Weakly mixing and rigid rank-one transformations preserving an infinite measure

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Abstract. In this paper we study the compatibility of rigidity with various notions of weak mixing in infinite ergodic theory. We prove that there exists an infinite measure-preserving transformation that is spectrally weakly mixing and rigid, but not doubly ergodic. We also construct an example to show that rigidity is compatible with rational ergodicity. At the end of the paper we explore the structure of rigidity sequences for infinite measure-preserving transformations that have ergodic Cartesian square, as well as the structure of rigidity sequences for infinite measure-preserving transformations that are rationally ergodic. All of our constructions are via the method of cutting and stacking.

Contents

1. Introduction

The generic transformation in the set of all invertible transformations that preserve a finite measure is both rigid and weakly mixing [20]. There have been many recent developments toward characterizing the possible rigidity sequences for weakly mixing maps (see [4], [10], [16], [17]). In this paper, we are interested in what happens when you move from finite ergodic theory to infinite ergodic theory (for more information see [2]). It is well-known that there are many equivalent definitions of weak mixing for transformations that preserve a finite measure. In infinite ergodic theory, however, these notions are no longer equivalent. Thus, the

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question becomes, which notions of weak mixing are compatible with rigidity in infinite ergodic theory? If they are compatible, what do the constructions look like and can we analyze their rigidity times?

For the remainder of this paper, we will be working with invertible transformations that preserve an infinite measure. We will be concerned with three types of weak mixing: spectral weak mixing, double ergodicity, and ergodicity of the Cartesian square. In [11] it was shown that ergodicity of the Cartesian square implies double ergodicity, which in turn implies spectral weak mixing. However, the reverse implications do not hold (see [6] and [11]).

Even though there are many nonequivalent notions of weak mixing, many generic results still remain valid. Consider the set of all automorphisms preserving an infinite measure equipped with the weak topology. In 2000 Choksi and Nadkarni proved that the generic transformation in this space has infinite ergodic index [13]. Since transformations with infinite ergodic index also have ergodic Cartesian square, all of the types of weak mixing mentioned above are also generic. In 2001 Ageev and Silva proved that rigidity is a generic property within the same set of automorphisms [9]. Finally, in the recent work of Bozgan et al. they show that rank-one transformations are generic [12]. Thus, the typical infinite measure-preserving automorphism is rank-one, rigid, and has ergodic Cartesian square. We will construct explicit examples of transformations with these properties and use the constructions to take the first step toward characterizing rigidity times for weakly mixing transformations that preserve an infinite measure. Our constructions are via the method of cutting and stacking.

The first main theorem of the paper cannot be obtained with categorical methods.

Theorem A. *There exists an infinite measure-preserving rank-one transformation that is spectrally weakly mixing and rigid, but not doubly ergodic.*

The second main theorem of this paper can be obtained via the categorical arguments outlined above, but we give a constructive proof.

Theorem B. *There exists an infinite measure-preserving rank-one transformation that has ergodic Cartesian square and is rigid.*

In this paper, we also explore the compatibility of rigidity with rational ergodicity. Rational ergodicity and weak rational ergodicity were introduced in 1977 by Aaronson [1]. When $T : X \to X$ is an invertible transformation that preserves a probability measure, μ, the Birkhoff ergodic theorem states that ergodicity of T is equivalent to

$$\frac{1}{N}\sum_{k=0}^{N-1}\mu(T^kA\cap B) = \mu(A)\mu(B) \quad \text{as } N\to\infty, \tag{1.1}$$

for every pair of measurable sets $A, B \subset X$. On the other hand, if X has infinite measure, then the Cesaro averages above converge to 0 for all sets A, B of finite measure. In [1] Aaronson showed that there exists no sequence of normalizing constants so that (1.1) converges to $\mu(A)\mu(B)$ and introduced the definitions of rational ergodicity and weak rational ergodicity. Recently, in [12] and [14] these notions were explored in the setting of rank-one transformations. In [3] Aaronson proved that the set of weakly rationally ergodic transformations is a meager subset of the set of infinite measure-preserving transformations. In the same paper, it was shown that rational ergodicity implies weak rational ergodicity, and the validity of the reverse implication remains open. Thus, determining whether rational ergodicity is compatible with rigidity must come from a construction. With that in mind, we have the following theorem:

Theorem C. *There exists an infinite measure-preserving rank-one transformation that is rationally ergodic and rigid.*

Finally, we use the ideas from the constructions in Theorems B and C to prove the following theorems which give a set of conditions under which a given sequence can be realized as a rigidity sequence for a transformation with ergodic Cartesian square or a rationally ergodic transformation.

Theorem D. *Let (n_m) be an increasing sequence of natural numbers such that $\frac{n_{m+1}}{n_m} \to \infty$. There exists an infinite measure-preserving rank-one transformation that has ergodic Cartesian square and is rigid along (n_m).*

Theorem E. *Let (n_m) be an increasing sequence of natural numbers such that $\frac{n_{m+1}}{n_m} \to \infty$. Furthermore, assume that $n_{m+1} = 2^{k_m} n_m + r_m$ where $0 \leq r_m < n_m$. There exists an infinite measure-preserving rank-one transformation that is rationally ergodic and rigid along (n_m).*

In the next section, we provide definitions and briefly review the method of cutting and stacking. In Section 3, we discuss rank-one constructions that are rigid and weak mixing. In Section 4, we construct an example to show the compatibility of rigidity with rational ergodicity. Finally, in Section 5, we analyze rigidity sequences for weakly mixing transformations and rationally ergodic transformations.

2. Preliminaries

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. We assume throughout this paper that $\mu(X) = \infty$ and $T : X \to X$ is invertible and measure-preserving.

We begin with the definition of rigidity in this setting.

Definition 1. The transformation T is *rigid* if there exists an increasing sequence of natural numbers (n_m) such that

$$\lim_{m\to\infty} \mu(T^{n_m} A \triangle A) = 0$$

for all sets A of finite positive measure.

We now give three nonequivalent definitions of weak mixing for transformations that preserve an infinite measure.

Definition 2. The transformation T is *spectrally weakly mixing* if $f \in L^\infty$ and $f \circ T = \lambda f$ for some $\lambda \in \mathbb{C}$ implies that f is constant almost everywhere.

Definition 3. The transformation T is *doubly ergodic* if for every pair of positive measure sets A, B, there exists a time n such that

$$\mu(T^n A \cap A) > 0 \quad \text{ and } \quad \mu(T^n A \cap B) > 0.$$

Definition 4. The transformation T has *ergodic Cartesian square* if $T \times T$ is ergodic with respect to $\mu \times \mu$.

As was mentioned in the introduction, the following string of strict implications was shown in [11]:

$$\text{ergodic Cartesian square} \implies \text{double ergodicity} \implies \text{spectral weak mixing.}$$

The following notation will be used in the definition of weak rational ergodicity. Let $F \in \mathcal{B}$ be a set of finite positive measure. The *intrinsic weight sequence* of F is given by

$$u_k(F) = \frac{\mu(F \cap T^k F)}{\mu(F)^2}.$$

Furthermore, let

$$a_n(F) = \sum_{k=0}^{n-1} u_k(F).$$

The following definition introduces a property in the spirit of (1.1) that may also be satisfied by transformations that preserve an infinite measure.

Definition 5. A conservative ergodic transformation T is *weakly rationally ergodic* if there exists an $F \in \mathcal{B}$ with $0 < \mu(F) < \infty$ such that

$$\frac{1}{a_n(F)} \sum_{k=0}^{n-1} \mu(A \cap T^k B) \to \mu(A)\mu(B) \quad \text{as } n \to \infty,$$

for all measurable $A, B \subseteq F$.

Finally, rational ergodicity (defined below) is stronger than weak rational ergodicity and requires the transformation to satisfy a Renyi inequality on a finite measure set.

