UNIFORM RIGIDITY SEQUENCES FOR TOPOLOGICALLY WEAKLY MIXING HOMEOMORPHISMS

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Abstract. In 1989 Glasner, Maon, and Weiss showed that there exists a large family of topologically weakly mixing homeomorphisms of the two-torus that are uniformly rigid. In this paper we use their category argument to study uniform rigidity sequences for topologically weakly mixing homeomorphisms of the two-torus. We show that if an increasing sequence of odd natural numbers grows fast enough, then it can be realized as the uniform rigidity sequence for a topologically weakly mixing homeomorphism of the two-torus.

1. Introduction

Uniform rigidity was introduced in 1989 by Glasner and Maon in their paper entitled "Rigidity in Topological Dynamics" [6] and is the topological analogue of classical rigidity in the ergodic theory framework. For more information about classical rigidity see [2] and [3]. In [7] Glasner and Weiss show that there is a large family of topologically weakly mixing homeomorphisms of the two-torus and later in [6] Glasner and Maon couple this with another result to show that there is a large family of topologically weakly mixing homeomorphisms of the two-torus that are uniformly rigid.

The idea of constructing maps with varied behavior by conjugating rotations is due to Anosov and Katok in their seminal paper [1]. This idea of conjugating a rotation is exploited in [7] to produce a large family of topologically weakly mixing homeomorphisms of the two-torus.

Let α be an irrational number between 0 and 1 and σ be a homeomorphism of the two-torus defined as irrational rotation by α in the first coordinate and the identity in the second coordinate. Let $\mathcal{O}$ be the closure of the set of conjugations of σ by homeomorphisms of the two-torus (this closure is taken with respect to the topology of uniform convergence of homeomorphisms and their inverses). The result of Glasner and Weiss in [7] discussed above can be stated precisely as:

Theorem 1.1. *There exists a dense G_δ subset $\mathcal{R}$ of $\mathcal{O}$ such that every $T \in \mathcal{R}$ is topologically weakly mixing and uniformly rigid.*

We will use the above category argument by Glasner and Weiss to obtain information about the structure of uniform rigidity sequences for topologically weakly mixing homeomorphisms. The main result of this paper is the following theorem.

Key words and phrases. topologically weakly mixing; uniformly rigid; rigidity sequence.

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Theorem 1.2. *Suppose (n_m) is an increasing sequence of odd natural numbers and $\psi(n_m) = n_m^{m(4n_m^2+2)}$. If (n_m) satisfies*

$$\frac{n_{m+1}}{n_m} \geq \psi(n_m)$$

then there exists a topologically weakly mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to (n_m).

Previously the author studied uniform rigidity sequences for weakly mixing homeomorphisms of the two-torus equipped with Lebesgue measure [10]. The required growth rate is faster and a weakly mixing homeomorphism is produced, as opposed to a topologically weakly mixing homeomorphism. The main result is

Theorem 1.3. *Let $\psi(x) = x^{x^3}$. If (n_m) is an increasing sequence of natural numbers satisfying*

$$\frac{n_{m+1}}{n_m} \geq \psi(n_m)$$

there exists a Lebesgue measure preserving homeomorphism of $\mathbb{T}^2$ that is weakly mixing and uniformly rigid with respect to (n_m).

Before we proceed with the proof of Theorem 1.2, it is necessary to recall some standard definitions from topological dynamics (see [5], [9]). Let X be a compact metric space with metric d and $T : X \to X$ a homeomorphism of X. Let $\mathcal{H}(X)$ be the set of homeomorphisms of X.

Definition 1.4. *The homeomorphism T is topologically weakly mixing if for any open subsets U_1, U_2, U_3, U_4 of X there exists $t \in \mathbb{Z}$ such that*

$$T^t(U_1 \times U_2) \cap (U_3 \times U_4) \neq \emptyset.$$

Note that in the above definition, $T^t(U_1 \times U_2)$ is shorthand for $T^t(U_1) \times T^t(U_2)$.

Define the uniform distance between two homeomorphisms S, T by

$$d_u(S,T) = \sup_{x \in X} d(S(x), T(x)) + \sup_{x \in X} d(S^{-1}(x), T^{-1}(x)).$$

With this metric the space $\mathcal{H}(X)$ is a complete metric space and the topology induced by d_u is the topology of uniform convergence. To simplify notation in the proof of Theorem 1.2 define

$$\overline{d}(S,T) = \sup_{x \in X} d(S(x), T(x)).$$

Notice that even though d_u is not right-invariant, $\overline{d}$ is right-invariant. This fact will be exploited throughout the paper.

Definition 1.5. *The homeomorphism T is uniformly rigid if there exists an increasing sequence of natural numbers (n_m) such that*

$$d_u(T^{n_m}, Id) \to 0$$

as $m \to \infty$.

2. Uniform Rigidity Sequences

In [7] Glasner and Weiss produce a large family of homeomorphisms of the two-torus that are topologically weakly mixing. We will use the inherent structure of their category argument to determine a sufficient growth rate for a sequence of natural numbers that guarantees the existence of a topologically weakly mixing homeomorphism of the two-torus that is uniformly rigid with respect to the given sequence.

From now on we will be working on the two torus $\mathbb{T}^2$. We will view $\mathbb{T}^2$ as $[0,1)^2$ where the coordinates are taken modulo 1. We will be using additive notation and $|\cdot|$ will denote the distance to the nearest integer or absolute value (the distinction should be clear from context).

The first step in our construction is to choose an irrational rotation that we will then conjugate. In [4] Eggleston shows that if an increasing sequence of natural numbers (n_m) is such that $\lim_{m\to\infty} \frac{n_{m+1}}{n_m} = \infty$ then $\lim_{m\to\infty} |n_m x| = 0$ holds for an uncountable set of x values. In the following lemma we use a similar argument.

Lemma 2.1. *Suppose (n_m) is an increasing sequence of natural numbers and let $\psi(n_m) = n_m^{m(4n_m^2+2)}$. If (n_m) satisfies $\frac{n_{m+1}}{n_m} \geq \psi(n_m)$ and (h'_m) is an increasing sequence of natural numbers satisfying $\frac{1/h'_m}{n_m/n_{m+1}} \to \infty$ as $m \to \infty$ where $h'_m > n_m^2$ then there exists α such that*

$$\frac{1}{h'_m} < |n_m \alpha| < \frac{1}{2(n_m)^2}.$$

Proof. Our goal is to build a Cantor set using the n_m-th roots of unity. From this Cantor set we will be able to select α irrational such that the desired bounds hold.

Let $h_m = 2n_m^2$. In this case $\frac{1/h_m}{n_m/n_{m+1}} \to \infty$ as $m \to \infty$. Also, recall from our assumptions that the sequence (h'_m) satisfied similar properties. Let M be large enough so that for all $m \geq M$ we have $n_{m+1} \geq 10 n_m h_m$ and $n_{m+1} \geq 10 n_m h'_m$. Note that $M = 2$ is sufficient for this purpose.

