Topologically Weakly Mixing Homeomorphisms of the Klein bottle that are Uniformly Rigid

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Abstract

In this paper we prove that there is a large family of topologically weakly mixing homeomorphisms of the Klein bottle that are uniformly rigid. We do this by viewing the Klein bottle as the quotient of the two-torus by an appropriate group action and producing topologically weakly mixing homeomorphisms of the two-torus that are uniformly rigid and equivariant with respect to the action.

1 Introduction

In ergodic theory, transformations that are of particular interest are ones that are rigid and also weakly mixing, see [1] for more information. These two behaviors are very different, though not exclusive. It is well known that weakly mixing, rigid maps are typical in the sense that they form a dense G_δ subset of all invertible measure-preserving transformations of a Lebesgue space (with respect to the weak topology) [6].

Uniform rigidity in topological dynamics was introduced by Glasner and Maon in 1989 [3]. This is a stronger notion of rigidity for compact metric spaces and is the topological analogue of rigidity in ergodic theory. As with rigidity, generic properties hold: there are large families of weakly mixing, uniformly rigid homeomorphisms of certain compact metric spaces. However, unlike rigidity, which does not depend on the space, the existence of weakly mixing, uniformly rigid homeomorphisms does. For example, in 2009 Silva et al. [7] showed that on a Cantor space there are no weakly mixing, uniformly rigid, measure-preserving maps with respect to any metric compatible with the topology (expect when the measure is concentrated at one point). In contrast, Yancey [8] produced a large family of such maps for the two-torus using techniques from Glasner and Weiss [4].

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Let $\mathcal{Q}$ be the closure of the set of conjugations of an aperiodic rotation by Lebesgue measure-preserving homeomorphisms of $\mathbb{T}^2$. In [8] the following is proved:

Theorem 1.1. *There exists a dense G_δ subset $\mathcal{F}$ of $\mathcal{Q}$ such that for every $T \in \mathcal{F}$, $(\mathbb{T}^2, T, \mu)$ is weakly mixing, uniformly rigid, and strictly ergodic.*

While the author was presenting results from [8] in a seminar, Alica Miller asked if it would be possible to obtain similar generic results on the Klein bottle. This is an interesting question since constructing maps on spaces with specified topological properties has always presented difficulties on the Klein bottle. For example, the question of constructing a minimal homeomorphism of the Klein bottle presented mathematicians with trouble for some time, until 1965 when Robert Ellis produced such a map [2].

In Section 4 we produce a large family of topologically weakly mixing homeomorphisms that are uniformly rigid on the Klein bottle. We were not able to prove measure weak mixing, which was the original goal. Our approach is to view the Klein bottle as the quotient of $\mathbb{T}^2$ by an appropriate group action and produce homeomorphisms of $\mathbb{T}^2$ that are topologically weakly mixing, uniformly rigid, and equivariant with respect to the group action. Since these maps are equivariant and the desired properties are compatible with the projection to the Klein bottle, the maps induce homeomorphisms on the Klein bottle that are topologically weakly mixing and uniformly rigid.

Let $\mathcal{P}$ be the closure of the set of conjugations of an aperiodic rotation by homeomorphisms of the Klein Bottle. The main result of this paper is the following:

Theorem 1.2. *There exists a dense G_δ subset $\mathcal{S}$ of $\mathcal{P}$ such that every $T \in \mathcal{S}$ is topologically weakly mixing and uniformly rigid.*

In Section 2 we set notation and recall some definitions. Section 3 contains a description of the Klein bottle when viewed as the quotient of $\mathbb{T}^2$ by a group action and Section 4 contains a proof of Theorem 1.2. The final section of the paper, Section 5, discusses further questions related to uniform rigidity.