Definition 6. A conservative ergodic transformation T is *rationally ergodic* if there exists an $M < \infty$ and $F \in \mathcal{B}$ with $0 < \mu(F) < \infty$ such that

$$\int_F \left(\sum_{k=0}^{n-1} \mathbb{1}_F \circ T^k \right)^2 d\mu \leq M \left(\int_F \left(\sum_{k=0}^{n-1} \mathbb{1}_F \circ T^k \right) d\mu \right)^2, \tag{2.1}$$

for all $n \in \mathbb{N}$.

2.1. Basics of Cutting and Stacking. In this paper, we construct transformations that are rigid and exhibit each of the above types of weak mixing, as well as transformations that are rigid and rationally ergodic. All of our examples are obtained via cutting and stacking. That is, we inductively define a sequence of towers, C_n, each of height h_n. Each C_n is a column of h_n disjoint intervals with equal measure denoted by $\{I_{n,0}, ..., I_{n,h_n-1}\}$. The elements of C_n are called *levels*. We often refer to $I_{n,0}$ as the *bottom level* and I_{n,h_n-1} as the *top level* of C_n. A transformation, T_n, is defined on $\{I_{n,0}, ..., I_{n,h_n-2}\}$ by moving up one level. That is, $T_n(I_{n,i}) = I_{n,(i+1)}$ for all $0 \leq i < h_n - 1$. Note that T_n is not defined on the top level of C_n. Thus, we must define C_{n+1} by first cutting C_n into q_n subcolumns of equal width. That is, for each $0 \leq i \leq h_n - 1$ we cut the i^{th} level into q_n pieces, and we denote these pieces by $I_{n,i}^{[0]}, I_{n,i}^{[1]}, ..., I_{n,i}^{[q_n-1]}$. We may then add any number of new levels (called *spacers*) above each subcolumn. Now, we stack every subcolumn of C_n above the subcolumn to its left to form C_{n+1}. Thus, C_{n+1} consists of q_n copies of C_n which may be separated by spacers. Finally, we define $T = \lim_{n\to\infty} T_n$. The transformation T is called a *rank-one* map.

We now define a notion of rigidity related specifically to rank-one transformations.

Definition 7. The sequence (n_m) is *rigid for* C_N if

$$\lim_{m\to\infty} \mu(T^{n_m} E \triangle E) = 0$$

for every level E of C_N.

Remark 1. If (n_m) is rigid for every C_N, then the transformation T is rigid along (n_m). A proof of this fact can be found in [10] Lemma 3.14.

The following two approximation lemmas will be used in the proofs of the main theorems, so we state them here for completeness. Lemma 2.1 is a consequence of the double approximation lemma, and a proof can be found in [11].

Lemma 2.1. *Let $A, B \subset [0,\infty)$ be sets of positive measure and let the levels $I, J \subset C_n$ be such that*

$$\mu(I \cap A) + \mu(J \cap B) > \delta\mu(I)$$

with the level J distance d above the level I (that is, $T^d I = J$). If the levels I, J are cut into $n+2$ equal pieces, $I^{[0]}, \ldots, I^{[n+1]}$ and $J^{[0]}, \ldots, J^{[n+1]}$, then there exists $k \in \mathbb{N}$ such that

$$\mu\left(I^{[k]} \cap A\right) + \mu\left(J^{[k]} \cap B\right) > \delta\mu(I^{[k]})$$

and $J^{[k]}$ is distance d above $I^{[k]}$ in C_{n+1}.

A higher dimensional version of the double approximation lemma can be found in [15]. Using the same methods as [11] proves the following lemma.

Lemma 2.2. *Let $A, B \subset [0,\infty) \times [0,\infty)$ be sets of positive measure and let the levels $I_1, I_2, J_1, J_2 \subset C_n$ be such that*

$$\mu \times \mu((I_1 \times J_1) \cap A) + \mu \times \mu(I_2 \times J_2) \cap B) > \delta\mu(I_1)\mu(J_1)$$

where $T^{d_1} I_1 = I_2$ and $T^{d_2} J_1 = J_2$. If the levels I_1, I_2, J_1, J_2 are cut into $n+2$ equal pieces, $I_i^{[0]}, \ldots, I_i^{[n+1]}$ and $J_i^{[0]}, \ldots, J_i^{[n+1]}$ for $i = 1, 2$, then there exists $k, l \in \mathbb{N}$ such that

$$\mu \times \mu\left(\left(I_1^{[k]} \times J_1^{[l]}\right) \cap A\right) + \mu \times \mu\left(\left(I_2^{[k]} \times J_2^{[l]}\right) \cap B\right) > \delta\mu\left(I_1^{[k]}\right)\mu\left(J_1^{[l]}\right),$$

with $I_2^{[k]}$ distance d_1 above $I_1^{[k]}$ and $J_2^{[l]}$ distance d_2 above $J_1^{[l]}$ in C_{n+1}.

2.2. Hajian-Kakutani +1 Construction is Not Rigid. The Hajian-Kakutani transformation was originally constructed in [19], and it is a classical example of a rank-one map that preserves an infinite measure. A modified version called the Hajian-Kakutani +1 (denoted HK(+1) for short) arises from adding one additional spacer to the original construction. It was shown in [7] that the HK(+1) construction is spectrally weakly mixing. Thus, it is natural to ask if this transformation is also rigid. It was shown in [18] that the HK(+1) transformation is not multiply recurrent and hence not rigid. We give a different proof (via explicit calculation) that the HK(+1) transformation is not rigid along any sequence.

We begin by describing the steps in the construction of HK(+1). The first stage, C_0, consists of the interval $[0, 1)$. Thus, the initial height $h_0 = 1$. In general, suppose we have already constructed the C_n tower, which is a union of h_n levels. To construct C_{n+1} from C_n do the following:

(1) Cut C_n into 2 equal pieces.

(2) Stack the right subcolumn of C_n on top of the left subcolumn.

(3) Add $2h_n + 1$ spacers to the end to form C_{n+1}.

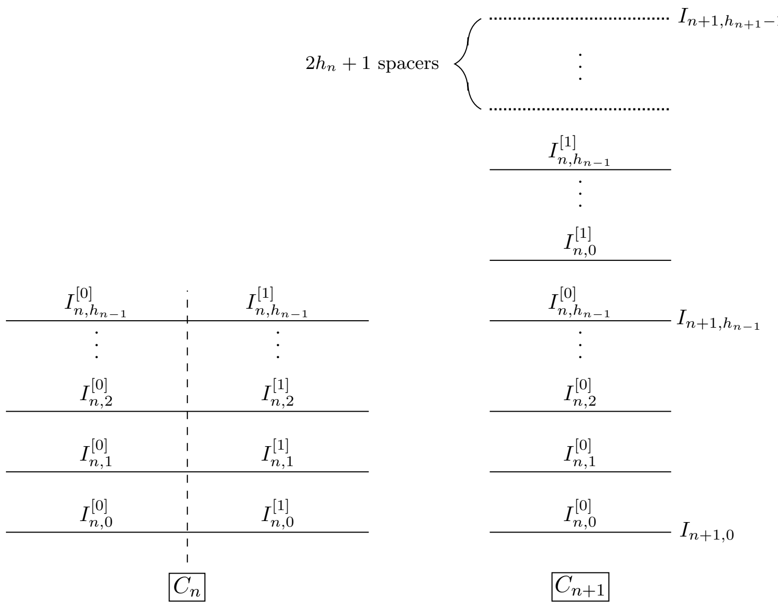

FIGURE 1. Construction of C_{n+1} for the HK(+1) transformation.

At each stage of the construction, notice that the total height of the tower is $h_{n+1} = 4h_n + 1$.

Proposition 2.3. *The HK(+1) construction is not rigid.*

Proof. To show that this construction is not rigid, it suffices to find a set E of positive finite measure such that $\liminf_{n\to\infty} \mu(T^n E \triangle E) > 0$.

Let $E = [0, 1)$. Suppose for a contradiction, that $\liminf_{n\to\infty} \mu(T^n E \triangle E) = 0$. Let $\epsilon > 0$, and let N be such that $N \in [h_n, h_{n+1} - 1]$ and $\mu(T^N E \triangle E) < \epsilon$.

