ON WEAKLY MIXING HOMEOMORPHISMS OF THE TWO-TORUS THAT ARE UNIFORMLY RIGID

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Abstract. In this paper we explore some properties of weakly mixing homeomorphisms of the two-torus that are uniformly rigid. Specifically we will show that there is a large family of weakly mixing, uniformly rigid, strictly ergodic homeomorphisms of the two-torus. Also, we will show that if a sequence of natural numbers satisfies a certain growth rate, then we can construct a weakly mixing homeomorphism of the two-torus that is uniformly rigid with respect to that sequence.

1. Introduction

In 1989 Glasner and Maon introduced the idea of uniform rigidity as a topological analogue of classical rigidity in ergodic theory (see [5]). This seems to be the correct topological analogue as similar generic properties hold in both settings.

It is well known that if you consider the group of all measure-preserving automorphisms of a Lebesgue probability space equipped with the weak topology, then the set of weakly mixing transformations and the set of rigid transformations each form a dense G_δ subset (see [7]). It is possible to prove a similar result for homeomorphisms of certain compact spaces. In Section 3 we will discuss some of the similarities and differences between uniform rigidity and rigidity. In Section 4 we will explore the following question posed by the authors of [8]:

Question 1.1. *Does there exist a measure-preserving homeomorphism which is both weakly mixing and uniformly rigid?*

As it turns out, this was not an open question. In [6], Glasner and Weiss showed that there is a large family of weakly mixing homeomorphisms on the infinite torus that are strictly ergodic. This result, coupled with an earlier result regarding uniform rigidity in [5], gives a positive answer to the above posed question. We will use their method of proof to show that you can do the same for the two torus. In fact, let $\mathcal{O}$ be the closure of the set of conjugations of an aperiodic rotation by measure-preserving homeomorphisms of $\mathbb{T}^2$. We will prove the following theorem:

Theorem 1.2. *There exists a dense G_δ subset $\mathcal{R}$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}$, $(\mathbb{T}^2, T, \mu)$ is weakly mixing, uniformly rigid, and strictly ergodic.*

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Finally, in Section 5 we give results in the direction of the following question posed by the authors of [8]:

Question 1.3. *Which zero density sequences occur as uniform rigidity sequences for an ergodic transformation?*

We were not able to answer this, undoubtedly difficult question, in full but we were able to prove that given a sufficient growth rate, the existence of a weakly mixing (ergodic) homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to your sequence is guaranteed. Specifically we prove the following two theorems:

Theorem 1.4. *If (n_m) is an increasing sequence of natural numbers satisfying*

$$\lim_{m\to\infty} \frac{n_{m+1}}{n_m} = \infty$$

there exists an ergodic homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to (n_m).

Theorem 1.5. *Let $\psi(x) = x^{x^3}$. If (n_m) is an increasing sequence of natural numbers satisfying*

$$\frac{n_{m+1}}{n_m} \geq \psi(n_m)$$

there exists a weakly mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to (n_m).

2. Preliminaries

Before we begin, it is necessary to recall some standard definitions from ergodic theory and topological dynamics (see [4] and [11] for more details). All spaces here will be considered simultaneously as topological spaces and as measure spaces. In particular, suppose (X, β, μ, T) is a dynamical system where (X, β, μ) is a Lebesgue probability space (ie. it is measure-theoretically isomorphic to the unit interval and has no atoms), X is a compact metric space with metric d, and T is a measure-preserving homeomorphism.

