho$ but neither of these contributions are necessary for the design of the causal theory. Instead, the particular solution

$$\varphi(\boldsymbol{r}, t) = -e \ln \rho \tag{2.22}$$

is sufficient to determine the causal potentials. A solution is also available by considering a first order equation in the scalar field ψ defined through the relation

$$\varphi(\rho) = \int^{\rho} \psi(\rho')\, d\rho' \tag{2.23}$$

The associated Green function for the first order problem is determined from

$$-\frac{\partial}{\partial \rho}\left[\rho^2\, G(\rho, s)\right] = \delta(\rho - s) \tag{2.24}$$

and seems to provide a neater solution.

2.3 Velocity Potentials and Velocity Fields

One way to solve equations (2.14) is in the rest frame followed by a Lorentz transformation to moving frame coordinates. However, it is a simple matter to recognize (2.14a) as the differential equation solved by the Liénard-Wiechert potentials inside the causality sphere. Moreover, equation (2.14b) can be solved by applying the operator ∂^ν to the scalar field φ in equation (2.22). Results for the causal potentials are

$$A_e^\nu = \frac{e}{\rho}\beta^\nu \cdot \vartheta \tag{2.25a}$$

$$A_\ell^\nu = \frac{e}{\rho}\mathcal{U}^\nu \cdot \vartheta \tag{2.25b}$$

These are complimentary timelike and spacelike potentials (un-identical twins) which may be added together immediately using (2.12) to determine the vacuum gauge velocity potentials given by

$$\boxed{A_v^\nu = \frac{eR^\nu}{\rho^2} \cdot \vartheta} \tag{2.26}$$

A graphical depiction of the potentials is shown in figure 2. For constant velocity

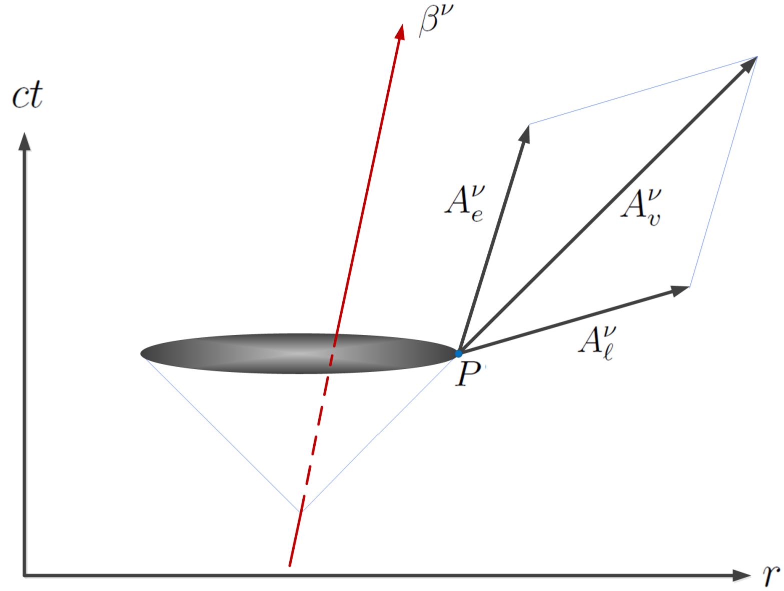

Figure 2: *Vacuum gauge velocity potentials at a point P. The light cone traces the potentials back to the world line of the charge at the retarded time.*

motion the effect of the vacuum gauge condition is to generate the gauge field A_ℓ^ν

which rotates the Lorentz gauge (Liénard-Wiechert) potentials onto the surface of the light cone. In Minkowski space both A_e^ν and A_ℓ^ν have equal lengths except for an overall sign. They are mutually orthogonal timelike and spacelike components of the vacuum gauge velocity potentials.