Before we proceed, we need some notation for how levels in C_n appear in C_{n+3}. Let $I_{n,0}$ be the bottom level of the C_n-th tower.

Define

$$e_j = \begin{cases} 0, & \text{if } j = 0, 2, 4, 6 \\ 2h_n + 1, & \text{if } j = 1, 5 \\ 10h_n + 4, & \text{if } j = 3. \end{cases}$$

Let k_j for $j = 0, 1, 2, \ldots, 7$ be defined by

$$k_0 = 0, \qquad k_{j+1} = k_j + h_n + e_j.$$

Let

$$Y_j = \bigcup_{i=0}^{h_n-1} T^{i+k_j} I_{n+3,0}$$

for $j = 0, 1, 2, \ldots, 7$. Then, $Y_{j+1} = T^{h_n+e_j} Y_j$ for $j = 0, 1, 2, \ldots, 6$.

Notice that $h_{n+1} = 4h_n + 1$. Thus $N = ah_n + b$ where $a = 1, 2, 3$ and $0 \le b \le h_n$. Let

$$e_{j,a} = \begin{cases} e_j, & \text{if } a = 1 \\ e_j + e_{j+1}, & \text{if } a = 2 \\ e_j + e_{j+1} + e_{j+2}, & \text{if } a = 3. \end{cases}$$

Then $T^N Y_j = T^{ah_n+b} Y_j = T^{b-e_{j,a}} Y_{j+a}$.

Suppose S_1, S_2 are two sets of positive measure. By $S_1 \overset{\epsilon}{\approx} S_2$ we will mean $\mu(S_1 \triangle S_2) < \epsilon$. Thus,

$$T^N(E \cap Y_j) = T^N E \cap T^N Y_j \overset{\epsilon}{\approx} E \cap T^{b-e_{j,a}} Y_{j+a}$$

for $0 \le j \le 7 - a$. Also notice that

$$T^N(E \cap Y_j) = T^{b-e_{j,a}}(E \cap Y_{j+a})$$

for $0 \le j \le 7 - a$. Putting these together we see that $E \cap Y_{j+a} \overset{\epsilon}{\approx} T^{-b+e_{j,a}} E \cap Y_{j+a}$. Thus $E \overset{8\epsilon}{\approx} T^{-b+e_{j,a}} E$. Since we can find j, j' such that $|e_{j,a} - e_{j',a}| = 8h_n + 3$, we have that $E \overset{16\epsilon}{\approx} T^{8h_n+3} E$.

Recall that $E = [0, 1)$ and thus the set E is a proper subset of Y_j's. Hence $E \cap T^{8h_n+3} E = \emptyset$, which contradicts $E \overset{16\epsilon}{\approx} T^{8h_n+3} E$.

□

3. Weakly Mixing Constructions

3.1. Spectral Weak Mixing and Rigidity. In this section, we explore transformations that are barely weakly mixing and rigid. We do this by constructing a transformation that is spectrally weakly mixing and rigid, but not doubly ergodic. The existence of such a transformation does not follow from categorical methods since the set of spectrally weakly mixing transformations that are not doubly ergodic is of first category.

Theorem A. *There exists an infinite measure-preserving rank-one transformation that is spectrally weakly mixing and rigid, but not doubly ergodic.*

We will begin by describing the construction. The first stage, C_0, consists of the interval $[0, 1)$. Thus, the initial height $h_0 = 1$. In general, suppose we have already constructed the C_n-th tower, which is a union of h_n levels. To construct C_{n+1} from C_n do the following:

(1) Cut C_n into $n + 2$ equal pieces.

(2) Compute the quantity $a_n = \lceil \frac{n+2}{3} \rceil$.

(3) Stack the subcolumns of C_n to form C_{n+1} in the following order: a_nh_n stacks, two spacers, $(n+2-a_n)h_n$ stacks, $2h_n-1$ spacers.

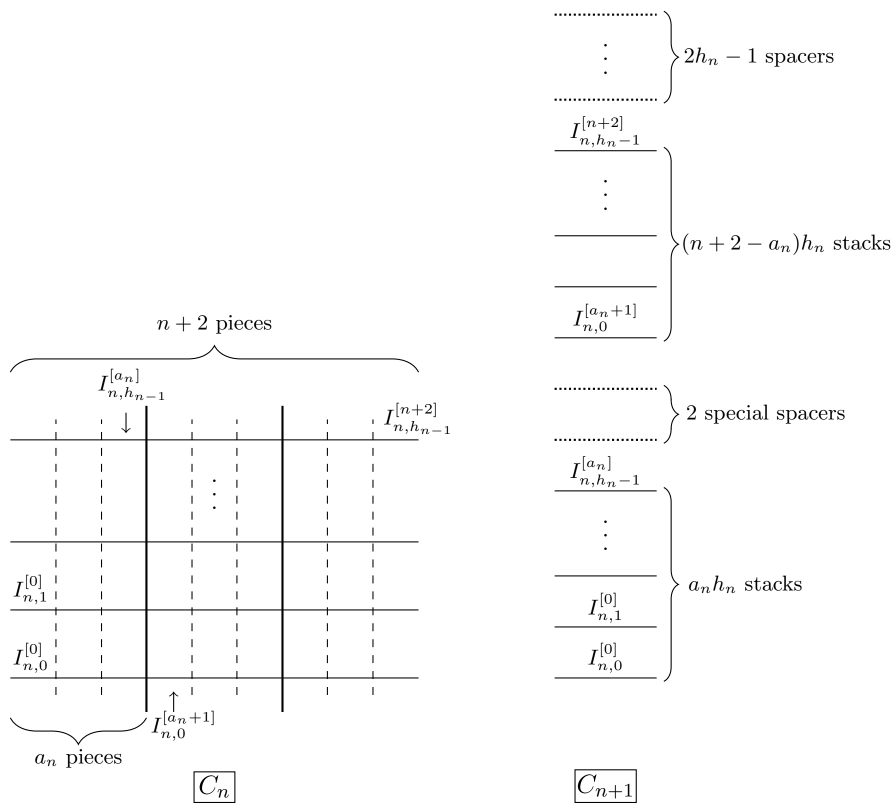

FIGURE 2. Construction of C_{n+1} for a transformation that is spectrally weakly mixing and rigid, but not doubly ergodic.

At each stage of the construction, notice that the total height of the tower is $h_{n+1} = (n+4)h_n + 1$. The total number of spacers added to form C_{n+1} is $2h_n + 1$, and this ensures that our transformation preserves an infinite measure. That is, $T : [0,\infty) \to [0,\infty)$. To see this, let S_n be the union of spacers that are added to form the n-th tower C_n. Let $\epsilon_n = \frac{2h_n+1}{(n+4)h_n+1}$. Then,

$$\frac{\mu(S_{n+1})}{\mu(C_{n+1})} = \frac{2h_n+1}{h_{n+1}} = \frac{2h_n+1}{(n+4)h_n+1} = \epsilon_n.$$

Since $\sum_{n=0}^{\infty} \epsilon_{n+1} = \infty$, T preserves an infinite measure.

Intuitively, the "special spacers" (i.e. the two spacers placed over the a_n-th subcolumn of C_n) are what allow us to prove spectral weak mixing, while at the same time not destroying rigidity. Also, the fact that the number of special spacers is two is what precludes double ergodicity. Before proving Theorem A, we need the following two lemmas.