2 Preliminaries

We will be considering homeomorphisms defined from X to X where X is a compact metric space with metric d. Since we will be discussing generic properties of these homeomorphisms we would like our space to be a complete metric space. Therefore we will be using the uniform distance between two homeomorphisms S, T of X given by

$$d_u(S,T) = \sup_{x\in X} d(S(x), T(x)) + \sup_{x\in X} d(S^{-1}(x), T^{-1}(x)).$$

The topology induced by d_u is the topology of uniform convergence and with this metric the group of homeomorphisms of X is a complete metric space. We will also call this the topology of uniform convergence of homeomorphisms and their inverses. To simplify notation, if S, T are two homeomorphisms defined on X, let $\overline{d}(S,T) = \sup_{x \in X} d(S(x), T(x))$. With this new notation, $d_u(S,T) = \overline{d}(S,T) + \overline{d}(S^{-1}, T^{-1})$. Notice that even though d_u is not right-invariant, $\overline{d}$ is right-invariant.

Now we recall some definitions from topological dynamics.

Definition 2.1. *A homeomorphism $T : X \to X$ is uniformly rigid if there exists a sequence of natural numbers (n_m) such that $d_u(T^{n_m}, Id) \to 0$ as $m \to \infty$.*

Remark 2.1. *The above definition is equivalent to $T^{n_m} \to Id$ uniformly on X (ie. $\overline{d}(T^{n_m}, Id) \to 0$).*

Definition 2.2. *A homeomorphism $T : X \to X$ is topologically weakly mixing if for any open subsets U_1, U_2, U_3, U_4 of X there exists $k \in \mathbb{Z}$ such that $T^k(U_1 \times U_2) \cap (U_3 \times U_4) \neq \emptyset$.*

3 The set-theoretic Klein bottle

In this section we will show how one obtains the Klein bottle from the quotient of the two-torus by a group action. For this discussion, let X be a topological space and H a discrete group. Suppose H acts on the space X on the left by $(x, h) \mapsto h.x$ where $x \in X$ and $h \in H$. We will be considering the quotient X/H.

The H-orbit of a point $x \in X$ is the set $\{h.x : h \in H\}$. The quotient map $\pi : X \to X/H$ sends $x \in X$ to the H-orbit of x. This means that we can think of X/H as the space X with the H-orbits collapsed to points.

Definition 3.1. *Suppose f is a function from X to X. The function f is H-equivariant if $f(h.x) = h.f(x)$ for all $h \in H$ and $x \in X$.*

If $f : X \to X$ is H-equivariant, then f carries the H-orbits of x to the H-orbits of $f(x)$. Thus f induces a well-defined map on the quotient X/H. Let $\overline{f} : X/H \to X/H$ be the map induced by f. In this case $\overline{f}$ commutes with the quotient map, ie. $\pi \circ f = \overline{f} \circ \pi$.

Definition 3.2. *The left action of H on X is continuous if for each $h \in H$ the map $x \mapsto h.x$ is continuous and is free if for each $x \in X$ the subgroup $\{h \in H : h.x = x\}$ is trivial.*

Definition 3.3. *The left action of H on X is properly discontinuous if it is continuous and for every $x \in X$ there exists an open neighborhood U_x of x such that the H-translates $h.U_x$ meet U_x for only finitely many $h \in H$.*

We will be studying actions that are free and properly discontinuous. In this case if X is a locally Hausdorff space then for every $x \in X$ we can find an open neighborhood U_x such that $U_x \cap h.U_x = \emptyset$ for all $h \in H \setminus \{1\}$. This means that when we identify points that lie in the same H-orbit to form X/H we are not squashing the space. Also, there is a unique topology on X/H called the quotient topology such that $\pi : X \to X/H$ is a continuous map that is a local homeomorphism. In the quotient topology, a subset Y of X/H is open if and only if its preimage under π is open in X. Finally, two points in the quotient X/H are close if the corresponding H-orbits in X contain points that are close.

To form the Klein bottle, let $X = \mathbb{T}^2$ where $\mathbb{T}^2$ is viewed as $[0,1)^2$ modulo one in each coordinate. Let $H = \{\mathbf{1}, -\mathbf{1}\}$ be a discrete group of order two and the action of H on $\mathbb{T}^2$ be defined by

$$\mathbf{1}.(x,y) = (x,y)$$

and

$$-\mathbf{1}.(x,y) = \left(x + \frac{1}{2}, 1 - y\right)$$

for all $(x,y) \in \mathbb{T}^2$. This action is easily seen to be free and properly discontinuous since you can think of it as rotation by π in the first coordinate and complex conjugation in the second. The quotient $\mathbb{T}^2/H$ will be called the set-theoretic Klein bottle and be denoted by $\mathbb{K}^2$.