Now we will build our Cantor set inductively. Suppose $m \geq M$. As part of the construction put two intervals close to "some" of the n_m-th roots of unity (determined as part of the induction) such that any point in either of the intervals is at most $\frac{1}{n_m h_m}$ away from the n_m-th root of unity and at least $\frac{1}{2n_m h'_m}$ away. In this stage of the construction note that each n_m-th root of unity that appears above has two symmetric intervals close to it, one on either side, each of length at least $\frac{1}{2n_m h'_m}$. Call the union of this collection of intervals C_m.

Since $\frac{1}{n_{m+1}}$ is much smaller than $\frac{1}{2n_m h'_m}$, there are many points of the form $\frac{j}{n_{m+1}}$ in each symmetric interval around the above mentioned n_m-th roots of unity. Now select in C_m pairs of symmetric intervals, each of size at least $\frac{1}{2n_{m+1}h'_{m+1}}$, close to each of the n_{m+1}-th roots of unity inside C_m in the same way as above. Call the union of this collection of intervals C_{m+1}.

Continue on in this manner and let the Cantor set C be defined as

$$C = \bigcap_{m=M}^{\infty} C_m.$$

For each point $x \in C$ we have that $n_m x$ is at most $\frac{1}{h_m}$ away from the closest integer and at least $\frac{1}{2h'_m}$ away from the closest integer. That is,

$$\frac{1}{2h'_m} < |n_m x| < \frac{1}{h_m}.$$

Hence, if $x \in C$ then $|n_m x| \to 0$ as $m \to \infty$.

Note that C is uncountable. Thus there exists $\alpha \in C$ that is irrational. Hence, α has the desired properties.

□

2.1. **Proof of Theorem 1.2.** Let (n_m) be a sequence of odd natural numbers satisfying

$$n_{m+1} \geq \psi(n_m) n_m$$

where $\psi(n_m) = n_m^{m(4n_m^2+2)}$. Let (h'_m) be a sequence that satisfies the conditions of Lemma 2.1 (this sequence will be easier to point out at each stage of our construction). From Lemma 2.1 we obtain an irrational α such that

$$\frac{1}{2h'_m} < |n_m \alpha| < \frac{1}{2(n_m)^2}.$$

We will need both of these bounds later in the proof. Let $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $\sigma(x, y) = (x + \alpha, y)$. By the nature of our choice of α, (n_m) is a uniform rigidity sequence for σ.

Define the set $O(\sigma)$ as

$$O(\sigma) = \left\{ G^{-1} \circ \sigma \circ G \ : \ G \in \mathcal{H}(\mathbb{T}^2) \right\}.$$

This set will be considered as a subset of all homeomorphisms of $\mathbb{T}^2$ with the topology of uniform convergence of homeomorphisms and their inverses. Let $\mathcal{O} = \overline{O(\sigma)}$ with the closure taken in the above topology.

Before we proceed, we need to define the set of topologically weakly mixing homeomorphisms of $\mathcal{O}$ as a dense G_δ set. Consider the countable collection of open dyadic cubes in $\mathbb{T}^2$ where a dyadic cube of order i has the form $\left(\frac{\ell}{2^i}, \frac{\ell+1}{2^i}\right) \times \left(\frac{m}{2^i}, \frac{m+1}{2^i}\right)$ where $\ell, m \in \{0, 1, \ldots, 2^i - 1\}$. Now, select open dyadic cubes $U_1^1, U_2^1, U_3^1, U_4^1$ such that each U_j^1 has order 1. For the second step select open dyadic cubes $U_1^2, U_2^2, U_3^2, U_4^2$ such that each cube still has order 1 and $U_1^1 \times U_2^1 \times U_3^1 \times U_4^1 \neq U_1^2 \times U_2^2 \times U_3^2 \times U_4^2$ as a subset of $\mathbb{T}^8$. We continue in this manner until we have exhausted all selections of four open dyadic cubes of order 1 and then proceed to dyadic cubes of order 2. In this way define $U_1^i, U_2^i, U_3^i, U_4^i$ for $i \geq 1$.

Define the set R_i as

$$R_i = \left\{ T \in \mathcal{O} \ : \ \textit{there exists an integer } t \textit{ with } T^t(U_1^i \times U_2^i) \cap (U_3^i \times U_4^i) \neq \emptyset \right\}.$$

Note that we are using shorthand notation when we write $T^t(U_1^i \times U_2^i)$. It is clear that $\mathcal{R} = \bigcap_i^\infty R_i$ is the set of topologically weakly mixing homeomorphisms of $\mathcal{O}$. Recall that in [7] Glasner and Weiss show that this set is a dense G_δ subset of $\mathcal{O}$.

We are going to show that successive conjugations of σ converge to a topologically weakly mixing homeomorphism that is uniformly rigid with respect to (n_m). We will form a nested sequence of closed balls B_i such that each $B_i \subseteq R_i$. Then, $\bigcap_{i=1}^{\infty} B_i$ will contain a homeomorphism T_0 that is topologically weakly mixing. The center of each B_i will be a conjugation of σ and will be chosen carefully so that in the end, T_0 will be the uniform limit of these conjugations and (n_m) will be a uniform rigidity sequence for T_0. We will use Lemma 2.1 to help us form this nested sequence of closed balls. This will be an inductive construction.

To begin, let $0 < \epsilon_1 < 1$. Let $U_j^1 = \left(\frac{a_j}{2}, \frac{a_j+1}{2}\right) \times \left(\frac{b_j}{2}, \frac{b_j+1}{2}\right)$, where $a_j, b_j \in \{0,1\}$ for $j = 1,2,3,4$. Notice that if $G \in \mathcal{H}(\mathbb{T}^2)$ then

$$GR_iG^{-1} = \left\{T \in \mathcal{O} \ : \ \textit{there exists an integer } t \textit{ with } T^t(GU_1^i \times GU_2^i) \cap (GU_3^i \times GU_4^i) \neq \emptyset\right\}$$

where GU_j^i should be interpreted as $G\left(U_j^i\right)$. Thus for notational purposes, if $G \in \mathcal{H}(\mathbb{T}^2)$ define

$$R_{G\circ i} = \left\{T \in \mathcal{O} \ : \ \textit{there exists an integer } t \textit{ with } T^t(GU_1^i \times GU_2^i) \cap (GU_3^i \times GU_4^i) \neq \emptyset\right\}.$$

Then, if $G \in \mathcal{H}(\mathbb{T}^2)$ we have $GR_iG^{-1} = R_{G\circ i}$. The first step is to find $G_1 \in \mathcal{H}(\mathbb{T}^2)$ such that

(1) $d_u(G_1^{-1} \circ \sigma \circ G_1, \sigma) < \epsilon_1$

(2) $G_1^{-1} \circ \sigma \circ G_1 \in R_1$.