Lemma 3.1. *Suppose I is an interval and A is a set of positive measure such that $\mu(A\cap I) > \frac{11}{12}\mu(I)$. Furthermore suppose that the interval I is divided into three equal pieces I_1, I_2, I_3. Then there exists a positive measure set A' such that $A', A'+\frac{1}{3}\mu(I) \subset A$ where $A' \subset I_1$. Moreover, $\mu(A') > \frac{1}{10}\mu(I)$.*

Proof. To begin, suppose that $\mu(I)=1$. Let $A_i = A\cap I_i$ for $i=1,2$. Since the $\mu(A\cap I) > \frac{11}{12}$ we have that

$$\mu(A_i) \geq \mu(A\cap I) - \mu(I_1) - \mu(I_2) > \frac{11}{12} - \frac{2}{3} = \frac{1}{4}.$$

Let $A' = A_1 \cap \left(A_2 - \frac{1}{3}\right)$. It remains to show that $\mu(A') > \frac{1}{10}$. Suppose for a contradiction that $\mu(A') \leq \frac{1}{10}$. Then,

$$\begin{aligned}\frac{1}{3} = \mu\left(I_1\right) &\geq \mu\left(A_1\right) + \mu\left(A_2 - \frac{1}{3}\right) - \mu\left(A_1 \cap \left(A_2 - \frac{1}{3}\right)\right) \\ &> \frac{1}{4} + \frac{1}{4} - \frac{1}{10} = \frac{2}{5}\end{aligned}$$

which is a contradiction. Therefore A' is our desired set.

□

Lemma 3.2. *The map T^2 is ergodic.*

Proof. Let A and B be subsets of $[0,\infty)$ of positive measure. To prove ergodicity of T^2 we must find a time m such that $\mu(T^{2m}A\cap B) > 0$. Let n be large enough so that $\mu(A\cap I) > \frac{23}{24}\mu(I)$ and $\mu(B\cap J) > \frac{23}{24}\mu(J)$ where I, J are levels of C_n. Suppose J is above I in C_n and specifically $T^d I = J$.

We now have two cases that depend on the parity of d.

Case 1: d is even

Let $d=2m$. Then $\mu(T^{2m}(A\cap I)\cap(B\cap J)) > 0$ since I and J are both $\frac{11}{12}$ full of A and B respectively.

Case 2: d is odd

Let $d = 2\ell+1$. Let C_{n+M} be the next stage in the construction such that $n+M+2$ is an odd multiple of 3. Apply the double approximation lemma (Lemma 2.1) to the levels I and J of C_n, M consecutive times to obtain levels $\overline{I}, \overline{J}$ of C_{n+M} such that $T^d\overline{I} = \overline{J}$ and

$$\mu(A\cap\overline{I}) + \mu(B\cap\overline{J}) > \left(2 - \frac{1}{12}\right)\mu(\overline{I}).$$

Note that $\mu(\overline{I}) = \mu(\overline{J})$. Then,

$$\begin{aligned}\mu(A\cap\overline{I}) &> \frac{11}{12}\mu(\overline{I}) \\ \mu(B\cap\overline{J}) &> \frac{11}{12}\mu(\overline{J}).\end{aligned}$$

Tower C_{n+M} is cut into $n+M+2 = 3a_{n+M}$ pieces and the subcolumn a_{n+M} has 2 special spacers added above it to form the tower C_{n+M+1}. Let $n+M+2=3k$ where k is odd. Since the levels $\overline{I}$ and $\overline{J}$ are $\frac{11}{12}$-full of the sets A and B respectively, by Lemma 3.1 there exists sets $A_1 \subset A$ and $B_1 \subset B$ of positive measure such that A_1 belongs to the left third of the level $\overline{I}$ and $T^d(A_1 + \frac{1}{3}\mu(\overline{I})) = B_1$.

Consider the quantity $kh_{n+M}+2+d$. Since k, h_{n+M}, and d are odd integers, the sum $kh_{n+M}+2+d$ is even. Let $kh_{n+M}+2+d=2m$. Then,

$$\mu(T^{2m}A_1 \cap B_1) = \mu(T^{kh_{n+M}+2+d}A_1 \cap B_1) = \mu(B_1) > 0.$$

□

The proof of Theorem A is given in the form of the following three lemmas, where each lemma highlights the individual properties that the transformation, T, exhibits.

Lemma 3.3. *The map T is spectrally weakly mixing.*

Proof. Suppose for a contradiction that $f \in L^\infty([0,\infty))$ is a nonconstant eigenfunction with eigenvalue λ. Thus $f \circ T = \lambda f$. Our plan is to restrict the eigenfunction to a set of finite measure and apply standard arguments.

Let $\overline{f} = f \restriction_{[0,1)}$, and let $x \in [0,1)$. If k is such that $T^k x \in [0,1)$, then

$$\overline{f}(T^k x) = f(T^k x) = \lambda^k f(x) = \lambda^k \overline{f}(x).$$

Note that $\overline{f} \in L^2([0,1))$. Thus, we may approximate $\overline{f}$ be a linear combination of characteristic functions. Let $\epsilon > 0$. Let $g \in L^2([0,1))$ be such that $\left\|g - \overline{f}\right\|_2 < \epsilon$ where g is a linear combination of characteristic functions of the levels of some C_n. Note that g is only defined on the levels of C_n that belong to $[0,1)$. Also, assume that $\|g\|_2 = 1$.

Now, let $E_1 \subset C_n$ be the $r_n := a_n h_n$ bottom stacks of tower C_{n+1} intersected with $[0,1)$. That is, $E_1 = \left(\bigcup_{i=0}^{a_n h_n - 1} I_{n+1,i}\right) \cap [0,1)$. Then $\mu(E_1) = \frac{a_n}{n+2} > \frac{1}{4}$. Notice that $g \restriction_{E_1} = g \circ T^{r_n+2} \restriction_{E_1}$ and

$$\begin{aligned}
\left\|g \circ T^{r_n+2} \restriction_{E_1} - \lambda^{r_n+2} g \restriction_{E_1}\right\|_2 &\leq \left\|g \circ T^{r_n+2} \restriction_{E_1} - \overline{f} \circ T^{r_n+2} \restriction_{E_1}\right\|_2 \\
&+ \left\|\overline{f} \circ T^{r_n+2} \restriction_{E_1} - \lambda^{r_n+2} \overline{f} \restriction_{E_1}\right\|_2 \\
&+ \left\|\lambda^{r_n+2} \overline{f} \restriction_{E_1} - \lambda^{r_n+2} g \restriction_{E_1}\right\|_2 \\
&\leq 2\left\|g - \overline{f}\right\|_2 \\
&< 2\epsilon.
\end{aligned}$$

Putting these together, we see that

$$\left\|g \restriction_{E_1} - \lambda^{r_n+2} g \restriction_{E_1}\right\|_2 < 2\epsilon$$

which implies

$$\left|\lambda^{r_n+2} - 1\right| \|g \restriction_{E_1}\|_2 < 2\epsilon.$$

Thus,

$$\left|\lambda^{r_n+2} - 1\right| < \frac{2\epsilon}{\|g \restriction_{E_1}\|_2} = \frac{2\epsilon}{\sqrt{\mu(E_1)}} < 4\epsilon.$$

Let $E_2 \subset C_n$ be the r_n stacks of C_{n+1} that follow the special spacer, intersected with $[0,1)$. In a similar manner, you can show that $|\lambda^{r_n} - 1| < 4\epsilon$. Therefore,

$$\left|\lambda^2 - 1\right| = \left|\lambda^{r_n+2} - \lambda^{r_n}\right| < 8\epsilon$$

and $\lambda^2 = 1$. Consider,

$$f(T^2 x) = \lambda^2 f(x) = f(x).$$

This is a contradiction since T^2 is ergodic by Lemma 3.2.

□

Lemma 3.4. *The map T is rigid along the height sequence.*

Proof. To see that T is rigid consider the sequence of heights (h_n). Let L be a level of C_N. Recall that L is cut into $N+2$ equal pieces before being stacked to form C_{N+1}. Observe, $\mu(T^{h_n}L \triangle L) \leq \frac{2\mu(L)}{n+2}$ for all $n \geq N$. Hence T is rigid along (h_n).

□

Lemma 3.5. *The map T is not doubly ergodic.*

Proof. To show that T is not doubly ergodic we need to find two sets of positive measure, A and B, such that there does not exist a time $n \in \mathbb{N}$ where $\mu(T^n A \cap A) > 0$ and $\mu(T^n A \cap B) > 0$ simultaneously.