Equation (2.26) can now be used to verify the causal field strength tensor in (1.4). To facilitate the calculation write the four-potentials as $A_v^\nu \to A_v^\nu \cdot \vartheta$ so that A_v^ν becomes the associated Maxwell potentials. Then it must be shown that

$$F_v^{\mu\nu} = \partial^\mu(A_v^\nu \cdot \vartheta) - \partial^\nu(A_v^\mu \cdot \vartheta) = (\partial^\mu A_v^\nu - \partial^\nu A_v^\mu) \cdot \vartheta \tag{2.27}$$

The validity of this result relies on the fact that $\partial^\nu \vartheta$ is a null vector which has already been determined in equation (1.6). To complete the calculation only requires differentiating the vacuum gauge potentials inside the causality sphere. Technically, this is not necessary since we know that velocity potentials differ from the Liénard-Wiechert potentials by a gauge transformation. However, this is not the whole story and it is worthwhile to perform the calculation:

$$F_v^{\mu\nu} = \frac{e}{\rho^2}\left(\partial^\mu R^\nu - \partial^\nu R^\mu\right) - \frac{2e}{\rho^3}\left(R^\nu \partial^\mu \rho - R^\mu \partial^\nu \rho\right) \tag{2.28}$$

But implicit derivatives are

$$\partial^\mu R^\nu = \mathrm{g}^{\mu\nu} - \frac{1}{\rho} R^\mu \beta^\nu \tag{2.29a}$$

$$\partial^\mu \rho = \beta^\mu - \frac{R^\mu}{\rho} + \frac{a^\lambda R_\lambda}{\rho} R^\mu \tag{2.29b}$$

and these can be inserted above to yield the causal tensor

$$F_v^{\mu\nu} = \frac{e}{\rho^3}\left(R^\mu \beta^\nu - R^\nu \beta^\mu\right) \cdot \vartheta \tag{2.30}$$

Of particular importance is how the calculation of the field strength tensor completely rejects all terms associated with the acceleration of the particle. This property is also visible in the calculation of the vacuum gauge condition where (2.29b) will be required along with $\partial_\nu R^\nu = 3$ to determine

$$\partial_\nu A_v^\nu = \frac{e}{\rho^2} \cdot \vartheta \tag{2.31}$$

The implication here is that the application of causality to the Maxwell-Lorentz field of the classical electron will require independent theories of the particles' velocity and acceleration fields.

Covariant Integral: An important application of Gauss' law begins with the integral of the vacuum gauge condition

$$I = \int_{\mathcal{V}} \partial_\nu A_v^\nu \, d^4x \tag{2.32}$$

where $\mathcal{V}$ represents the four-volume of the causal light cone accessible to the particle. Since the integrand and the volume element are both Lorentz scalars, this means that I is a scalar invariant. It is most easily evaluated in the proper frame where

$$I = \int_0^{ct} \left[\int_0^{ct'} \frac{e}{r^2} r^2 dr \, d\Omega \right] cdt' = 2\pi e c^2 \tau^2 \tag{2.33}$$

Since this result is proportional to the interval, in a frame moving with velocity $\boldsymbol{\beta}$ it generalizes to

$$x^\mu x_\mu = \frac{1}{2\pi e} \int_{\mathcal{V}} \partial_\nu A^\nu_v d^4x \tag{2.34}$$

where $x^\mu = (ct, \boldsymbol{\beta} ct)$ is the coordinate vector of the particle.