The homeomorphism G_1 will have a similar form as the homeomorphism G in the generic argument in [7]. However, this construction is more technical because we need explicit constants in order to use the given growth rate to form our first closed ball B_1.

Let y_1 be a point in $\left(\frac{32b_1+9}{64}, \frac{32b_1+11}{64}\right)$ and choose β_1 irrational such that $y_3 := y_1 + \beta_1 \in \left(\frac{32b_3+13}{64}, \frac{32b_3+15}{64}\right)$. Define $h_1 : \mathbb{T} \to \mathbb{T}$ by $h_1(y) = y + \beta_1$.

Similarly, let y_2 be a point in $\left(\frac{32b_2+17}{64}, \frac{32b_2+19}{64}\right)$ and choose β_2 irrational such that $y_4 := y_2 + \beta_2 \in \left(\frac{32b_4+21}{64}, \frac{32b_4+23}{64}\right)$. Define $h_2 : \mathbb{T} \to \mathbb{T}$ by $h_2(y) = y + \beta_2$. Without loss of generality, assume $\beta_1 > \beta_2$.

Now choose $x_1 \in \left(\frac{32a_1+9}{64}, \frac{32a_1+11}{64}\right)$, $x_2 \in \left(\frac{32a_2+17}{64}, \frac{32a_2+19}{64}\right)$, $x_3 \in \left(\frac{32a_3+13}{64}, \frac{32a_3+15}{64}\right)$, and $x_4 \in \left(\frac{32a_4+21}{64}, \frac{32a_4+23}{64}\right)$.

We are now ready to start building our desired function $G_1 \in \mathcal{H}(\mathbb{T}^2)$. Let $x \to g_x^1$ be a continuous function from $[0,1)$ to $\mathcal{H}(\mathbb{T})$ such that $g_0^1, g_{\frac{3}{4}}^1, g_1^1 = Id$, $g_{\frac{1}{4}}^1 = h_1$, and $g_{\frac{1}{2}}^1 = h_2$ with linear interpolation in between. Thus

$$\begin{aligned}
&g_x^1(y) = 4x\beta_1 + y \ ; \ 0 \leq x \leq \frac{1}{4}\\
&g_x^1(y) = \beta_1(2-4x) + \beta_2(4x-1) + y \ ; \ \frac{1}{4} \leq x \leq \frac{1}{2}\\
&g_x^1(y) = \beta_2(3-4x) + y \ ; \ \frac{1}{2} \leq x \leq \frac{3}{4}\\
&g_x^1(y) = y \ ; \ \frac{3}{4} \leq x \leq 1.
\end{aligned}$$

The modulus of continuity of g^1 is $\omega_{g^1}(\delta) = \sup_{|x-x'|<\delta} d_u(g_x^1, g_{x'}^1) \leq 8\beta_1\delta$.

By the choice of the y_j's above we know that they are all distinct. Thus we may place non-overlapping tent maps around each y_j. To that end, let p_1 be a tent map such that

$p_1(y_1) = 1$ and $p_1\left(\left(\frac{8b_1+2}{16}, \frac{8b_1+3}{16}\right)^c\right) = 0$. Similarly, define p_2, p_3, p_4 by

$$p_2(y_2) = 1\ ;\ p_2\left(\left(\frac{8b_2+4}{16}, \frac{8b_2+5}{16}\right)^c\right) = 0$$
$$p_3(y_3) = 1\ ;\ p_3\left(\left(\frac{8b_3+3}{16}, \frac{8b_3+4}{16}\right)^c\right) = 0$$
$$p_4(y_4) = 1\ ;\ p_4\left(\left(\frac{8b_4+5}{16}, \frac{8b_4+6}{16}\right)^c\right) = 0.$$

In this case the modulus of continuity of each p_j is $\omega_{p_j}(\delta) \le 32 \cdot 2^1\delta$. Let $M_1 = 32 \cdot 2^1$ and $C_1 = 26M_1$.

Let $\eta_1 = \frac{\epsilon_1}{16M_1}$ and $\delta_1 = \frac{\eta_1}{16}$. Then, if $|x - x'| < \delta_1$ we have $d_u(g^1_x, g^1_{x'}) < \frac{\eta_1}{2}$. Since (n_m) is an increasing sequence, there exists M such that $n_M > \max(\frac{1}{\delta_1}, 8192 \cdot 32 \cdot 2^{15}C_1)$. WLOG, suppose that $n_1 > \max(\frac{1}{\delta_1}, 8192 \cdot 32 \cdot 2^{15}C_1)$.

Define $c_j \in [0, 1)$ such that

$$x_1 + c_1 = \frac{1}{4n_1},\ \ x_2 + c_2 = \frac{3}{4n_1},\ \ x_3 + c_3 = \frac{1}{4n_1} + \frac{1}{2},\ \ x_4 + c_4 = \frac{3}{4n_1} + \frac{1}{2}$$

all taken modulo one. Let f_1 be defined by

$$f_1(y) = \sum_{j=1}^{4} c_j p_j(y).$$

Then, if $|y - y'| < \eta_1$ we have

$$|f_1(y) - f_1(y')| \le \sum_{j=1}^{4} c_j\,|p_j(y) - p_j(y')| < 4M_1\eta_1 = \frac{\epsilon_1}{4}.$$

Now we are ready to define G_1. Let $G_1 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$G_1(x, y) = \left(x + f_1(y), g^1_{n_1(x+f_1(y))}(y)\right).$$

Then,

$$G_1^{-1}(x, y) = \left(x - f_1((g^1_{n_1x})^{-1}(y)), (g^1_{n_1x})^{-1}(y)\right)$$

and

$$G_1^{-1} \circ \sigma \circ G_1(x, y) = (x + \alpha + f_1(y) - f_1(y_*), y_*)$$

where $y_* = (g^1_{n_1(x+\alpha+f_1(y))})^{-1} g^1_{n_1(x+f_1(y))}(y)$. The modulus of continuity of G_1 is given by $\omega_{G_1}(\delta) \le C_1 n_1 \delta$. It should also be noted that the modulus of continuity of G_1^{-1} is bounded by the same number.

We will first check that condition (1) is satisfied, that is $d_u(G_1^{-1} \circ \sigma \circ G_1, \sigma) < \epsilon_1$. To begin, notice that

$$G_1^{-1} \circ \sigma \circ G_1\,(x, y) - \sigma\,(x, y) = (f_1(y) - f_1(y_*), y - y_*)\,.$$

Since $|n_1(x + \alpha + f_1(y)) - n_1(x + f_1(y))| = |n_1\alpha| < \frac{1}{2(n_1)^2} < \frac{1}{n_1} < \delta_1$, we must have

$$\overline{d}\left(\left(g^1_{n_1(x+\alpha+f_1(y))}\right)^{-1} g^1_{n_1(x+f_1(y))}, Id\right) < \frac{\eta_1}{2}.$$

This implies that $|y-y_*|<\eta_1$ and therefore, $|f(y)-f(y_*)|<\frac{\epsilon_1}{4}$. Thus $\overline{d}\left(G_1^{-1}\circ\sigma\circ G_1,\sigma\right)<\frac{\epsilon_1}{2}$. In a similar fashion, $\overline{d}\left(G_1^{-1}\circ\sigma^{-1}\circ G_1,\sigma^{-1}\right)<\frac{\epsilon_1}{2}$. Therefore, $d_u(G_1^{-1}\circ\sigma\circ G_1,\sigma)<\epsilon_1$ and (1) is verified.