Let $A = I_{1,0}$ and $B = I_{1,1}$, that is A and B are the bottom two levels of C_1. Define $N_{A,A}$ and $N_{A,B}$ by

$$N_{A,A} = \{n \in \mathbb{N} : \mu(T^n A \cap A) > 0\}$$
$$N_{A,B} = \{n \in \mathbb{N} : \mu(T^n A \cap B) > 0\}.$$

Thus, we need to show that $N_{A,A} \cap N_{A,B} = \emptyset$. Since A and B are one level apart, that is $TA = B$, it suffices to prove that the set of differences of $N_{A,A}$ does not contain the element one. Hence, we need to show $1 \notin (N_{A,A} - N_{A,A})$.

Let $N^n_{A,A} = N_{A,A} \cap \{1, 2, \ldots, h_n - 1\}$. Then $N_{A,A} = \bigcup_{n=2}^{\infty} N^n_{A,A}$. We will show that 2 is the smallest positive number in $N_{A,A} - N_{A,A}$ by inductively analyzing $N^n_{A,A} - N^n_{A,A}$.

First consider the tower C_2. Let $d_1 = h_1 + 2$ and $d_2 = h_1$. Note that d_1 and d_2 are consecutive differences between the levels of A in C_2. Then,

$$N^2_{A,A} = \left\{ \sum_{i=k}^{k'} d_i : 1 \leq k \leq k' \leq 2 \right\}.$$

Clearly the smallest possible positive difference between elements of $N^2_{A,A}$ is 2.

Now consider the tower C_n for some $n \geq 2$. Let $d_1, d_2, \ldots, d_{m_n}$ be consecutive differences between the levels of A in C_n, where $m_n = \frac{(n+1)!}{2} - 1$. Then

$$N^n_{A,A} = \left\{ \sum_{i=k}^{k'} d_i : 1 \leq k \leq k' \leq m_n \right\}.$$

By our inductive hypothesis, we can assume that the smallest positive difference between elements of $N^n_{A,A}$ is 2.

Consider the tower C_{n+1}. There are $m_{n+1} + 1$ levels of A in C_{n+1}. Let $\overline{d}_1, \ldots, \overline{d}_{m_{n+1}}$ represent consecutive differences between the levels of A in C_{n+1}. Notice that

$$\overline{d}_{1+j} = d_1, \quad \overline{d}_{2+j} = d_2, \quad \ldots \quad \overline{d}_{m_n+j} = d_{m_n},$$

where j has the form $j = k(m_n+1)$ (k a nonnegative integer) and $m_n + j \leq m_{n+1}$. Also,

$$\overline{d}_{k(m_n+1)} = \begin{cases} h_n - (d_1 + \cdots + d_{m_n}) + 2, & \text{if } k = a_n = \lceil \frac{n+2}{3} \rceil \\ h_n - (d_1 + \cdots + d_{m_n}), & \text{otherwise.} \end{cases}$$

Thus, the smallest positive difference between elements of $N_{A,A}^{n+1}$ is also 2. This completes our inductive argument.

$\square$

Remark 2. In the above construction, if we instead added only one special spacer after the first $a_n h_n$ stacks and $2h_n$ spacers at the end, then an argument almost identical to the one in Lemma 3.3 shows that the resulting transformation is spectrally weakly mixing. It can also be shown that this construction is doubly ergodic on the levels. That is, if I, J are levels of some C_N, then there exists a time n such that $\mu(T^n I \cap I) > 0$ and $\mu(T^n I \cap J) > 0$. It was, however, shown in [11] that there exist transformations that are doubly ergodic on intervals but not doubly ergodic, and the question of whether this transformation is doubly ergodic remains open.

3.2. Ergodicity of the Cartesian Square and Rigidity. In this section, we give a constructive proof of the following theorem.

Theorem B. *There exists an infinite measure-preserving rank-one transformation that has ergodic Cartesian square and is rigid.*

Let us begin by describing the construction. The first stage, C_0, consists of the interval $[0, 1)$. Thus, the initial height is $h_0 = 1$. Define the sequence (s_n) by $s_0 = 1, s_1 = 2, s_2 = 1, s_3 = 2, s_4 = 3, s_5 = 1, \ldots$. It is clear that (s_n) cycles through every natural number infinitely often. In general, suppose we have already constructed the C_n tower, which is a union of h_n levels. To construct C_{n+1} from C_n do the following:

(1) Cut C_n into $n + 2$ equal pieces.

(2) Compute the quantity $a_n = \lceil \frac{n+2}{3} \rceil$.

(3) Stack the subcolumns of C_n to form C_{n+1} in the following order: $a_n h_n$ stacks, s_n spacers, $(n + 2 - a_n)h_n$ stacks, $2h_n + 1 - s_n$ spacers.

This construction is very similar to the previous construction in Section 3.1. In particular, the number of total number of spacers added to form C_{n+1} is still $2h_n + 1$. Thus, as before, $h_{n+1} = (n+4)h_n + 1$, and T preserves an infinite measure. The two constructions differ in the amount of special spacers that we place over the a_n-th subcolumn. Intuitively, allowing the number of spacers added above the a_n-th subcolumn to cycle through every number infinitely often is what allows us to prove ergodicity of the Cartesian square, while at the same time not destroying rigidity.

Before we prove Theorem B, we prove the following proposition.

Proposition 3.6. *The transformation T is doubly ergodic and rigid.*

Proposition 3.6 is indeed implied by our main result (Theorem B), as ergodicity of the Cartesian square implies double ergodicity. We have, however, chosen to include a separate statement and proof because the proof technique is similar to that in Theorem B but much cleaner. Upon conclusion of the proof of Proposition 3.6, we will prove Theorem B.

The following counting lemma will be used in the proof of Proposition 3.6.

Lemma 3.7. *Suppose I is an interval and A is a set of positive measure such that $\mu(A \cap I) > \frac{11}{12}\mu(I)$. Furthermore suppose that the interval I is divided into three equal pieces I_1, I_2, I_3. Then there exists a positive measure set A' such that $A', A' + \frac{1}{3}\mu(I), A' + \frac{2}{3}\mu(I) \subset A$.*

Proof. To begin, suppose that $\mu(I) = 1$. Let $A_i = A \cap I_i$ for $i = 1, 2, 3$. Since the $\mu(A \cap I) > \frac{11}{12}$ we have that

$$\mu(A_i) \geq \mu(A \cap I) - \mu(I_1) - \mu(I_2) > \frac{11}{12} - \frac{2}{3} = \frac{1}{4}.$$

Let $A' = (A_1) \cap \left(A_2 - \frac{1}{3}\right) \cap \left(A_3 - \frac{2}{3}\right)$. It remains to show that A' has positive measure. Suppose for a contradiction that $\mu(A') = 0$. First observe that

$$\mu\left(\left(A_2 - \frac{1}{3}\right) \cap \left(A_3 - \frac{1}{3}\right)\right) \leq \mu\left(I_1\right) - \mu\left(A_1\right) < \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Now,

$$\begin{aligned}\frac{6}{12} &< \mu\left(A_2 - \frac{1}{3}\right) + \mu\left(A_3 - \frac{2}{3}\right) \\ &\leq \mu\left(\left(A_2 - \frac{1}{3}\right) \cup \left(A_3 - \frac{2}{3}\right)\right) + \mu\left(\left(A_2 - \frac{1}{3}\right) \cap \left(A_3 - \frac{2}{3}\right)\right) \\ &< \mu\left(I_1\right) + \frac{1}{12} = \frac{5}{12}\end{aligned}$$

which is a contradiction. Therefore A' is our desired set.