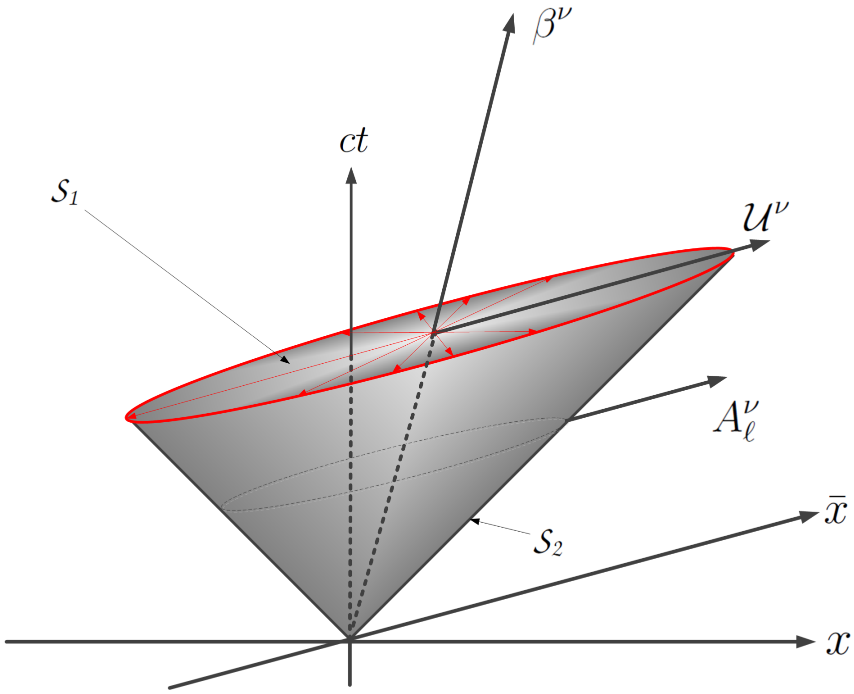

Figure 3: *The gauge field A^ν_ℓ can be integrated over the hyper-surfaces enclosing the four-volume of the light cone. There is no contribution from the integral over the hyper-ellipse.*

Another approach to evaluate I is to apply Gauss' law and integrate over the surface of the light cone shown in figure 3. First note that the volume integral can be written

$$I = \int_{\mathcal{V}} \partial_\nu A^\nu_e d^4x + \int_{\mathcal{V}} \partial_\nu A^\nu_\ell d^4x \tag{2.35}$$

The first integral is zero and this implies that the surface integrals of the Liénard-Wiechert potentials enclosing the volume $\mathcal{S}$ add to zero:

$$\int_{\mathcal{S}} A_e^\nu d_\nu s = 0 \tag{2.36}$$

More specifically, non-zero integrals over each of two hypersurfaces are

$$\int_{\mathcal{S}_1} A_e^\nu \beta_\nu \, d^3\sigma + \int_{\mathcal{S}_2} A_e^\nu R_\nu \, d^2\omega = 0 \tag{2.37}$$

As always, these integrals are most easily evaluated in the rest frame having values $\pm 2\pi e c^2 \tau^2$.

All that remains is to apply Gauss' law to the second integral in equation (2.35), but as indicating by figure 3, the gauge field only contributes along the light cone producing the result

$$I = \int_{\mathcal{S}_2} A_\ell^\nu R_\nu \, d^2\omega \tag{2.38}$$

2.4 Lorentz Transformation of Potentials and Fields

The transformation equations for the velocity potentials mirror those of the coordinate transformation:

$$A'_v = \gamma(A_v + \boldsymbol{\beta} \cdot \boldsymbol{A}_v) \tag{2.39a}$$

$$\boldsymbol{A}'_v = \boldsymbol{A}_v + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{A}_v)\boldsymbol{\beta} + \gamma A_v \boldsymbol{\beta} \tag{2.39b}$$

However, velocity potentials are composed of timelike and spacelike components—each of which may be transformed by itself. Component transformations from the rest frame are particularly simple.

$$A'_e = \gamma A_e \qquad \boldsymbol{A}'_e = \gamma \boldsymbol{\beta} A_e \tag{2.40a}$$

$$A'_\ell = \gamma \boldsymbol{\beta} \cdot \boldsymbol{A}_\ell \qquad \boldsymbol{A}'_\ell = \boldsymbol{A}_\ell + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{A}_\ell)\boldsymbol{\beta} \tag{2.40b}$$

The norms of both four-vectors are easily shown to be Lorentz scalars. Moreover, their norms are the same to within sign which will enforce a zero norm for the velocity potential

$$A'^\nu_v = A'^\nu_e + A'^\nu_\ell \tag{2.41}$$

The importance of the spacelike potentials emerge when the velocity field strength tensor is written

$$eF_v^{\mu\nu} = [A'^\mu_\ell, \, A'^\nu_e] \tag{2.42}$$

The anti-symmetric tensor loses its traditional identity as a fundamental object—being replaced by an interaction among four-potentials. The second order transformation law for $F_v^{\mu\nu}$ is then a combination of first order transformations

$$[A'^\mu_\ell, \, A'^\nu_e] = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} \, [A^\alpha_\ell, \, A^\beta_e] \tag{2.43}$$