4 Generic Results

In this section we will show that there is a large family of topologically weakly mixing homeomorphisms of the Klein bottle that are uniformly rigid. In [5] Glasner and Weiss produced a large family of topologically weakly mixing homeomorphisms of the two-torus that were uniformly rigid. We use their maps on the two-torus as a starting point for our constructions that eventually get pushed down to the Klein bottle.

Our plan is to produce homeomorphisms of $\mathbb{T}^2$ that are topologically weakly mixing, uniformly rigid, and H-equivariant. Since these maps are H-equivariant, they will induce homeomorphisms on $\mathbb{K}^2$. Notice that topological weak mixing and uniform rigidity are compatible with the projection to the Klein bottle. Thus these induced homeomorphisms of $\mathbb{K}^2$ will be topologically weakly mixing and uniformly rigid.

The model of $\mathbb{T}^2$ that we will be using is the unit interval model where it is viewed as $[0,1)^2$ and the coordinates are taken modulo 1. We will be using additive notation and $|\cdot|$ will denote the distance to the nearest integer or absolute value (the distinction should be clear from context). Let $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $\sigma(x,y) = (x+\alpha, y)$ where α is irrational and $\{n\alpha : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$. In this case it is easy to see that σ is H-equivariant.

Let $\mathcal{H}(X)$ be the set of homeomorphisms of a compact metric space X. Define the set $O(\sigma)$ as follows:

$$O(\sigma) = \left\{G^{-1} \circ \sigma \circ G \;:\; G \in \mathcal{H}(\mathbb{T}^2) \textit{ and } G \textit{ is } H-\textit{equivariant}\right\}.$$

We will be considering $O(\sigma)$ as a subset of all homeomorphisms of $\mathbb{T}^2$ with the topology of uniform convergence of homeomorphisms and their inverses. Let $\mathcal{O} = \overline{O(\sigma)}$ with the closure taken in the above topology.

In a similar fashion, define the set $P(\sigma)$ as follows:

$$P(\sigma) = \left\{ G^{-1} \circ \overline{\sigma} \circ G \ : \ G \in \mathcal{H}(\mathbb{K}^2) \right\}.$$

Recall that $\overline{f}$ denotes the induced map on $\mathbb{K}^2$ of the original map $f : \mathbb{T}^2 \to \mathbb{T}^2$. The set $P(\sigma)$ will be considered as a subset of all homeomorphisms of $\mathbb{K}^2$ with the topology of uniform convergence of homeomorphisms and their inverses. Let $\mathcal{P} = \overline{P(\sigma)}$ with the closure taken in the above topology.

4.1 Topological Weak Mixing

The main goal of this subsection is to prove the following theorem and corollary:

Theorem 4.1. *There exists a dense G_δ subset $\mathcal{R}_1$ of $\mathcal{O}$ such that every $T \in \mathcal{R}_1$ is topologically weakly mixing.*

Corollary 4.1. *There exists a dense G_δ subset $\mathcal{S}_1$ of $\mathcal{P}$ such that every $T \in \mathcal{S}_1$ is topologically weakly mixing.*

Proof of Theorem 4.1:
Let $\{U_i\}$ be a countable basis for $\mathbb{T}^2$. Define the set $R_{U_i,\, U_j,\, U_l,\, U_m}$ as follows:

$$\begin{aligned} R_{U_i,\, U_j,\, U_l,\, U_m} =& \{T \in \mathcal{O} \ : \ \textit{there exists an integer } k \textit{ with} \\ & T^k(U_i \times U_j) \cap (U_l \times U_m) \neq \emptyset\}. \end{aligned}$$

We will show that $\mathcal{R}_1 = \bigcap_{i,\, j,\, l,\, m} R_{U_i,\, U_j,\, U_l,\, U_m}$ is our desired dense G_δ subset of $\mathcal{O}$.