□

Proof of Proposition 3.6. Similar to the construction in Section 3.1, T is rigid by Lemma 3.4. Thus, we need only show double ergodicity. To that end, let $A, B \subset [0, \infty)$ be sets of positive measure. Our goal is show that there exists a time m such that $\mu(T^m A \cap A) > 0$ and $\mu(T^m A \cap B) > 0$. Let n be large enough so that $\mu(A \cap I) \geq \frac{23}{24}\mu(I)$ and $\mu(B \cap J) \geq \frac{23}{24}\mu(J)$ where I, J are levels of C_n. Without loss of generality, suppose that the level J is d levels above I in C_n. That is, $T^d I = J$.

Let C_{n+M} be the next stage in the construction such that the tower C_{n+M} has d special spacers added above the a_{n+M}-th subcolumn to form C_{n+M+1}. Apply the double approximation lemma (Lemma 2.1) to the levels I and J of C_n, M consecutive times to obtain levels $\overline{I}, \overline{J}$ of C_{n+M} such that $T^d \overline{I} = \overline{J}$ and

$$\mu(A \cap \overline{I}) + \mu(B \cap \overline{J}) > (2 - \frac{1}{12})\mu(\overline{I}).$$

Note that $\mu(\overline{I}) = \mu(\overline{J})$, so

$$\mu(A \cap \overline{I}) > \frac{11}{12}\mu(\overline{I})$$
$$\mu(B \cap \overline{J}) > \frac{11}{12}\mu(\overline{J}).$$

To make the picture more clear, assume that $n + M + 2$ is divisible by 3. That is, the tower C_{n+M} is cut into $3a_{n+M}$ pieces, and the a_{n+M}-th subcolumn has d special spacers added above it to form the tower C_{n+M+1}. Since the levels $\overline{I}$ and $\overline{J}$ are $\frac{11}{12}$-full of the sets A and B respectively, by Lemma 3.7 there exists sets $\overline{A_1} \subset A$,

$\overline{A_2} \subset A$, and $\overline{B} \subset B$ of positive measure such that $\overline{A_1}$ belongs to the left third of the level $\overline{I}$, $\overline{A_1} + \frac{1}{3}\mu(\overline{I}) = \overline{A_2}$, and $T^d(\overline{A_2} + \frac{1}{3}\mu(\overline{I})) = \overline{B}$.

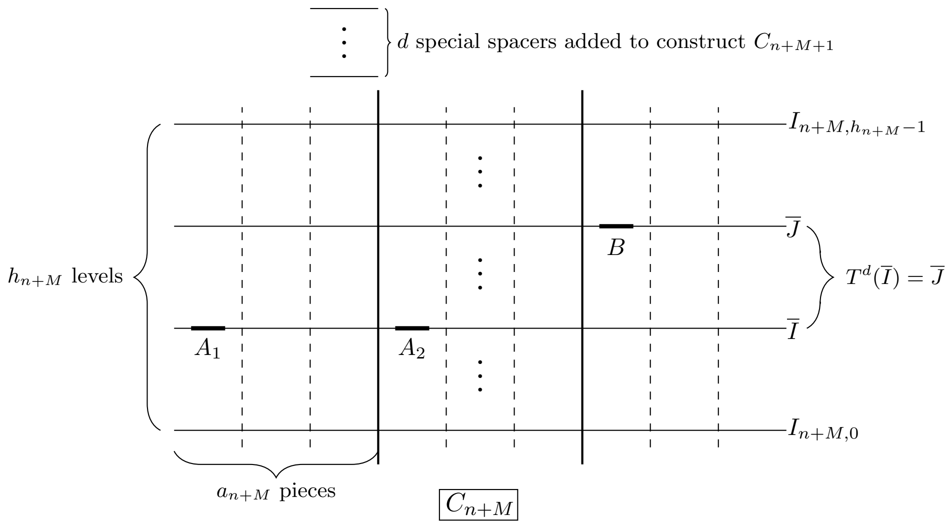

FIGURE 3. Illustration of $\overline{I}$ and $\overline{J}$ in C_{n+M} along with the placement of sets A_1, A_2, and B.

Now observe the following

$$T^{a_{n+M}h_{n+M}+d}\overline{A_1} = \overline{A_2}$$

$$T^{a_{n+M}h_{n+M}+d}\overline{A_2} = T^d(\overline{A_2} + \frac{1}{3}\mu(\overline{I})) = \overline{B}.$$

If we let $m = a_{n+M}h_{n+M} + d$, then we have the result.

$\square$

We are now ready to prove Theorem B. Similar to the proof of double ergodicity, finding levels in an advantageous location to approximate arbitrary sets is a critical element of the proof.

Proof of Theorem B. Again, the argument in Lemma 3.4 shows that T is rigid. Thus, our goal is to show that the map $T \times T$ is ergodic. That is, given sets of positive measure, $E_1, E_2 \subset [0,\infty) \times [0,\infty)$, there exists a time m such that $\mu \times \mu((T \times T)^m E_1 \cap E_2) > 0$.

Let n be large enough so that $\mu \times \mu(E_1 \cap (I_1 \times J_1)) > \frac{199}{200}\mu(I_1)\mu(J_1)$ and $\mu \times \mu(E_2 \cap (I_2 \times J_2)) > \frac{199}{200}\mu(I_2)\mu(J_2)$ where I_1, I_2, J_1, J_2 are levels of C_n. Without loss of generality, suppose that I_2 is above I_1 and J_2 is above J_1 in C_n. Let d_1, d_2 be such that $T^{d_1} I_1 = I_2$ and $T^{d_2} J_1 = J_2$. Suppose that $d_1 > d_2$, and let k be such that $d_1 = d_2 + k$.

Let C_{n+M} be the next stage in the construction such that the tower C_{n+M} has k special spacers added above the a_{n+M}-th subcolumn to form C_{n+M+1}. Apply Lemma 2.2 to the squares $I_1 \times J_1$ and $I_2 \times J_2$, M consecutive times to obtain squares $\overline{I_1} \times \overline{J_1}$ and $\overline{I_2} \times \overline{J_2}$ where $\overline{I_1}, \overline{I_2}, \overline{J_1}, \overline{J_2}$ are levels in C_{n+M}, $T^{d_1}\overline{I_1} = \overline{I_2}$ and $T^{d_2}\overline{J_1} = \overline{J_2}$, and

$$\mu \times \mu \left(E_1 \cap \left(\overline{I_1} \times \overline{J_1}\right)\right) + \mu \times \mu \left(E_2 \cap \left(\overline{I_2} \times \overline{J_2}\right)\right) > \left(2 - \frac{1}{100}\right) \mu\left(\overline{I_1}\right) \mu\left(\overline{J_1}\right).$$

Then,

$$\mu \times \mu \left(E_1 \cap \left(\overline{I_1} \times \overline{J_1}\right)\right) > \frac{99}{100} \mu\left(\overline{I_1}\right) \mu\left(\overline{J_1}\right)$$
$$\mu \times \mu \left(E_2 \cap \left(\overline{I_2} \times \overline{J_2}\right)\right) > \frac{99}{100} \mu\left(\overline{I_1}\right) \mu\left(\overline{J_1}\right).$$

Let π_i be the projection map onto the i-th coordinate. Notice that since $\mu \times \mu(E_i \cap (\overline{I_i} \times \overline{J_i})) > \frac{99}{100}\mu(\overline{I_i})\mu(\overline{J_i})$, we have that $\mu(\pi_1(E_i) \cap \overline{I_i}) > \frac{99}{100}\mu(\overline{I_i}) > \frac{11}{12}\mu(\overline{I_i})$ and $\mu(\pi_2(E_i) \cap \overline{J_i}) > \frac{99}{100}\mu(\overline{J_i}) > \frac{11}{12}\mu(\overline{J_i})$ for $i = 1, 2$.

To make the picture more clear, assume that $n+M+2$ is divisible by 3. That is, the tower C_{n+M} is cut into $3a_{n+M}$ pieces, and the subcolumn a_{n+M} has k special spacers added above it to form the tower C_{n+M+1}. Since the levels $\overline{J_1}$ and $\overline{J_2}$ are $\frac{11}{12}$-full of the sets $\pi_2(E_1)$ and $\pi_2(E_2)$ respectively, by Lemma 3.1 there exists sets $A_2 \subset \pi_2(E_1)$ and $B_2 \subset \pi_2(E_2)$ of measure at least $\frac{1}{10}\mu(\overline{J_1})$ such that A_2 belongs to the left third of the level $\overline{J_1}$ and $T^{d_2}(A_2 + \frac{1}{3}\mu(\overline{J_1})) = B_2$.