Transformation Law for Velocity Fields: As an example of the use of the gauge field, velocity electric and magnetic field vectors in a moving frame are

$$\mathbf{E}'_v = \frac{\gamma}{\rho}\left(\boldsymbol{A}'_\ell - A'_\ell \boldsymbol{\beta}\right) \qquad \mathbf{B}'_v = \frac{\gamma}{\rho}\left(\boldsymbol{\beta} \times \boldsymbol{A}'_\ell\right) \tag{2.44}$$

Now suppose the transformation of the gauge field in equation (2.40b) is inserted into (2.44). In terms of rest frame potentials one finds

$$\mathbf{E}'_v = \frac{\gamma}{\rho}\left[\boldsymbol{A}_\ell - \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\boldsymbol{A}_\ell)\boldsymbol{\beta}\right] \tag{2.45a}$$

$$\mathbf{B}'_v = \frac{\gamma}{\rho}(\boldsymbol{\beta}\times\boldsymbol{A}_\ell) \tag{2.45b}$$

In the rest frame, $\boldsymbol{A}_\ell/\rho = \mathbf{E}_v$ which derives the transformation law for the electric and magnetic field vectors of the particle

$$\mathbf{E}'_v = \gamma\mathbf{E}_v - \frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{E}_v)\boldsymbol{\beta} \tag{2.46a}$$

$$\mathbf{B}'_v = \gamma\boldsymbol{\beta}\times\mathbf{E}_v \tag{2.46b}$$

3 Vacuum Gauge Acceleration Potentials

As already discussed, the rejection of particle accelerations by the velocity potentials imply that the acceleration potentials will satisfy their own independent differential equation. Once derived, its solution and subsequent calculation of the field strength tensor $F_a^{\mu\nu}$ are straight-forward.

3.1 Differential Equation for Acceleration Potentials

Working in the Maxwell limit, the form of the vacuum gauge source equation—inclusive of accelerated motions—will be

$$\Box^2 A^\nu - \partial^\nu\partial_\mu A^\mu = \frac{4\pi}{c}J_e^\nu \tag{3.1}$$

As a trial solution, A_v^ν can be inserted into the left side of this equation. Excluding the location of the charge, the calculation shows that

$$\Box^2 A_v^\nu = \frac{2e}{\rho^3}\mathcal{U}^\nu \tag{3.2a}$$

$$\partial^\nu\partial_\mu A_v^\mu = \frac{2e}{\rho^3}\mathcal{U}^\nu - \frac{2e}{\rho^4}(a^\lambda R_\lambda)R^\nu \tag{3.2b}$$

The trial solution therefore differs from the actual solution by a single term involving the acceleration of the particle. It is convenient to write (3.2b) as

$$\partial^\nu \partial_\mu A_v^\mu = \frac{4\pi}{c}[J_\ell^\nu + J_a^\nu] \tag{3.3}$$

which defines the ***acceleration current***

$$J_a^\nu \equiv -\frac{ec}{2\pi\rho^4}(a^\lambda R_\lambda)R^\nu \tag{3.4}$$

Now assume that the potentials during accelerated motions can be written as

$$A^\nu = A_v^\nu + A_a^\nu \tag{3.5}$$

Inserting this into (3.1) along with the simultaneous appearance of J_a^ν shows that

$$\Box^2 A_v^\nu + \Box^2 A_a^\nu - \partial^\nu \partial_\mu A_v^\mu - \partial^\nu \partial_\mu A_a^\mu = \frac{4\pi}{c} J_e^\nu + \frac{4\pi}{c} J_a^\nu \tag{3.6}$$

Velocity and acceleration terms can now be de-coupled resulting in two independent equations which obey

$$\Box^2 A_v^\nu - \partial^\nu \partial_\mu A_v^\mu = \frac{4\pi}{c} J_e^\nu \tag{3.7a}$$

$$\Box^2 A_a^\nu - \partial^\nu \partial_\mu A_a^\mu = \frac{4\pi}{c} J_a^\nu \tag{3.7b}$$