We will be considering measure-preserving homeomorphisms defined from X to X. If S, T are two measure-preserving homeomorphisms from X to X, then in the sup metric the distance from S to T is defined by $\sup_{x\in X} d(S(x), T(x))$. When endowed with this metric, which induces the topology of uniform convergence, the group becomes a topological group that is not complete. To see this, notice that you can construct a sequence of measure-preserving homeomorphisms which converge uniformly to a continuous function with no inverse. In this paper, some of the issues that we will be considering are generic issues and therefore we need our space to be complete. To that end, we define a new metric where the uniform distance is given by

$$d_u(S, T) = \sup_{x\in X} d(S(x), T(x)) + \sup_{x\in X} d(S^{-1}(x), T^{-1}(x)).$$

The topology induced by d_u is still the topology of uniform convergence and with this metric the group of measure-preserving homeomorphisms on X is a complete metric space. We will also call this the topology of uniform convergence of homeomorphisms and their inverses. To simplify notation, if S, T are two homeomorphisms defined on X, let

$$\overline{d}(S,T) = \sup_{x \in X} d(S(x), T(x)).$$

With this new notation,

$$d_u(S,T) = \overline{d}(S,T) + \overline{d}(S^{-1}, T^{-1}).$$

Notice that even though d_u is not right-invariant, $\overline{d}$ is right-invariant. This fact will be exploited throughout the paper.

Definition 2.1. *The homeomorphism T is uniformly rigid if there exists a sequence of natural numbers (n_m) such that*

$$d_u(T^{n_m}, Id) \to 0$$

as $m \to \infty$.

Remark 2.2. *The above definition is equivalent to $T^{n_m} \to Id$ uniformly on X (ie. $\overline{d}(T^{n_m}, Id) \to 0$).*

Definition 2.3. *The homeomorphism T is rigid if there exists a sequence of natural numbers (n_m) such that the powers (T^{n_m}) converge to the identity in the strong operator topology as $m \to \infty$. That is,*

$$\|f \circ T^{n_m} - f\|_2 \to 0$$

as $m \to \infty$ for all $f \in L_2(X, \mu)$.

For the purpose of this paper, let $\langle f \mid g \rangle$ denote the standard inner-product on $L_2(X, \mu)$ given by

$$\langle f \mid g \rangle = \int_X f \cdot \overline{g} \, d\mu.$$

Let U_T be the unitary operator on $L_2(X, \mu)$ induced by T, called the Koopman operator, defined by $U_T(f) = f \circ T$ for all $f \in L_2(X, \mu)$.

Definition 2.4. *The homeomorphism T is ergodic if*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle U_T^n f | g \rangle - \langle f | 1 \rangle \langle 1 | g \rangle = 0$$

for all $f, g \in L_2(X, \mu)$.

Definition 2.5. *The homeomorphism T is weakly mixing if*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U_T^n f | g \rangle - \langle f | 1 \rangle \langle 1 | g \rangle| = 0$$

for all $f, g \in L_2(X, \mu)$.

Definition 2.6. *The homeomorphism T is strongly mixing if*

$$\lim_{n\to\infty} |\langle U_T^n f|g\rangle - \langle f|1\rangle\langle 1|g\rangle| = 0$$

for all $f, g \in L_2(X, \mu)$.

Definition 2.7. *The homeomorphism T is called uniquely ergodic if there is only one T-invariant probability measure μ on (X, T).*

Definition 2.8. *The homeomorphism T is strictly ergodic if it is uniquely ergodic and the unique T-invariant probability measure on X has full topological support (ie. $supp(\mu) = X$).*

Definition 2.9. *The homeomorphism T is minimal if every orbit of T is dense in X.*

Remark 2.10. *The homeomorphism T is strictly ergodic if and only if it is uniquely ergodic and minimal.*

We also need some notation before we go further. Let $C(X)$ be the continuous functions defined on X and let $C^0(X)$ be those functions that are mean zero and L_2-norm one. That is, if $f \in C^0(X)$ then $\int_X f(x)d\mu = 0$ and $\left(\int_X |f(x)|^2 \, d\mu\right)^{\frac{1}{2}} = 1$.

3. Rigidity and Uniform Rigidity

As noted in the introduction, uniform rigidity seems to be the right topological analogue of rigidity. It is natural to question whether they are distinct properties of a dynamical system. In [8] the authors show that uniform rigidity and rigidity coincide on the unit interval (or circle $\mathbb{S}^1$) when μ is a finite Oxtoby-Ulam measure (ie. μ is nonatomic, zero on the boundary, and locally positive). Their proof of this fact relies heavily on the notion that connected sets with large diameter have large measure. Since this is not always the case in higher dimensions, a different approach is needed.