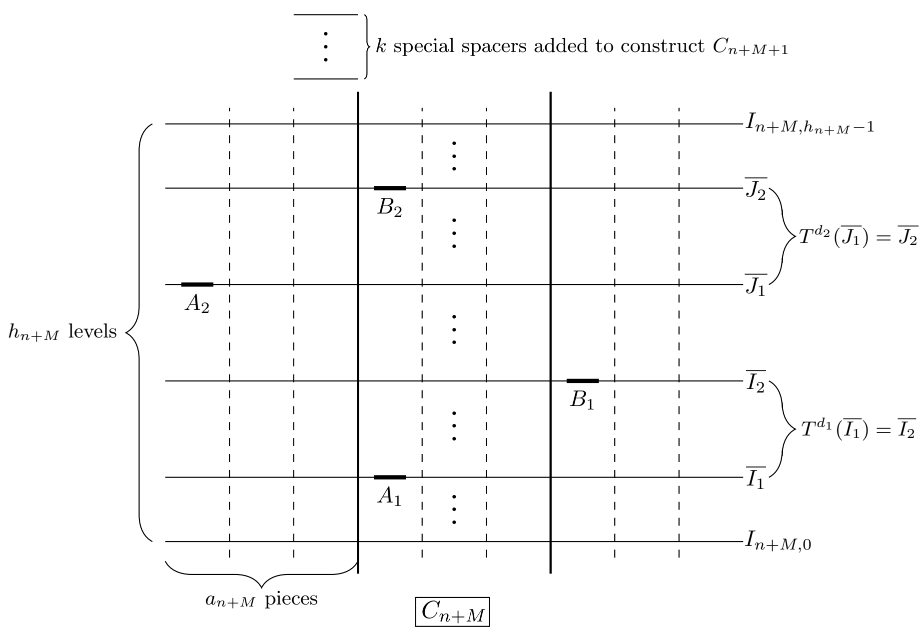

FIGURE 4. Illustration of $\overline{I_1}$, $\overline{I_2}$, $\overline{J_1}$, and $\overline{J_2}$ in C_{n+M} along with an example placement of sets A_1, A_2, B_1, and B_2.

A similar calculation can be carried out with levels $\overline{I_1}$ and $\overline{I_2}$ to obtain sets $A_1 \subset \pi_1(E_1)$ and $B_1 \subset \pi_1(E_2)$ of measure at least $\frac{1}{10}\mu(\overline{I_1})$ such that A_1 belongs to the middle third of the level $\overline{I_1}$ and $T^{d_1}(A_1 + \frac{1}{3}\mu(\overline{I_1})) = B_1$.

Now observe the following

$$T^{a_{n+M}h_{n+M}+d_1}A_1 = T^{d_1}\left(A_1 + \frac{1}{3}\mu(\overline{I_1})\right) = B_1$$
$$T^{a_{n+M}h_{n+M}+d_1}A_2 = T^{d_2}\left(A_2 + \frac{1}{3}\mu(\overline{J_1})\right) = B_2.$$

Let $m = a_{n+M}h_{n+M} + d_1$. Note that $\mu \times \mu(A_1 \times A_2) > \frac{1}{100}\mu(\overline{I_1})\mu(\overline{J_1})$ and $(T \times T)^m(A_1 \times A_2) = B_1 \times B_2$. Since E_i is at least $\frac{199}{200}$-full of $\overline{I_i} \times \overline{J_i}$ we have that $\mu \times \mu((A_1 \times A_2) \cap E_1) > 0$ and $\mu \times \mu((B_1 \times B_2) \cap E_2) > 0$. Hence, $\mu \times \mu((T \times T)^m E_1 \cap E_2) > 0$.

□

Recently, we learned of the work of Adams and Silva who produced an example of an infinite measure-preserving rank-one transformation that has ergodic Cartesian square and is rigid [8].

4. Rationally Ergodic Construction

In this section, we will show that rigidity is compatible with rational ergodicity. Recall that rational ergodicity implies weak rational ergodicity, which is a meager subset of the set of infinite measure-preserving transformations [3]. Thus, we cannot establish compatibility through categorical arguments, and instead prove the following theorem.

Theorem C. *There exists an infinite measure-preserving transformation that is rationally ergodic and rigid.*

A recent theorem proved independently by Aaronson et al. [5] and Bozgan et al. [12] shows that all rank-one transformations with a bounded number of cuts are boundedly rationally ergodic. Bounded rational ergodicity is stronger than rational ergodicity, and we will appeal to this theorem to prove rational ergodicity of our construction. Note that the rigidity of our previous constructions hinged on the number of cuts going to infinity. Thus, in an effort to reconcile two seemingly contradictory properties, we will realize a rank-one construction with an unbounded number of cuts as a rank-one construction with a bounded number of cuts by rarely adding spacers.

Now, let us describe our construction. Begin with the interval $[0, 1)$ as the first stage, C_0. The initial height is then $h_0 = 1$. Define the sequence (s_n) by

$$s_n = \begin{cases} 2h_n & \text{if } n = 2^k \text{ for some } k \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

In general, suppose we have already constructed the C_n-th tower. To construct C_{n+1} do the following:

(1) Cut C_n into 2 equal pieces.

(2) Stack the subcolumns of C_n to form C_{n+1} in the following order: $2h_n$ stacks, s_n spacers.

The construction resembles the Hajian-Kakutani transformation from 1970 [19], but spacers are only added when the stage of the construction is a power of 2.

Proof of Theorem C. It is clear from the description that T is a rank-one transformation with 1 cut. Appealing to the above mentioned theorem in [5] or [12], we immediately obtain that T is rationally ergodic.

We now show that T is rigid along the sequence (n_m) where $n_m = h_{2^{m-1}+1}$. This sequence was obtained from the heights of the towers that directly follow the addition of spacers. Let E be a level of C_N. Let M be the smallest positive integer such that $h_N \leq n_M$. Notice that to obtain $C_{2^{M-1}+1}$ from $C_{2^{M-1}}$ we cut the $h_{2^{M-1}}$ levels of $C_{2^{M-1}}$ into 2 equal pieces, stack the two subcolumns right on left, and then add $s_{2^{M-1}} = 2h_{2^{M-1}}$ spacers at the top. We wish to observe what happens to the set E under n_M iterations of T. Recall that n_M is the height of tower $C_{2^{M-1}+1}$. Since there are no spacers added again until the 2^M stage of the construction, to analyze $T^{n_M}E \setminus E$ it is best to consider E in the C_{2^M+1} tower. Here we observe,

$$\mu(T^{n_M}E \setminus E) \leq \frac{1}{2^{2^M - 2^{M-1}}}\mu(E) = \frac{1}{2^{2^{M-1}}}\mu(E).$$

Similarly, for all m such that $h_N \leq n_m$ we have

$$\mu(T^{n_m}E \triangle E) \leq \frac{2}{2^{2^m - 2^{m-1}}}\mu(E) = \frac{2}{2^{2^{m-1}}}\mu(E) = \frac{1}{2^{2^{m-1}-1}}\mu(E).$$

Since $\frac{1}{2^{2^{m-1}-1}} \to 0$ as $m \to \infty$ we have rigidity.

□

Remark 3. Intuitively, the above construction can be thought of as a rank-one construction where the C_m-th tower is of height n_m. To obtain C_{m+1} from C_m, you cut C_m into $2^{2^{m-1}}$ equal pieces and then stack as follows: $2^{2^{m-1}}n_m$ stacks and $2h_{2^m}$ spacers.