While the first equation has already been investigated, the mathematical form of the acceleration equation implies that the acceleration fields will satisfy their own set of Maxwell-like equations

$$\partial_\mu F_a^{\mu\nu} = \frac{4\pi}{c} J_a^\nu \tag{3.8}$$

Moreover, J_a^ν is a null vector and will satisfy the continuity equation $\partial_\nu J_a^\nu = 0$ as it must since $F_a^{\mu\nu}$ is anti-symmetric. This is easily proved by noting that

$$\partial_\nu (a^\lambda R_\lambda) R^\nu = a^\lambda R_\lambda \tag{3.9}$$

The fact that J_a^ν points in the direction of R^ν gives it a physical interpretation as a massless $1/r^2$ electrical current which radiates from the instantaneous retarded position of the charge. This means that every point in space containing non-zero fields $\mathbf{E}_a$ and $\mathbf{B}_a$ can be associated with a local vacuum current density at that same point. Except for an overall constant, J_a^ν is nothing more than the projection of the particle four-acceleration along the direction of the velocity potentials.

3.2 Acceleration Potentials and Acceleration Fields

It may be possible to solve the differential equation in (3.7b) by a brute force calculation, but it is easier to determine its solution by referring directly to the gauge transformation in equation (2.17). Including particle accelerations this is

$$A^\nu = A_e^\nu + \partial^\nu \varphi \tag{3.10a}$$

$$= A_e^\nu - \frac{e}{\rho}\left[\beta^\nu - \frac{R^\nu}{\rho}(1 - a^\lambda R_\lambda)\right] \tag{3.10b}$$

The gauge transformation eliminates the Liénard-Wiechert potentials altogether and replaces them with null potentials. In addition, both the velocity and acceleration potentials are easily recognizeable in the brackets. Including the requirement to enforce causality, the acceleration potentials and the acceleration current are

$$\boxed{A_a^\nu = -\frac{e}{\rho^2}(a^\lambda R_\lambda)R^\nu \cdot \vartheta} \qquad \boxed{J_a^\nu \equiv -\frac{ec}{2\pi\rho^4}(a^\lambda R_\lambda)R^\nu \cdot \vartheta} \tag{3.11}$$

As with the velocity theory, the vacuum gauge is requiring radial null potentials such that $A_\nu^a A_a^\nu = 0$. Note also that A_a^ν does not diminish for large R.

The ability of the causal potentials in (3.11) to render causal acceleration fields is determined by their direction along the light cone. Writing $A_a^\nu \to A_a^\nu \cdot \vartheta$, then the simple calculation verifies

$$F_a^{\mu\nu} = \partial^\mu(A_a^\nu \cdot \vartheta) - \partial^\nu(A_a^\mu \cdot \vartheta) = (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)\cdot\vartheta \tag{3.12}$$

Inside the causal region the acceleration potentials may be written

$$A_a^\nu = -(a^\lambda R_\lambda)A_v^\nu \tag{3.13}$$

which can be differentiated to produce

$$F_a^{\mu\nu} = -\partial^\mu(a^\lambda R_\lambda)A_v^\nu + \partial^\nu(a^\lambda R_\lambda)A_v^\mu - (a^\lambda R_\lambda)F_v^{\mu\nu} \tag{3.14}$$

A rigorous calculation of $\partial^\mu(a^\lambda R_\lambda)$ requires some computational stamina but leads to the simple result

$$\partial^\nu(a^\lambda R_\lambda) = a^\nu + \frac{\dot{a}^\lambda R_\lambda}{\rho}R^\nu \tag{3.15}$$

where $\dot{a}^\lambda = \partial a^\lambda/\partial\tau$ and where τ is the proper time. Now define the scalar field[2] $\xi \equiv a^\lambda R_\lambda/\rho$ and easily determine the final form of the field strength tensor:

$$F_a^{\mu\nu} = \frac{e}{\rho^2}\left[R^\mu a^\nu - R^\nu a^\mu - \xi(R^\mu\beta^\nu - R^\nu\beta^\mu)\right]\cdot\vartheta \tag{3.16}$$

[2] This definition of ξ is similar to Rohrlich's definition $a_u \equiv a^\nu \mathcal{U}_\nu$. The new definition was chosen to avoid confusion with the covariant four-acceleration.