At this point our first conjugation of σ remains close to σ. Now we need to check that the conjugation of σ we chose belongs to R_1. Our goal is to find $t_1\in\mathbb{N}$ such that

$$G_1^{-1}\circ\sigma^{t_1}\circ G_1\left(\left(\frac{a_1}{2},\frac{a_1+1}{2}\right)\times\left(\frac{b_1}{2},\frac{b_1+1}{2}\right)\right)\cap\left(\left(\frac{a_3}{2},\frac{a_3+1}{2}\right)\times\left(\frac{b_3}{2},\frac{b_3+1}{2}\right)\right)\neq\emptyset$$
$$G_1^{-1}\circ\sigma^{t_1}\circ G_1\left(\left(\frac{a_2}{2},\frac{a_2+1}{2}\right)\times\left(\frac{b_2}{2},\frac{b_2+1}{2}\right)\right)\cap\left(\left(\frac{a_4}{2},\frac{a_4+1}{2}\right)\times\left(\frac{b_4}{2},\frac{b_4+1}{2}\right)\right)\neq\emptyset.$$

Recall that in the generic argument in [7] the existence of such a t_1 is shown. For this proof we need however to explicitly calculate t_1. This is where the upper and lower bounds on $|n_m\alpha|$ come into play.

We have chosen α so that
$$\frac{512n_1-1}{1024n_1^3}<|n_1\alpha|<\frac{1}{2n_1^2}.$$

Note that $h_1'=\frac{1024n_1^3}{512n_1-1}$. Let $t_1=n_1^2$. In this case, $\frac{512n_1-1}{1024n_1}<|t_1n_1\alpha|<\frac{1}{2}$ and $\left|t_1n_1\alpha-\frac{1}{2}\right|<\frac{1}{1024n_1}$. Notice that

$$G_1\left(x_1,y_1\right)=\left(x_1+c_1,g^1_{n_1(x_1+c_1)}(y_1)\right)=\left(\frac{1}{4n_1},y_3\right)$$

and

$$G_1^{-1}\circ\sigma^{t_1}\circ G_1(x_1,y_1)=\left(\frac{1}{4n_1}+t_1\alpha-f_1\left(\left(g^1_{n_1\left(\frac{1}{4n_1}+t_1\alpha\right)}\right)^{-1}(y_3)\right),\left(g^1_{n_1\left(\frac{1}{4n_1}+t_1\alpha\right)}\right)^{-1}(y_3)\right).$$

Therefore,

$$\begin{aligned}&G_1^{-1}\circ\sigma^{t_1}\circ G_1(x_1,y_1)-(x_3,y_3)\\&\qquad=\left(\frac{1}{4n_1}+t_1\alpha-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)-x_3,\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)-y_3\right)\end{aligned}$$

Consider the first coordinate above:

$$\begin{aligned}&\frac{1}{4n_1}+t_1\alpha-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)-x_3\\&\quad=\left(\frac{1}{4n_1}+\frac{1}{2}\right)+\left(t_1\alpha-\frac{1}{2}\right)-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)-x_3\\&\quad=c_3+\left(t_1\alpha-\frac{1}{2}\right)-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)\\&\quad=\left(t_1\alpha-\frac{1}{2}\right)+f_1\left(\left(g^1_{n_1\left(\frac{1}{4n_1}+\frac{1}{2}\right)}\right)^{-1}(y_3)\right)-f_1\left(\left(g^1_{\frac{1}{4}+t_1n_1\alpha}\right)^{-1}(y_3)\right)\end{aligned}$$

Since $\left|t_1 n_1 \alpha - \frac{1}{2}\right| < \frac{1}{1024 n_1}$, we have

$$\left|\left(g^1_{n_1\left(\frac{1}{4n_1}+\frac{1}{2}\right)}\right)^{-1}(y_3) - \left(g^1_{\frac{1}{4}+t_1 n_1 \alpha}\right)^{-1}(y_3)\right| < 4\beta_1\left(\frac{1}{1024 n_1}\right) < \frac{4}{1024 n_1},$$

which implies

$$\left|f_1\left(\left(g^1_{n_1\left(\frac{1}{4n_1}+\frac{1}{2}\right)}\right)^{-1}(y_3)\right) - f_1\left(\left(g^1_{\frac{1}{4}+t_1 n_1 \alpha}\right)^{-1}(y_3)\right)\right| < 4M_1\left(\frac{4}{1024 n_1}\right) = \frac{M_1}{64 n_1}.$$

Since n_1 is odd, we have

$$\left|\frac{1}{4n_1} + t_1\alpha - f_1\left(\left(g^1_{\frac{1}{4}+t_1 n_1 \alpha}\right)^{-1}(y_3)\right) - x_3\right| < \frac{1}{1024} + \frac{M_1}{64 n_1} < \frac{1}{64} + \frac{1}{64} = \frac{1}{32}.$$

Similarly, for the second coordinate we obtain

$$\left|\left(g^1_{\frac{1}{4}+t_1 n_1 \alpha}\right)^{-1}(y_3) - y_3\right| < \frac{4}{1024 n_1} < \frac{1}{32}.$$

Thus, $G_1^{-1} \circ \sigma^{t_1} \circ G_1\left(\left(\frac{a_1}{2}, \frac{a_1+1}{2}\right) \times \left(\frac{b_1}{2}, \frac{b_1+1}{2}\right)\right) \cap \left(\left(\frac{a_3}{2}, \frac{a_3+1}{2}\right) \times \left(\frac{b_3}{2}, \frac{b_3+1}{2}\right)\right) \neq \emptyset$. In a similar manner, we obtain $G_1^{-1} \circ \sigma^{t_1} \circ G_1\left(\left(\frac{a_2}{2}, \frac{a_2+1}{2}\right) \times \left(\frac{b_2}{2}, \frac{b_2+1}{2}\right)\right) \cap \left(\left(\frac{a_4}{2}, \frac{a_4+1}{2}\right) \times \left(\frac{b_4}{2}, \frac{b_4+1}{2}\right)\right) \neq \emptyset$. Therefore (2) is satisfied.