5. Rigidity Sequences

In this section we will explore which sequences can be realized as rigidity sequences for two different types of transformations preserving an infinite measure. First, we prove a theorem on rigidity sequences for weakly mixing transformations (inspired by a proposition in [10]). Second, we prove a theorem on rigidity sequences for rationally ergodic transformations.

Theorem D. *Let (n_m) be an increasing sequence of natural numbers such that $\frac{n_{m+1}}{n_m} \to \infty$. There exists a rank-one, infinite measure-preserving transformation that has ergodic Cartesian square and is rigid along (n_m).*

Proof. Let (n_m) be an increasing sequence of natural numbers such that $\frac{n_{m+1}}{n_m} \to \infty$. Without loss of generality, define $n_0 = 1$ and suppose that $\frac{n_{m+1}}{n_m} \geq 4$ for all $m \geq 0$. Our construction will be a variant of the construction in Section 3.2.

Write n_{m+1} as $n_{m+1} = q_m n_m + r_m$ where $0 \leq r_m < n_m$. Let p_m be the least positive integer such that $\frac{p_m n_m + r_m}{n_{m+1}} \geq \frac{1}{m+1}$ for all $m \geq 0$. If we let $\epsilon_m = \frac{p_m n_m + r_m}{n_{m+1}}$, then $\sum_{m=0}^{\infty} \epsilon_m = \infty$ and $\epsilon_m \to 0$. Also notice that n_{m+1} can be written as $n_{m+1} = (q_m - p_m)n_m + p_m n_m + r_m$ where $q_m - p_m \geq 2$ for all $m \geq 0$ and $q_m - p_m \to \infty$.

Now we will describe the construction. The sequence n_m will be the height of the C_m tower, that is $h_m = n_m$. The first stage, C_0, consists of the interval $[0,1)$. Thus the initial height $h_0 = 1 = n_0$. Define the sequence (s_m) by $s_0 = 1, s_1 = 2, s_2 = 1, s_3 = 2, s_4 = 3, s_5 = 1, \ldots$. It is clear that (s_m) cycles through every natural number infinitely often. In general, suppose we have already constructed the C_m tower, which is a union of $h_m = n_m$ levels. To construct C_{m+1} from C_m do the following:

(1) Cut C_m into $q_m - p_m$ equal pieces.

(2) Compute the quantity $a_m = \lceil \frac{q_m - p_m}{3} \rceil$.

(3) Stack the subcolumns of C_m to form C_{m+1} in the following order: $a_m h_m$ stacks, s_m spacers, $(q_m - p_m - a_m)h_m$ stacks, $p_m h_m + r_m - s_m$ spacers.

Notice that since $p_m h_m = p_m n_m > m+1$ for all $m \geq 0$ and $s_m \leq m+1$, there are a positive number of spacers placed at the end of the C_{m+1} tower. A proof similar to that of Theorem B shows that this construction has ergodic Cartesian square and is rigid along the sequence of heights, which is (n_m). Also similar to before, the fact that $\sum_{m=0}^{\infty} \epsilon_m = \infty$ guarantees that T preserves an infinite measure. □

Remark 4. Is there an example of an infinite measure-preserving rank-one transformation that is weakly mixing and rigid along a sequence (n_m) where the ratios do not tend to infinity? More specifically, does there exist a rank-one transformation preserving an infinite measure that is weakly mixing and rigid along $n_m = 2^m$?

Motivated by the construction in Theorem C, we now generalize possible rigidity sequences for rationally ergodic transformations.

Theorem E. *Let (n_m) be an increasing sequence of natural numbers such that $\frac{n_{m+1}}{n_m} \to \infty$. Furthermore, assume that $n_{m+1} = 2^{k_m} n_m + r_m$ where $0 \leq r_m < n_m$. Then there exists an infinite measure-preserving transformation that is rationally ergodic and rigid along (n_m).*

Proof. Let (n_m) be an increasing sequence of natural numbers such that $\frac{n_{m+1}}{n_m} \to \infty$. Without loss of generality, define $n_0 = 1$ and suppose that $\frac{n_{m+1}}{n_m} \geq 5$ for all $m \geq 0$. Suppose that n_{m+1} can be written as $n_{m+1} = 2^{k_m} n_m + r_m$ where $0 \leq r_m < n_m$.

Let $p_m = 2^{k_m - 1}$ and $\epsilon_m = \frac{p_m n_m + r_m}{n_{m+1}}$. Then $\epsilon_m > \frac{1}{4}$ for all $m \geq 0$. Therefore, $\sum_{m=0}^{\infty} \epsilon_m = \infty$. Before we explicitly describe the construction, we will try to give the intuition. If we were to proceed in a manner similar to the previous theorem, then we would cut the n_m levels of C_m into $2^{k_m} - p_m = 2^{k_m - 1}$ equal pieces, stack them left to right and add $p_m n_m + r_m = 2^{k_m - 1} n_m + r_m$ spacers to the end. However, this would not produce a rank-one transformation with a bounded number of cuts, which is what we need in order to guarantee rational ergodicity. With that in mind, we will cut the tower into 2 equal pieces and stack left to right. This process will continue $k_m - 1$ times, at which point we will add the appropriate number of spacers to double the measure. Now we will describe the construction explicitly.

Begin with the interval $[0,1)$ as the first stage, C_0. The initial height is then $h_0 = 1$. Define the sequence (s_n) by

$$s_n = \begin{cases} 2^{k_m-1}n_m + r_m & \text{if } n = k_m + k_{m-1} + \cdots + k_0 - (m+2) \\ 0 & \text{otherwise.} \end{cases}$$

In general, suppose we have already constructed the C_n tower. To construct C_{n+1} do the following:

(1) Cut C_n into 2 equal pieces.

(2) Stack the subcolumns of C_n to form C_{n+1} in the following order: $2h_n$ stacks, s_n spacers.

The first thing to notice about the construction is that most of the time you are simply cutting the existing tower into two pieces and stacking them, much like an odometer. However at certain times, namely along the sequence of times when s_n is nonzero, spacers are being added. The fact that $\sum_{m=0}^{\infty} \epsilon_m = \infty$ guarantees that T preserves an infinite measure. Also, it is clear that the number of cuts is bounded and thus by a theorem in [5] or [12], T is rationally ergodic.

We now show that T is rigid along the sequence (n_m). Note that $n_0 = 1 = h_0$ and in general $n_m = h_{k_{m-1}+\cdots+k_0-m}$ for $m \geq 1$. Let E be a level of C_N. Let M be the smallest positive integer such that $h_N \leq n_M$. We wish to observe what happens to the set E under n_M iterations of T. Similar to the proof of rigidity in Section 4, since there are no spacers added after the stage that has height n_M until the $k_M + \cdots + k_0 - (M+2)$ stage of the construction, it is best to view E as a union of levels in $C_{k_M+\cdots+k_0-(M+1)}$. Observe,

$$\mu(T^{n_M}E \setminus E) \leq \frac{1}{2^{k_M-1}}\mu(E).$$

Similarly, for all m such that $h_N \leq n_m$ we have

$$\mu(T^{n_m}E \triangle E) \leq \frac{2}{2^{k_m-1}}\mu(E).$$

Since $\frac{1}{2^{k_m-1}} \to 0$ as $m \to \infty$ we have rigidity.

$\square$

Remark 5. In the above proof the key ingredients are that the ratios of the sequence go to infinity and that we can realize the corresponding rank-one construction with the number of cuts tending to infinity as a rank-one construction with a bounded number of cuts. This observation leads to a generalization of the above theorem.

Theorem 5.1. *Suppose (n_m) is an increasing sequence of natural numbers such that $\frac{n_{m+1}}{n_m} \to \infty$ and write $n_{m+1} = q_m n_m + r_m$. If there exists $K > 0$ such that for all $m \geq 0$ we can find numbers $a_1, \ldots, a_{\ell_m}$ with $a_i \leq K$ where $\lceil \frac{q_m}{2} \rceil = a_1 \cdots a_{\ell_m}$ then there exists an infinite measure-preserving transformation that is rationally ergodic and rigid along (n_m).*

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