From the definition of topological weak mixing we see that $\mathcal{R}_1$ is precisely the set of topologically weakly mixing homeomorphisms in $\mathcal{O}$. Also, clearly $\mathcal{R}_1$ is open in $\mathcal{O}$. Thus it remains to show that each $R_{U_i,\, U_j,\, U_l,\, U_m}$ is dense. For simplicity of notation, we will show that $R_{U_1,\, U_2,\, U_3,\, U_4}$ is dense in $\mathcal{O}$. Since $R_{U_1,\, U_2,\, U_3,\, U_4} \subseteq \mathcal{O}$ and $\mathcal{O}$ is closed, it suffices to show that if $G_0 \in \mathcal{H}(\mathbb{T}^2)$ and G_0 is H-equivariant then $G_0^{-1} \circ \sigma \circ G_0 \in \overline{R_{U_1,\, U_2,\, U_3,\, U_4}}$.

Suppose that (G_m) is a sequence in $\mathcal{H}(\mathbb{T}^2)$ of H-equivariant homeomorphisms such that $d_u(G_m^{-1} \circ \sigma \circ G_m, \sigma) \to 0$ as $m \to \infty$ and for all m, $G_m^{-1} \circ \sigma \circ G_m \in R_{G_0 U_1,\, G_0 U_2,\, G_0 U_3,\, G_0 U_4}$. Notice that

$$G_0 R_{U_1,\, U_2,\, U_3,\, U_4} G_0^{-1} = R_{G_0 U_1,\, G_0 U_2,\, G_0 U_3,\, G_0 U_4}.$$

Since G_0, G_0^{-1} are continuous, $d_u(G_0^{-1} \circ G_m^{-1} \circ \sigma \circ G_m \circ G_0, G_0^{-1} \circ \sigma \circ G_0) \to 0$ as $m \to \infty$. Thus we can write

$$\begin{aligned} G_0^{-1} \circ \sigma \circ G_0 &= \lim_{m \to \infty} (G_0^{-1} \circ G_m^{-1} \circ \sigma \circ G_m \circ G_0) \\ &\in \overline{G_0^{-1} R_{G_0 U_1,\, G_0 U_2,\, G_0 U_3,\, G_0 U_4} G_0} \\ &= \overline{R_{U_1,\, U_2,\, U_3,\, U_4}}. \end{aligned}$$

Therefore we have reduced the rest of the proof of Theorem 4.1 to the following lemma:

Lemma 4.1. *Let $\epsilon > 0$ and U_1, U_2, U_3, U_4 be open sets in $\mathbb{T}^2$. Then there exists an H-equivariant $G \in \mathcal{H}(\mathbb{T}^2)$ such that the following two properties hold:*

1. $d_u(G^{-1} \circ \sigma \circ G, \sigma) < \epsilon$

2. $G^{-1} \circ \sigma \circ G \in R_{U_1,\, U_2,\, U_3,\, U_4}$ *(ie.* $\sigma \in R_{GU_1,\, GU_2,\, GU_3,\, GU_4}$*).*

Remark 4.1. *The U_1, U_2, U_3, U_4 that appear in the above lemma are $G_0U_1, G_0U_2, G_0U_3, G_0U_4$ from the proof of Theorem 4.1.*

Proof of Lemma 4.1. Let $\epsilon > 0$ and U_1, U_2, U_3, U_4 be open sets in $\mathbb{T}^2$. Let $p : \mathbb{T}^2 \to \mathbb{T}$ be a projection onto the second coordinate. Let h_1, h_2 be homeomorphisms of $\mathbb{T}$ such that the following two properties hold:

- $h_1(pU_1) \cap (pU_3) \neq \emptyset$ and $h_2(pU_2) \cap (pU_4) \neq \emptyset$

- $h_i(1-y) = 1 - h_i(y)$ for $i = 1, 2$.