Now that we have $G_1^{-1} \circ \sigma \circ G_1 \in R_1$, we proceed with finding a closed ball, which we will call B_1, centered at $G_1^{-1} \circ \sigma \circ G_1$ such that $B_1 \subseteq R_1$. We need to explicitly calculate the radius of B_1 to ensure that $B_1 \subseteq R_1$. Let

$$\kappa_1 = \frac{1}{16 \cdot 2^1 (C_1 n_1)^{2n_1^2 - 1}}$$

and

$$B_1 = \left\{T \in \mathcal{O} : d_u(G_1^{-1} \circ \sigma \circ G_1, T) \leq \kappa_1\right\}.$$

Notice that for any $n \in \mathbb{N}$ and $T \in \mathcal{O}$, we have $d_u(T^n, G_1^{-1} \circ \sigma^n \circ G_1) = \overline{d}(T^n, G_1^{-1} \circ \sigma^n \circ G_1) + \overline{d}(T^{-n}, G_1^{-1} \circ \sigma^{-n} \circ G_1)$. Consider the following:

$$\begin{aligned}
\overline{d}(T^n, G_1^{-1} \circ \sigma^n \circ G_1) &= \overline{d}((G_1^{-1} \circ \sigma \circ G_1)(G_1^{-1} \circ \sigma^{n-1} \circ G_1), T(T^{n-1})) \\
&\leq \overline{d}((G_1^{-1} \circ \sigma \circ G_1)(G_1^{-1} \circ \sigma^{n-1} \circ G_1), (G_1^{-1} \circ \sigma \circ G_1)(T^{n-1})) \\
&\quad + \overline{d}((G_1^{-1} \circ \sigma \circ G_1)(T^{n-1}), T(T^{n-1})) \\
&\leq \omega_{G_1^{-1}\circ\sigma\circ G_1}(\overline{d}(G_1^{-1} \circ \sigma^{n-1} \circ G_1, T^{n-1})) + \overline{d}(G_1^{-1} \circ \sigma \circ G_1, T) \\
&\leq \sum_{i=0}^{n-1} \omega^i_{G_1^{-1}\circ\sigma\circ G_1}(\overline{d}(G_1^{-1} \circ \sigma \circ G_1, T)) \\
&\leq \overline{d}(G_1^{-1} \circ \sigma \circ G_1, T) \sum_{i=0}^{n-1} [(C_1 n_1)^2]^i \\
&= \overline{d}(G_1^{-1} \circ \sigma \circ G_1, T) \frac{(C_1 n_1)^{2n} - 1}{(C_1 n_1)^2 - 1} \\
&\leq \overline{d}(G_1^{-1} \circ \sigma \circ G_1, T)(C_1 n_1)^{2n-1}
\end{aligned}$$

where $\omega^0 = Id$. A similar calculation can be carried out to yield $\overline{d}(T^{-n}, G_1^{-1} \circ \sigma^{-n} \circ G_1) \leq \overline{d}(G_1^{-1} \circ \sigma^{-1} \circ G_1, T^{-1})(C_1 n_1)^{2n-1}$. Thus,

$$d_u(T^n, G_1^{-1} \circ \sigma^n \circ G_1) \leq d_u(G_1^{-1} \circ \sigma \circ G_1, T)(C_1 n_1)^{2n-1}.$$

We will show that $B_1 \subseteq R_1$. Let $T \in B_1$. In this case,

$$\begin{aligned} d\left(T^{t_1}\left(x_1, y_1\right), \left(x_3, y_3\right)\right) &\leq d\left(T^{t_1}\left(x_1, y_1\right), G_1^{-1} \circ \sigma^{t_1} \circ G_1\left(x_1, y_1\right)\right) \\ &+ d\left(G_1^{-1} \circ \sigma^{t_1} \circ G_1\left(x_1, y_1\right), \left(x_3, y_3\right)\right) \\ &\leq d_u(G_1^{-1} \circ \sigma \circ G_1, T)(C_1 n_1)^{2t_1 - 1} + \frac{1}{16} \\ &\leq \kappa_1\left(C_1 n_1\right)^{2t_1 - 1} + \frac{1}{16} \\ &= \frac{1}{32} + \frac{1}{16} \\ &< \frac{1}{8}. \end{aligned}$$

Thus $T^{t_1}\left(\left(\frac{a_1}{2}, \frac{a_1+1}{2}\right) \times \left(\frac{b_1}{2}, \frac{b_1+1}{2}\right)\right) \cap \left(\left(\frac{a_3}{2}, \frac{a_3+1}{2}\right) \times \left(\frac{b_3}{2}, \frac{b_3+1}{2}\right)\right) \neq \emptyset$. In a similar manner, $T^{t_1}\left(\left(\frac{a_2}{2}, \frac{a_2+1}{2}\right) \times \left(\frac{b_2}{2}, \frac{b_2+1}{2}\right)\right) \cap \left(\left(\frac{a_4}{2}, \frac{a_4+1}{2}\right) \times \left(\frac{b_4}{2}, \frac{b_4+1}{2}\right)\right) \neq \emptyset$. Hence, we have the desired result i.e. $B_1 \subseteq R_1$.

Thus far we have constructed the closed ball B_1 centered at $G_1^{-1} \circ \sigma \circ G_1$ such that $B_1 \subseteq R_1$. The next step in our inductive procedure is to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in R_2 \cap B_1$ and then construct the closed ball B_2 centered at $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1$ such that $B_2 \subseteq R_2 \cap B_1$. Notice that in the second step of the induction, the dyadic cubes still have order 1. To that end, let $\epsilon_2 = \frac{\kappa_1}{2C_1 n_1} < \epsilon_1$. Now similar to before, we want to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that

(1) $d_u(G_2^{-1} \circ \sigma \circ G_2, \sigma) < \epsilon_2$

(2) $G_2^{-1} \circ \sigma \circ G_2 \in R_{G_1 \circ 2}$.

Let $U_j^2 = \left(\frac{a_j}{2}, \frac{a_j+1}{2}\right) \times \left(\frac{b_j}{2}, \frac{b_j+1}{2}\right)$, where $a_j, b_j \in \{0, 1\}$ for $j = 1, 2, 3, 4$. Let $U_j^{2'}$ be an open dyadic sub-cube of U_j^2 such that any point in $U_j^{2'}$ is at least $\frac{1}{8}$ from the boundary of U_j^2. Since we need to construct $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_2^{-1} \circ \sigma \circ G_2 \in R_{G_1 \circ 2}$, we will consider dyadic cubes inside each $G_1 U_j^{2'}$ and repeat a similar argument.

Observe that G_1 is a bi-Lipschitz map such that

$$\frac{1}{C_1 n_1} d\left(\left(x, y\right), \left(x', y'\right)\right) \leq d\left(G_1\left(x, y\right), G_1\left(x', y'\right)\right) \leq C_1 n_1 d\left(\left(x, y\right), \left(x', y'\right)\right).$$

Let k_1 be the smallest integer such that $n_1 \leq 2^{k_1}$. Then each $G_1 U_j^{2'}$ contains a dyadic cube of order $15 + k_1$. To see this, use the bi-Lipschitz property of G_1 to obtain a lower bound on the size ball that each $G_1 U_j^{2'}$ contains and then place a dyadic cube inside the ball. Now let $\left(\frac{c_j}{2^{15+k_1}}, \frac{c_j+1}{2^{15+k_1}}\right) \times \left(\frac{d_j}{2^{15+k_1}}, \frac{d_j+1}{2^{15+k_1}}\right)$, where $c_j, d_j \in \{0, 1, \ldots, 2^{15+k_1} - 1\}$, denote the dyadic cube inside $G_1 U_j^{2'}$ for $j = 1, 2, 3, 4$.