3.3 Total Potentials

Combining the velocity potentials in (2.26) with the acceleration potentials in (3.11), the general vacuum gauge solution for arbitrary motions of a charged particle can be written

$$\boxed{A^\nu(\boldsymbol{r},t) = \frac{e(1-a^\lambda R_\lambda)}{\rho^2} R^\nu \cdot \vartheta} \tag{3.17}$$

Based on previous discussion, the mathematical structure of this formula also allows the potentials to be divided as

$$A^\nu = A^\nu_e + A^\nu_\ell + A^\nu_a \tag{3.18}$$

where

$$A^\nu_e = \frac{e}{\rho}\beta^\nu \cdot \vartheta \qquad A^\nu_\ell = \frac{e}{\rho}\mathcal{U}^\nu \cdot \vartheta \qquad A^\nu_a = -\frac{e}{\rho}\,\xi R^\nu \cdot \vartheta \tag{3.19}$$

Each of these potentials can be associated with its own vector current density, and each current density points in the same direction as its associated potentials. Two of the currents are conserved but one is not:

$$\partial_\nu J^\nu_e = 0 \qquad \partial_\nu J^\nu_\ell = -\frac{ec}{2\pi\rho^4} \qquad \partial_\nu J^\nu_a = 0 \tag{3.20}$$

Finally, it is important to compare the magnitudes of the velocity and acceleration potentials which become equal when $a^\lambda R_\lambda \sim c^2$. Unless accelerations are extremely large, it can be assumed that acceleration potentials represent only a small correction.

Causality and the Scalar Field: The development of vacuum gauge theory begins from the assertion that the velocity field of a charged particle is a causal field. However, causality can (and probably should) be introduced by constraining the scalar field φ as

$$\varphi = -e\ln\rho \cdot \vartheta \tag{3.21}$$

A first derivative for general accelerated motions picks up a delta function

$$\partial^\nu\varphi = [A^\nu_v + A^\nu_a - A^\nu_e]\cdot\vartheta - e\ln\rho \cdot \partial^\nu\vartheta \tag{3.22}$$

but the second derivative can be written

$$\partial^\mu\partial^\nu\varphi = [\partial^\mu A^\nu_v + \partial^\mu A^\nu_a - \partial^\mu A^\nu_e]\cdot\vartheta + S^{\mu\nu} \tag{3.23}$$

where $S^{\mu\nu}$ is the collection of remaining terms symmetric in the two indicies. Now form the object

$$\partial_\mu\left[\partial^\mu\partial^\nu\varphi - \partial^\nu\partial^\mu\varphi\right] = 0 \tag{3.24}$$

This set of operations can be written in terms of the potentials as

$$[\Box^2 A^\nu_v + \Box^2 A^\nu_a - \Box^2 A^\nu_e - \partial^\nu\partial_\mu A^\mu_v - \partial^\nu\partial_\mu A^\mu_a]\cdot\vartheta + \partial_\mu\vartheta(F^{\mu\nu} - F^{\mu\nu}) = 0 \tag{3.25}$$

But the velocity and acceleration terms in the brackets can be separated with the introduction of the acceleration current. Moreover, the point charge four-current J_e^ν replaces the wave operator acting on the Liénard-Wiechert potentials. What remains is independent equations of motion given by

$$[\,\Box^2 A_v^\nu - \partial^\nu \partial_\mu A_v^\mu - \frac{4\pi}{c} J_e^\nu\,] \cdot \vartheta = 0 \tag{3.26a}$$

$$[\,\Box^2 A_a^\nu - \partial^\nu \partial_\mu A_a^\mu - \frac{4\pi}{c} J_a^\nu\,] \cdot \vartheta = 0 \tag{3.26b}$$

Both are automatically constrained by casuality without the presence of the null current. This calculation effectively summarizes the entire theory of vacuum gauge potentials previously introduced.