Now choose points in the above intersections such that

- $y_i \in pU_i$ for $i = 1, 2, 3, 4$

- $h_1(y_1) = y_3$ and $h_2(y_2) = y_4$

- $y_1, y_2, y_3, y_4, 1-y_1, 1-y_2, 1-y_3, 1-y_4$ are all distinct points.

Also choose distinct x_i such that the point (x_i, y_i) belongs to U_i for $i = 1, 2, 3, 4$.

We are now ready to start building our desired function G. Let $x \to g_x$ be a continuous function from $[0, 1)$ to $\mathcal{H}(\mathbb{T})$ such that $g_0, g_{\frac{1}{4}}, g_{\frac{1}{2}}, g_{\frac{3}{4}} = Id$, $g_{\frac{1}{8}}, g_{\frac{5}{8}} = h_1$, and $g_{\frac{3}{8}}, g_{\frac{7}{8}} = h_2$ with linear interpolation in between. By the choice of the y_i's above we know that y_i and $1 - y_i$ are distinct. Let V_i and V_{-i} be pairwise distinct, symmetric neighborhoods around y_i and $1-y_i$ respectively that are all equal in length. Define continuous bump functions b_i, b_{-i} on $\mathbb{T}$ such that

- b_i is symmetric about y_i on V_i.

- $b_i(V_i^c) = 0$ and $b_{-i}(V_{-i}^c) = 0$

- $b_i(y_i) = 1$, $b_{-i}(1-y_i) = 1$, and $b_i(y) = b_{-i}(1-y)$ for all $y \in \mathbb{T}$

Let $\eta > 0$ be such that if $|y - y'| < \eta$ then $\max_{1 \leq i \leq 4} |b_i(y) - b_i(y')| < \frac{\epsilon}{32}$. Let $\delta > 0$ be such that if $|x - x'| < \delta$ then $\overline{d}(g_x^{-1} g_{x'}, Id) < \min(\eta, \frac{\epsilon}{4})$. Now we will use a rational approximation of α. Choose $q \in \mathbb{Z}\backslash\{0\}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$ with $\frac{1}{q} < \delta$ for some $p \in \mathbb{Z}$.

Define $c_i \in [0,1)$ such that

$$x_1 + c_1 = \frac{1}{8q}, \quad x_2 + c_2 = \frac{3}{8q}, \quad x_3 + c_3 = \frac{1}{4q}, \quad x_4 + c_4 = \frac{1}{2q}$$

all taken modulo one. Let f be defined by $f(y) = \sum_{i=1}^{4} [c_i b_i(y) + c_i b_{-i}(y)]$. Then it is easy to see that $f(1-y) = f(y)$. Now let $G : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $G(x,y) = \big(x + f(y), g_{q(x+f(y))}(y)\big)$. Notice that $G^{-1}(x,y) = (x - f(g_{qx}^{-1}(y)), g_{qx}^{-1}(y))$ and $G^{-1} \circ \sigma \circ G(x,y) = (x + \alpha + f(y) - f(y_*), y_*)$ where $y_* = g^{-1}_{q(x+\alpha+f(y))} g_{q(x+f(y))}(y)$. We claim that this is our desired $G \in \mathcal{H}(\mathbb{T}^2)$. The first thing to check is that G is H-equivariant. To see this observe the following:

$$\begin{aligned}
G\left(-\mathbf{1}.(x,y)\right) &= G\left(x + \frac{1}{2}, 1 - y\right) \\
&= \left(x + \frac{1}{2} + f(1-y), g_{q(x+\frac{1}{2}+f(1-y))}(1-y)\right) \\
&= \left(x + \frac{1}{2} + f(y), g_{q(x+f(y))+\frac{q}{2}}(1-y)\right) \\
&= \left(x + \frac{1}{2} + f(y), g_{q(x+f(y))}(1-y)\right) \\
&= \left(x + \frac{1}{2} + f(y), 1 - g_{q(x+f(y))}(y)\right) \\
&= -\mathbf{1}.G\left(x,y\right).
\end{aligned}$$

Thus G is H-equivariant.