Upon analyzing an example in [1] we observed that there exists a homeomorphism of $\mathbb{T}^2$ that is weakly mixing and rigid, but not uniformly rigid. Thus, rigidity and uniform rigidity do not coincide on $\mathbb{T}^2$. We will now explain this example further.

Example 3.1 In [1] the authors use the Gaussian Measure Space construction (for details regarding this construction see [2] page 188) to build an example of a weakly mixing transformation that is rigid along powers of 2. Since rigid transformations have zero measure-theoretic entropy, we may apply a result of Lind and Thouvenot (see [10]) to obtain a hyperbolic toral automorphism of $\mathbb{T}^2$ that is weakly mixing and rigid along powers of 2 with respect to an Oxtoby-Ulam measure. Now, recall that hyperbolic toral automorphisms of $\mathbb{T}^2$ are strongly mixing with respect to Lebesgue measure. Thus, this example is not uniformly rigid.

The above example illustrates that rigidity does not imply uniform rigidity. However, if T is uniformly rigid, then T is also rigid with respect to the same sequence. A question that remains open, is:

Question 3.1. *In which situations do rigidity and uniform rigidity coincide?*

Another natural question to ask is:

Question 3.2. *Which sequences of natural numbers can be realized as uniform rigidity sequences and as rigidity sequences (not necessarily for the same dynamical system)?*

Recall that rigidity sequences for weakly mixing transformations have gaps that tend to infinity and have zero density (see [1]). Since uniform rigidity implies rigidity, this is also the case for uniform rigidity sequences. Even though this is the case, we give a direct proof here.

Proposition 3.3. *Let (n_m) be an increasing sequence of natural numbers. Let T be weakly mixing and uniformly rigid with respect to (n_m) (ie. $d_u(T^{n_m}, Id) \to 0$ as $m \to \infty$). Then the sequence (n_m) has gaps tending to infinity and has zero density.*

Proof. Our goal is to show that (n_m) has gaps that tend to infinity. Thus we want to show that $n_{m+1} - n_m \to \infty$. Suppose for a contradiction that there exists $d \geq 1$ such that $d = n_{m+1} - n_m$ infinitely often. Thus infinitely often we have

$$\begin{aligned} d_u(T^{n_{m+1}}, Id) &= d_u(T^{d+n_m}, Id) \\ &= d_u(T^{n_m}(T^d), Id) \\ &= \overline{d}(T^{n_m}(T^d), Id) + \overline{d}(T^{-d}(T^{-n_m}), Id) \\ &= \overline{d}(T^{n_m}, T^{-d}) + \overline{d}(T^{-n_m}, T^d) \\ &= d_u(T^{n_m}, T^{-d}) \end{aligned}$$

Thus $T^d = Id$ since $d_u(T^{n_{m+1}}, Id) \to 0$. Let $f \in L_2(X, \mu)$ be nonconstant. Then $f \circ T^d = f$ which is a contradiction since T is weakly mixing and hence totally ergodic. Therefore, (n_m) has gaps that tend to infinity and hence zero density.

□

4. Generic Properties of Homeomorphisms of $\mathbb{T}^2$

We will show, using a category argument, that there is a large family of homeomorphisms on the two torus $\mathbb{T}^2$ each of which are weakly mixing, uniformly rigid, and strictly ergodic (thus minimal).

In [6] Glasner and Weiss produced a large family of homeomorphisms on the infinite torus that are weakly mixing and strictly ergodic. We will use their method of proof to show that this can be accomplished on the two-torus. We will also show that there is a large family of uniformly rigid homeomorphisms on $\mathbb{T}^2$. Then in the next section, we will use the inherent structure of the category argument to determine a sufficient growth rate for a sequence of natural numbers that guarantees the existence of a weakly mixing homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to the given sequence.