Pair Production/Annihilation: An excellent example of the use of total vacuum gauge potentials is for the description of the creation and annihilation of an $e^+ e^-$ pair. The spacetime diagram in figure 4 shows the pair created at the origin of coordinates and annihilated at a later time $c\tau_o$. First, write the position vector of each particle as

$$\boldsymbol{w}_+ = \boldsymbol{w}_+(\tau) \tag{3.27a}$$

$$\boldsymbol{w}_- = \boldsymbol{w}_-(\tau) \tag{3.27b}$$

Since the pair created and annihilated itself, their positions should be identical at time $\tau = 0$ and $\tau = \tau_o$:

$$\boldsymbol{w}_+(0) = \boldsymbol{w}_-(0) \tag{3.28a}$$

$$\boldsymbol{w}_+(\tau_o) = \boldsymbol{w}_-(\tau_o) \tag{3.28b}$$

Add to these constraints the retardation condition, which is generally different for each particle

$$R_+^\nu = x^\nu - w_+^\nu(t_r) \tag{3.29a}$$

$$R_-^\nu = x^\nu - w_-^\nu(t_r) \tag{3.29b}$$

In the frame S the vacuum gauge potentials $A_{tot}^\nu(ct, \boldsymbol{r})$ are the sum of vacuum gauge potentials associated with each particle. In addition, since both particles share the same causality requirements at both ends of their life span, the form of the potentials is

$$A_{tot}^\nu(ct, \boldsymbol{r}) = \left[A_{v+}^\nu + A_{a+}^\nu + A_{v-}^\nu + A_{a+}^\nu\right] \cdot \vartheta(ct_r) \cdot \vartheta(\gamma c \tau_o - ct_r) \tag{3.30}$$

Derivatives then determine the appropriate form of the causal field stength tensor

$$F_{tot}^{\mu\nu}(ct, \boldsymbol{r}) = [F_+^{\mu\nu} + F_-^{\mu\nu}] \cdot \vartheta(ct_r) \cdot \vartheta(\gamma c \tau_o - ct_r) \tag{3.31}$$

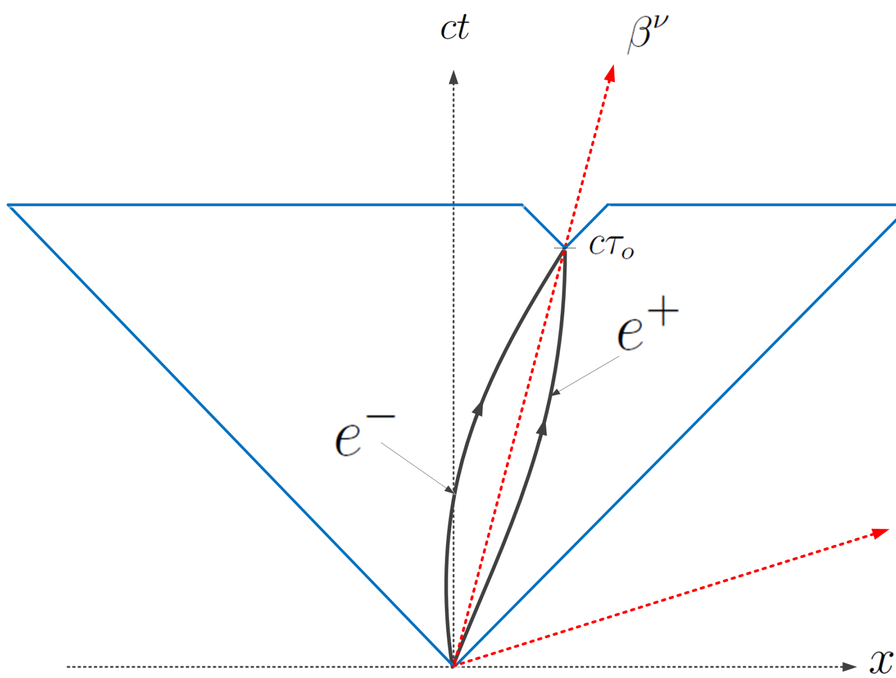

Figure 4: *Spacetime diagram exhibiting a two particle* $e^+ e^-$ *system in the vacuum gauge. The blue line marks the limit of the causal fields as viewed in the frame* S.