Now we need to check that (2) is verified, that is $\sigma \in R_{GU_1,\, GU_2,\, GU_3,\, GU_4}$. To see this observe

$$\begin{aligned}
G\left(x_1, y_1\right) &= \left(x_1 + c_1, g_{q(x_1+c_1)}(y_1)\right) = \left(\frac{1}{8q}, y_3\right) \\
G\left(x_2, y_2\right) &= \left(x_2 + c_2, g_{q(x_2+c_2)}(y_2)\right) = \left(\frac{3}{8q}, y_4\right) \\
G\left(x_3, y_3\right) &= \left(x_3 + c_3, g_{q(x_3+c_3)}(y_3)\right) = \left(\frac{1}{4q}, y_3\right) \\
G\left(x_4, y_4\right) &= \left(x_4 + c_4, g_{q(x_4+c_4)}(y_4)\right) = \left(\frac{1}{2q}, y_4\right).
\end{aligned}$$

Thus $G\left(x_3, y_3\right) - G\left(x_1, y_1\right) = \left(\frac{1}{8q}, 0\right)$ and $G\left(x_4, y_4\right) - G\left(x_2, y_2\right) = \left(\frac{1}{8q}, 0\right)$. Hence there exists $k \in \mathbb{Z}$ such that $\sigma^k\left(GU_1\right) \cap \left(GU_3\right) \neq \emptyset$ and $\sigma^k\left(GU_2\right) \cap \left(GU_4\right) \neq \emptyset$. Therefore $\sigma \in R_{GU_1,\, GU_2,\, GU_3,\, GU_4}$ as desired.

It remains to check that (1) is verified, that is $d_u(G^{-1} \circ \sigma \circ G, \sigma) < \epsilon$. To begin notice that

$$G^{-1} \circ \sigma \circ G\left(x,y\right) - \sigma\left(x,y\right) = \left(f(y) - f(y_*), y_* - y\right).$$

Since $|q(x+\alpha+f(y)) - q(x+f(y))| = |q\alpha| < \frac{1}{q} < \delta$ we must have $\overline{d}\left(g^{-1}_{q(x+\alpha+f(y))} g_{q(x+f(y))}, Id\right) < \min\left(\eta, \frac{\epsilon}{4}\right)$. This implies that $|y_* - y| < \eta$ and therefore

$$\begin{aligned}
|f(y) - f(y_*)| &= \left| \sum_{i=1}^{4} [c_i(b_i(y) - b_i(y_*)) + c_i(b_{-i}(y) - b_{-i}(y_*))] \right| \\
&\leq \sum_{i=1}^{4} c_i \left|b_i(y) - b_i(y_*)\right| + \sum_{i=1}^{4} c_i \left|b_{-i}(y) - b_{-i}(y_*)\right| \\
&< 4\left(\frac{\epsilon}{32}\right) + 4\left(\frac{\epsilon}{32}\right) \\
&= \frac{\epsilon}{4}.
\end{aligned}$$

Thus $\overline{d}\left(G^{-1} \circ \sigma \circ G, \sigma\right) < \frac{\epsilon}{2}$. In a similar fashion, $\overline{d}\left(G^{-1} \circ \sigma^{-1} \circ G, \sigma^{-1}\right) < \frac{\epsilon}{2}$. Therefore, $d_u(G^{-1} \circ \sigma \circ G, \sigma) < \epsilon$.

□

Proof of Corollary 4.1:

Let $\{U_i\}$ be a countable basis for $\mathbb{T}^2$. We may assume that for each U_i we have $U_i \cap h.U_i = \emptyset$ for all $h \in H$. Let $\overline{U_i}$ be the image of U_i under the quotient map π. Then $\{\overline{U_i}\}$ is a countable basis for $\mathbb{K}^2$. Define the set $S_{\overline{U_i},\, \overline{U_j},\, \overline{U_l},\, \overline{U_m}}$ as follows:

$$\begin{aligned}
S_{\overline{U_i},\, \overline{U_j},\, \overline{U_l},\, \overline{U_m}} = &\{T \in \mathcal{P} \ : \textit{ there exists an integer } k \textit{ with} \\
&T^k(\overline{U_i} \times \overline{U_j}) \cap (\overline{U_l} \times \overline{U_m}) \neq \emptyset\}.
\end{aligned}$$

We will show that $\mathcal{S}_1 = \bigcap_{i,\, j,\, l,\, m} S_{\overline{U_i},\, \overline{U_j},\, \overline{U_l},\, \overline{U_m}}$ is our desired dense G_δ subset of $\mathcal{P}$.