Now we will pick new points x_j, y_j and new functions h_1, h_2 for the second step in the induction. We abuse notation here to avoid excessive use of superscripts. Let y_1 be a point in $\left(\frac{32d_1+9}{2^{20+k_1}}, \frac{32d_1+11}{2^{20+k_1}}\right)$ and choose β_1 irrational such that $y_3 := y_1 + \beta_1 \in \left(\frac{32d_3+13}{2^{20+k_1}}, \frac{32d_3+15}{2^{20+k_1}}\right)$. Define $h_1 : \mathbb{T} \to \mathbb{T}$ by $h_1(y) = y + \beta_1$.

Similarly, let y_2 be a point in $\left(\frac{32d_2+17}{2^{20+k_1}}, \frac{32d_2+19}{2^{20+k_1}}\right)$ and choose β_2 irrational such that $y_4 := y_2 + \beta_2 \in \left(\frac{32d_4+21}{2^{20+k_1}}, \frac{32d_4+23}{2^{20+k_1}}\right)$. Define $h_2 : \mathbb{T} \to \mathbb{T}$ by $h_2(y) = y + \beta_2$. Without loss of generality, assume $\beta_1 > \beta_2$.

Now choose $x_1 \in \left(\frac{32c_1+9}{2^{20+k_1}}, \frac{32c_1+11}{2^{20+k_1}}\right)$, $x_2 \in \left(\frac{32c_2+17}{2^{20+k_1}}, \frac{32c_2+19}{2^{20+k_1}}\right)$, $x_3 \in \left(\frac{32c_3+13}{2^{20+k_1}}, \frac{32c_3+15}{2^{20+k_1}}\right)$, and $x_4 \in \left(\frac{32c_4+21}{2^{20+k_1}}, \frac{32c_4+23}{2^{20+k_1}}\right)$.

We are now ready to start building our desired function $G_2 \in \mathcal{H}(\mathbb{T}^2)$. Let $x \to g_x^2$ be a continuous function from $[0, 1)$ to $\mathcal{H}(\mathbb{T})$ such that $g_0^2, g_{\frac{3}{4}}^2, g_1^2 = Id$, $g_{\frac{1}{4}}^2 = h_1$, and $g_{\frac{1}{2}}^2 = h_2$ with linear interpolation in between. Thus as before, the modulus of continuity of g^2 is $\omega_{g^2}(\delta) = \sup_{|x-x'|<\delta} d_u(g_x^2, g_{x'}^2) \leq 8\beta_1\delta$.

By the choice of the y_j's above we know that they are all distinct. Thus, we may place non-overlapping tent maps p_j around each y_j as before, where the modulus of continuity of each p_j is $\omega_{p_j}(\delta) \leq 32 \cdot 2^{15+k_1}\delta$. Let $M_2 = 32 \cdot 2^{15+k_1}$ and $C_2 = 26M_2$.

Let $\eta_2 = \frac{\epsilon_2}{16M_2}$ and $\delta_2 = \frac{\eta_2}{16}$. Then, if $|x - x'| < \delta_2$ we have $d_u(g_x^2, g_{x'}^2) < \frac{\eta_2}{2}$. If $n_2 \geq 8192 \cdot 32 \cdot 2^{40}C_2$ then we proceed similar to before and define new $c_j \in [0, 1)$ such that

$$x_1 + c_1 = \frac{1}{4n_2}, \quad x_2 + c_2 = \frac{3}{4n_2}, \quad x_3 + c_3 = \frac{1}{4n_2} + \frac{1}{2}, \quad x_4 + c_4 = \frac{3}{4n_2} + \frac{1}{2}$$

all taken modulo one. Let f_2 be defined by

$$f_2(y) = \sum_{j=1}^{4} c_j p_j(y).$$

Then, if $|y - y'| < \eta_2$ we have

$$|f_2(y) - f_2(y')| \leq \sum_{j=1}^{4} c_j \, |p_j(y) - p_j(y')| < 4M_2\eta_2 = \frac{\epsilon_2}{4}.$$

Now we are ready to define G_2. Let $G_2 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$G_2(x, y) = \left(x + f_2(y), g^2_{n_2(x+f_2(y))}(y)\right).$$

Then,

$$G_2^{-1}(x, y) = \left(x - f_2((g^2_{n_2x})^{-1}(y)), (g^2_{n_2x})^{-1}(y)\right)$$

and

$$G_2^{-1} \circ \sigma \circ G_2(x, y) = (x + \alpha + f_2(y) - f_2(y_*), y_*)$$

where $y_* = (g^2_{n_2(x+\alpha+f_2(y))})^{-1} g^2_{n_2(x+f_2(y))}(y)$. The modulus of continuity of G_2 is given by $\omega_{G_2}(\delta) \leq C_2 n_2 \delta$, where $C_2 = 26M_2$. It should also be noted that the modulus of continuity of G_2^{-1} is bounded by the same number.

If it is not the case that $n_2 \geq 8192 \cdot 2^2 C_2^2$, then we use U_j^1 in place of U_j^i and $G_i = Id$ at each stage of the induction until a term of the sequence (n_m) exceeds $8192 \cdot 32 \cdot 2^{40}C_2$.

We now proceed with the induction under the assumption that $n_2 \geq 8192 \cdot 32 \cdot 2^{40} C_2$. Similar to before, to show that $d_u(G_2^{-1} \circ \sigma \circ G_2, \sigma) < \epsilon_2$ we need to check that $n_2 > \frac{1}{\delta_2}$. Observe

$$\begin{aligned}
\frac{1}{\delta_2} &= \frac{16}{\eta_2} \\
&= \frac{256 M_2}{\epsilon_2} \\
&= \frac{512\,(C_1 n_1)\, M_2}{\kappa_1} \\
&= 512\left(16 \cdot 2^1\right) M_2 \,(C_1 n_1)^{2n_1^2} \\
&\leq 8192 C_1\,(32) \cdot 2^{15} n_1^2\,(n_1)^{4n_1^2} \\
&< (n_1)^{4n_1^2+2} \cdot n_1 \\
&\leq \psi\,(n_1)\, n_1.
\end{aligned}$$

Therefore, $d_u(G_2^{-1} \circ \sigma \circ G_2, \sigma) < \epsilon_2$.