The four-current density associated with this system is determined by the two point charges and also by two causality induced delta-currents. It would be delightful if the delta-currents canceled for the two equal and opposite charges, but this is not quite the case since each particle has a different velocity during creation and annihilation. For example, using $e = -|e|$ the first delta current may be written

$$J^\nu_{N1} = -\frac{ec}{4\pi}\frac{(1,\hat{\boldsymbol{r}})}{r^2}\left[\frac{1}{\gamma_-^3(1-\hat{\boldsymbol{r}}\cdot\boldsymbol{\beta}_-)^3} - \frac{1}{\gamma_+^3(1-\hat{\boldsymbol{r}}\cdot\boldsymbol{\beta}_+)^3}\right]\cdot\delta(ct-r) \tag{3.32}$$

where all velocities are evaluated at $ct_r = 0$.

Certainly the time $c\tau_o$ can be anything, but there exists a real possibility that a model such as this can be linked to the brief appearance of $e^+ e^-$ pairs described by quantum field theory. In support of this idea the two particle system already has two causality functions serving as classical creation and annihilation operators. If an appropriate classical Hamiltonian can be constructed, it would be a tremendous accomplishment to develop a simple quantization procedure leading directly to the quantized Dirac field Hamiltonian operator.

A Derivatives of the Null Vector

The covariant derivative of R^ν is

$$\partial^\mu R^\nu = \mathrm{g}^{\mu\nu} - \frac{R^\mu \beta^\nu}{\rho} \tag{A.1}$$

The trace of the resulting matrix gives the 4-divergence $\partial_\nu R^\nu = 3$. In terms of individual components–and with the inclusion of a sign–a useful construction is:

$$-\,\partial^\mu R^\nu = \begin{bmatrix} -\dfrac{\partial R}{\partial t} & -\dfrac{\partial \boldsymbol{R}}{\partial t} \\ \\ \boldsymbol{\nabla} R & \boldsymbol{\nabla} \boldsymbol{R} \end{bmatrix} \tag{A.2}$$

where individual components are given by

$$\frac{\partial R}{\partial t} = 1 - \frac{\gamma R}{\rho} \qquad\qquad \frac{\partial \boldsymbol{R}}{\partial t} = \frac{-\gamma R \boldsymbol{\beta}}{\rho} \tag{A.3}$$

$$\boldsymbol{\nabla} R = \frac{\gamma \boldsymbol{R}}{\rho} \qquad\qquad \boldsymbol{\nabla} \boldsymbol{R} = \mathbf{1} + \frac{\gamma \boldsymbol{R}\boldsymbol{\beta}}{\rho} \tag{A.4}$$

The determinant of (A.2) can be written $\det[\partial^\mu R^\nu] = 0$. The divergence of $\boldsymbol{R}$ follows from $Tr[\boldsymbol{\nabla}\boldsymbol{R}]$ and has a value

$$\boldsymbol{\nabla} \cdot \boldsymbol{R} = 3 + \frac{\gamma \boldsymbol{R} \cdot \boldsymbol{\beta}}{\rho} \tag{A.5}$$

Let $\boldsymbol{w}(ct_r)$ be the retarded position of a charged particle at time ct_r. The light cone condition is defined by

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{w}(ct_r) \qquad\qquad R \equiv ct - ct_r \tag{A.6}$$

Suppose that retarded coordinates are viewed collectively as $x_r^\mu = (ct_r, \boldsymbol{R})$. A transformation to present time coordinates $x^\mu = (ct, \boldsymbol{r})$ is then $x_r^\nu = x_r^\nu(x^\mu)$ and it follows that

$$dx_r^\nu = \frac{\partial x_r^\nu}{\partial x^\mu}\, dx^\mu \tag{A.7}$$

The matrix generated by this transformation can be written

$$\frac{\partial x_r^\nu}{\partial x^\mu} = \begin{bmatrix} \dfrac{\partial ct_r}{\partial t} & \dfrac{\partial \boldsymbol{R}}{\partial t} \\ \\ -\boldsymbol{\nabla} ct_r & -\boldsymbol{\nabla}\boldsymbol{R} \end{bmatrix}$$

Derivatives of R and $\boldsymbol{R}$ have already been evaluated while derivatives of the retarded time are

$$\frac{\partial t_r}{\partial t} = \frac{\gamma R}{\rho} \qquad\qquad \boldsymbol{\nabla} ct_r = \frac{-\gamma \boldsymbol{R}}{\rho} \tag{A.8}$$