From the definition of topological weak mixing we see that $\mathcal{S}_1$ is precisely the set of topologically weakly mixing homeomorphisms in $\mathcal{P}$. Also, clearly $\mathcal{S}_1$ is open in $\mathcal{P}$. Thus it remains to show that each $S_{\overline{U_i},\, \overline{U_j},\, \overline{U_l},\, \overline{U_m}}$ is dense. As in the previous theorem, everything reduces to proving a key lemma.

Lemma 4.2. *Let $\epsilon > 0$ and $\overline{U_1}, \overline{U_2}, \overline{U_3}, \overline{U_4}$ be open sets in $\mathbb{K}^2$. Then there exists $G \in \mathcal{H}(\mathbb{K}^2)$ such that the following two properties hold:*

1. $d_u(G^{-1} \circ \overline{\sigma} \circ G, \overline{\sigma}) < \epsilon$

2. $G^{-1} \circ \overline{\sigma} \circ G \in S_{\overline{U_1},\, \overline{U_2},\, \overline{U_3},\, \overline{U_4}}$ *(ie.* $\overline{\sigma} \in S_{G\overline{U_1},\, G\overline{U_2},\, G\overline{U_3},\, G\overline{U_4}}$*).*

Proof. First you apply Lemma 4.1 to obtain $H \in \mathcal{H}(\mathbb{T}^2)$ that is H-equivariant and satisfies

- $d_u(H^{-1} \circ \sigma \circ H, \sigma) < \epsilon$

- $H^{-1} \circ \sigma \circ H \in R_{U_1,\, U_2,\, U_3,\, U_4}$ (ie. $\sigma \in R_{HU_1,\, HU_2,\, HU_3,\, HU_4}$).

Then $\overline{H}$ is the induced map of H on $\mathbb{K}^2$. Take $G = \overline{H}$.

□

4.2 Uniform Rigidity

In this subsection we will show that uniform rigidity is generic in the set $\mathcal{O}$ and in the set $\mathcal{P}$. Similar to a theorem found in [3] we have

Theorem 4.2. *There exists a dense G_δ subset $\mathcal{R}_2$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}_2$, $(\mathbb{T}^2, T)$ is uniformly rigid.*

A simply corollary is the following:

Corollary 4.2. *There exists a dense G_δ subset $\mathcal{S}_2$ of $\mathcal{P}$ such that for every $T \in \mathcal{S}_2$, $(\mathbb{K}^2, T)$ is uniformly rigid.*

4.3 Main Result

We now put the previous two theorems together to prove Theorem 1.2.

Proof of Theorem 1.2. Let

$$\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2.$$

□

5 More Questions on Uniform Rigidity

Thus far we have discussed a generic result for topologically weakly mixing homeomorphisms of the Klein bottle that are uniformly rigid. We are also interested in exploring a similar generic result for weakly mixing homeomorphisms of the Klein bottle.

Question 5.1. *Is there a large family of weakly mixing homeomorphisms of the Klein bottle that are uniformly rigid?*

If so, then this would imply our result. We believe the best way to approach this problem is again by viewing the Klein bottle as the quotient of $\mathbb{T}^2$ by a group action and then producing weakly mixing homeomorphisms of $\mathbb{T}^2$ that are uniformly rigid and equivariant with respect to the group action.

Another question is the following:

Question 5.2. *Does every connected compact metric space admit a large family of (topologically) weakly mixing homeomorphisms that are uniformly rigid?*

The answer to this question would shed some light on how the topology of a space affects the dynamical properties defined there.

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