Next we need to show that $G_2^{-1} \circ \sigma \circ G_2 \in R_{G_1 \circ 2}$. Our goal is to find $t_2 \in \mathbb{N}$ such that

$$\begin{aligned}
G_2^{-1} \circ \sigma^{t_2} \circ G_2 &\left(\left(\frac{c_1}{2^{15+k_1}}, \frac{c_1+1}{2^{15+k_1}}\right) \times \left(\frac{d_1}{2^{15+k_1}}, \frac{d_1+1}{2^{15+k_1}}\right)\right) \\
&\cap \left(\left(\frac{c_3}{2^{15+k_1}}, \frac{c_3+1}{2^{15+k_1}}\right) \times \left(\frac{d_3}{2^{15+k_1}}, \frac{d_3+1}{2^{15+k_1}}\right)\right) \neq \emptyset
\end{aligned}$$

$$\begin{aligned}
G_2^{-1} \circ \sigma^{t_2} \circ G_2 &\left(\left(\frac{c_2}{2^{15+k_1}}, \frac{c_2+1}{2^{15+k_1}}\right) \times \left(\frac{d_2}{2^{15+k_1}}, \frac{d_2+1}{2^{15+k_1}}\right)\right) \\
&\cap \left(\left(\frac{c_4}{2^{15+k_1}}, \frac{c_4+1}{2^{15+k_1}}\right) \times \left(\frac{d_4}{2^{15+k_1}}, \frac{d_4+1}{2^{15+k_1}}\right)\right) \neq \emptyset.
\end{aligned}$$

We have chosen α so that

$$\frac{\left(16 \cdot 2^{18+k_1}\right) n_2 - 1}{\left(16 \cdot 2^{19+k_1}\right) n_2^3} < |n_2 \alpha| < \frac{1}{2n_2^2}.$$

Note that $h_2' = \frac{\left(16 \cdot 2^{19+k_1}\right) n_2^3}{\left(16 \cdot 2^{18+k_1}\right) n_2 - 1}$. Let $t_2 = n_2^2$. It follows that $\frac{\left(16 \cdot 2^{18+k_1}\right) n_2 - 1}{\left(16 \cdot 2^{19+k_1}\right) n_2} < |t_2 n_2 \alpha| < \frac{1}{2}$ and $\left|t_2 n_2 \alpha - \frac{1}{2}\right| < \frac{1}{\left(16 \cdot 2^{19+k_1}\right) n_2}$. Similar to the earlier calculation, we obtain

$$\begin{aligned}
&G_2^{-1} \circ \sigma^{t_2} \circ G_2(x_1, y_1) - (x_3, y_3) \\
&\qquad = \left(\frac{1}{4n_2} + t_2 \alpha - f_2\left(\left(g^2_{\frac{1}{4}+t_2 n_2 \alpha}\right)^{-1}(y_3)\right) - x_3, \left(g^2_{\frac{1}{4}+t_2 n_2 \alpha}\right)^{-1}(y_3) - y_3\right)
\end{aligned}$$

and

$$\left|\frac{1}{4n_2} + t_2 \alpha - f_2\left(\left(g^2_{\frac{1}{4}+t_2 n_2 \alpha}\right)^{-1}(y_3)\right) - x_3\right| < \frac{1}{16 \cdot 2^{19+k_1}} + \frac{M_2}{2^{19+k_1} n_2} < \frac{1}{2^{18+k_1}}.$$

Similarly, for the second coordinate we obtain

$$\left|\left(g^2_{\frac{1}{4}+t_2 n_2 \alpha}\right)^{-1}(y_3) - y_3\right| < \frac{4}{(16 \cdot 2^{19+k_1})\, n_2} < \frac{1}{2^{18+k_1}}.$$

Thus, $G_2^{-1} \circ \sigma \circ G_2 \in R_{G_1 \circ 2}$.

Recall that our goal for the second step in the inductive procedure is to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in R_2 \cap B_1$. Thus far we have constructed $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in R_2$. We now need to check that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in B_1$. To that end, observe

$$\begin{aligned}
\overline{d}(G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1, G_1^{-1} \circ \sigma \circ G_1) &= \overline{d}(G_1^{-1}(G_2^{-1} \circ \sigma \circ G_2), G_1^{-1}(\sigma)) \\
&\leq C_1 n_1 \left(\frac{\epsilon_2}{2}\right) \\
&= C_1 n_1 \left(\frac{\kappa_1}{4 C_1 n_1}\right) \\
&= \frac{\kappa_1}{4}.
\end{aligned}$$

Similarly, $\overline{d}(G_1^{-1} \circ G_2^{-1} \circ \sigma^{-1} \circ G_2 \circ G_1, G_1^{-1} \circ \sigma^{-1} \circ G_1) \leq \frac{\kappa_1}{4}$ and $d_u(G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1, G_1^{-1} \circ \sigma \circ G_1) \leq \frac{\kappa_1}{2}$, which implies that $G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1 \in B_1 \subseteq R_1$.

Let $\overline{G_2} := G_2 \circ G_1$. With this new notation we have shown that $(\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2} \in R_2 \cap B_1$. Now we need to find a closed ball, call it B_2, centered at $(\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2}$ that is a subset of $R_2 \cap B_1$. Let

$$\kappa_2 = \frac{1}{16 \cdot 2^2 (C_1 C_2 n_1 n_2)^{2n_2^2 - 1}}$$

and

$$B_2 = \left\{T \in \mathcal{O} : d_u((\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2}, T) \leq \kappa_2\right\}.$$

We will first show that $B_2 \subseteq R_2$. Let $\left(x_j', y_j'\right) = G_1^{-1}(x_j, y_j) \in U_j^{2'}$ for $j = 1, 2, 3, 4$. Let $T \in B_2$ and consider

$$\begin{aligned}
d\left(T^{t_2}(x_1', y_1'), (x_3', y_3')\right) &\leq d\left(T^{t_2}(x_1', y_1'), G_1^{-1} \circ G_2^{-1} \circ \sigma^{t_2} \circ G_2 \circ G_1(x_1', y_1')\right) \\
&\quad + d\left(G_1^{-1} \circ G_2^{-1} \circ \sigma^{t_2} \circ G_2 \circ G_1(x_1', y_1'), (x_3', y_3')\right) \\
&\leq d_u(G_1^{-1} \circ G_2^{-1} \circ \sigma \circ G_2 \circ G_1, T)(C_1 C_2 n_1 n_2)^{2t_2 - 1} \\
&\quad + d\left(G_1^{-1} \circ G_2^{-1} \circ \sigma^{t_2} \circ G_2(x_1, y_1), G_1^{-1}(x_3, y_3)\right) \\
&\leq \kappa_2 (C_1 C_2 n_1 n_2)^{2t_2 - 1} + C_1 n_1 \left(\frac{1}{2^{17+k_1}}\right) \\
&\leq \frac{1}{32} + \frac{1}{32} \\
&= \frac{1}{16}.
\end{aligned}$$

Hence, we have the desired result, that is $B_2 \subseteq R_2$.

Next we will show that $B_2 \subseteq B_1$. Let $T \in B_2$ and consider

$$\begin{aligned} d_u(T, G_1^{-1} \circ \sigma \circ G_1) &\leq d_u(T, (\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2}) \\ &\quad + d_u((\overline{G_2})^{-1} \circ \sigma \circ \overline{G_2}, G_1^{-1} \circ \sigma \circ G_1) \\ &\leq \kappa_2 + \frac{\kappa_1}{2} \\ &< \frac{\kappa_1}{2} + \frac{\kappa_1}{2} \\ &= \kappa_1. \end{aligned}$$

Therefore, $B_2 \subseteq R_2 \cap B_1$.

Thus far in our inductive procedure, we have constructed two closed nested balls $B_1 \supseteq B_2$ centered at conjugations of σ such that $B_1 \subseteq R_1$ and $B_2 \subseteq R_2$. The general inductive step can be carried out in the same way.

In the end, this inductive procedure produces a nested sequence of closed balls (B_m) and a sequence $(\overline{G_m})$ of homeomorphisms where each $\overline{G_m} = G_m \circ G_{m-1} \circ \cdots \circ G_1$ and each G_m is of the form

$$G_m(x, y) = (x + f_m(y), g^m_{n_m(x+f_m(y))}(y)).$$

After the m-th stage of the construction has been completed, we have a homeomorphism G_m that satisfies

(1) $d_u(G_m^{-1} \circ \sigma \circ G_m, \sigma) < \epsilon_m$ where $\epsilon_m = \frac{\kappa_{m-1}}{2C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}}$

(2) $G_m^{-1} \circ \sigma \circ G_m \in R_{G_1 \circ \cdots \circ G_{m-1} \circ m}$.

At the end of this stage we also have a closed ball B_m centered at $\left(\overline{G_m}\right)^{-1} \circ \sigma \circ \overline{G_m}$ with radius

$$\kappa_m = \frac{1}{16 \cdot 2^m \left(C_1 \cdots C_m n_1 \cdots n_m\right)^{2n_m^2 - 1}}$$

such that $B_m \subseteq R_m$. Recall that we are working in a complete metric space. Let $T_0 = \bigcap_{m=1}^{\infty} B_m$. Therefore, T_0 is topologically weakly mixing. Also, $\left(\overline{G_m}\right)^{-1} \circ \sigma \circ \overline{G_m}$ converges uniformly to T_0 since $\left(\overline{G_m}\right)^{-1} \circ \sigma \circ \overline{G_m}$ is the center of B_m.

Now that we have T_0 which is topologically weakly mixing, we need to show that it is uniformly rigid with respect to (n_m). To do this, we need to make a preliminary estimate. First notice that

$$G_m^{-1} \circ \sigma^{n_m} \circ G_m(x, y) - \sigma^{n_m}(x, y) = (x + f_m(y) - f_m(y_*), y_*)$$

where $y_* = \left(g^m_{n_m(x+n_m\alpha+f_m(y))}\right)^{-1} g^m_{n_m(x+f_m(y))}(y)$. In either case

$$\left|n_m^2 \alpha\right| < \frac{1}{n_m} < \delta_m$$

and we can conclude that $d_u(G_m^{-1} \circ \sigma^{n_m} \circ G_m, \sigma^{n_m}) < \epsilon_m$. Now observe the following:

$$\bar{d}\left(\left(\overline{G_m}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_m}, \left(\overline{G_{m-1}}\right)^{-1} \circ \sigma^{n_m} \circ \overline{G_{m-1}}\right) =$$

$$
\begin{aligned}
&\overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1}\left(G_m^{-1}\circ\sigma^{n_m}\circ G_m\right),\left(\overline{G_{m-1}}\right)^{-1}\left(\sigma^{n_m}\right)\right)\\
&\le \left(C_1\cdots C_{m-1}n_1\cdots n_{m-1}\right)\left(\frac{\epsilon_m}{2}\right)\\
&=\frac{\kappa_{m-1}}{4}.
\end{aligned}
$$

Hence, $d_u(\left(\overline{G_m}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_m},\left(\overline{G_{m-1}}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_{m-1}})\le\frac{\kappa_{m-1}}{2}$.

The final estimate will show that T_0 is uniformly rigid with respect to (n_m). Indeed

$$
\begin{aligned}
d_u\left(T_0^{n_m},Id\right)\le{}& d_u\left(T_0^{n_m},\left(\overline{G_m}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_m}\right)\\
&+d_u\left(\left(\overline{G_m}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_m},\left(\overline{G_{m-1}}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_{m-1}}\right)\\
&+d_u\left(\left(\overline{G_{m-1}}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_{m-1}},Id\right)\\
&=d_u\left(T_0^{n_m},\left(\overline{G_m}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_m}\right)+d_u\left(\left(\overline{G_m}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_m},\left(\overline{G_{m-1}}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_{m-1}}\right)\\
&+\overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_{m-1}},Id\right)+\overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1}\circ\sigma^{-n_m}\circ\overline{G_{m-1}},Id\right)\\
&=d_u\left(T_0^{n_m},\left(\overline{G_m}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_m}\right)+d_u\left(\left(\overline{G_m}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_m},\left(\overline{G_{m-1}}\right)^{-1}\circ\sigma^{n_m}\circ\overline{G_{m-1}}\right)\\
&+\overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1}(\sigma^{n_m}),\left(\overline{G_{m-1}}\right)^{-1}(Id)\right)+\overline{d}\left(\left(\overline{G_{m-1}}\right)^{-1}(\sigma^{-n_m}),\left(\overline{G_{m-1}}\right)^{-1}(Id)\right)\\
&\le\kappa_m(C_1\cdots C_mn_1\cdots n_m)^{2n_m-1}+\frac{\kappa_{m-1}}{2}+2\left(\frac{C_1\cdots C_{m-1}n_1\cdots n_{m-1}}{n_m^2}\right)\\
&\le\left(\frac{1}{16\cdot 2^m(C_1\cdots C_mn_1\cdots n_m)^{2n_m^2-1}}\right)(C_1\cdots C_mn_1\cdots n_m)^{2n_m-1}+\frac{\kappa_{m-1}}{2}\\
&+2\left(\frac{n_1n_2\cdots n_{m-1}}{n_m}\right)^2\\
&\le\frac{1}{2^m}+\frac{\kappa_{m-1}}{2}+2\left(\frac{\psi(n_{m-1})}{n_m}\right)^2\\
&\le\frac{1}{2^m}+\frac{\kappa_{m-1}}{2}+2\left(\frac{1}{n_{m-1}}\right)^2.
\end{aligned}
$$

Thus $d_u(T_0^{n_m},Id)\to 0$ as $m\to\infty$ and T_0 is uniformly rigid with respect to (n_m). Therefore we have constructed a topologically weakly mixing homeomorphism that is uniformly rigid with respect to (n_m).

□

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