From now on we will be working on the two torus $\mathbb{T}^2$. We will be using the model of $\mathbb{T}^2$ where it is viewed as $[0, 1)^2$ and the coordinates are taken modulo 1. We will be using additive notation and $|\cdot|$ will denote the distance to the nearest integer or absolute value (the distinction should be clear from context). Also, let $\|\cdot\|$ denote the fractional part. If $0 \leq x \leq \frac{1}{2}$ then $|x| = \|x\|$ and if $\frac{1}{2} < x < 1$ then $|x| = 1 - \|x\|$. Note that $\mathbb{T}^2$ is a compact monothetic group. Thus we may choose $\alpha = (\alpha_1, \alpha_2)$ such that the set $\{n\alpha \ : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$. Let $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ be a rotation homeomorphism defined by $\sigma(x_1, x_2) = (x_1+\alpha_1, x_2+\alpha_2)$. Suppose that $\mathbb{T}$ is equipped with the usual Lebesgue measure and μ is the corresponding product measure on $\mathbb{T}^2$. Note that σ preserves μ.

Let $\mathcal{H}(\mathbb{T}^2)$ be the set of measure-preserving homeomorphisms of $\mathbb{T}^2$. Define the set $O(\sigma)$ as follows:

$$O(\sigma) = \{G \circ \sigma \circ G^{-1} : G \in \mathcal{H}(\mathbb{T}^2)\}.$$

Throughout this paper we will be considering $O(\sigma)$ as a subset of all homeomorphisms of $\mathbb{T}^2$ with the topology of uniform convergence of homeomorphisms and their inverses. As before, if S, T are two homeomorphisms in $\mathcal{H}(\mathbb{T}^2)$ then,

$$\begin{aligned} d_u(S,T) &= \sup_{x \in \mathbb{T}^2} d(S(x), T(x)) + \sup_{x \in \mathbb{T}^2} d(S^{-1}(x), T^{-1}(x)) \\ &= \overline{d}(S,T) + \overline{d}(S^{-1}, T^{-1}). \end{aligned}$$

From now on we will be considering generic properties inside $\mathcal{O} = \overline{O(\sigma)}$.

4.1. **Genericity of Weakly Mixing Homeomorphisms.** The goal of this subsection is to prove the following theorem:

Theorem 4.1. *There exists a dense G_δ subset $\mathcal{R}_1$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}_1$, $(\mathbb{T}^2, T, \mu)$ is weakly mixing.*

We will need several lemmas to prove this theorem. The first lemma below will help us write the set of weakly mixing homeomorphisms in $\mathcal{O}$ as a G_δ set, see [6] for the proof of this lemma.

Lemma 4.2. *Let T be a measure-preserving homeomorphism of X. Then T is weakly mixing if and only if there exists a dense subset $\{\phi_i\}$ of $C^0(X)$ such that for all i there exists n with*

$$|\langle U_T^n \phi_i | \phi_i \rangle| < 0.99.$$

Let $\phi \in C^0(\mathbb{T}^2)$ and $0 < \eta < 1$. Define

$$R_\phi(\eta) = \{T \in \mathcal{O} : \textit{there exists } n \textit{ with } |\langle U_T^n \phi | \phi \rangle| < \eta\}.$$

Lemma 4.3. *Let $\phi(x_1, x_2) = \sum c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}$ be an element of $L_2(\mathbb{T}^2, \mu)$ with $\|\phi\|_2 = 1$ (ie. $\sum |c_{\ell_1,\ell_2}|^2 = 1$). Suppose that there exists an η, $0 < \eta < 1$, such that for every index $(\ell_1, \ell_2) \in \mathbb{Z}^2$ we have $|c_{\ell_1,\ell_2}| < \eta$. Then $\sigma \in R_\phi(\eta)$.*

Proof. Let $\phi(x_1, x_2) = \sum c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}$ be an element of $L_2(\mathbb{T}^2, \mu)$ with $\|\phi\|_2 = 1$. Suppose that there exists an η, $0 < \eta < 1$, such that for every index $(\ell_1, \ell_2) \in \mathbb{Z}^2$ we have $|c_{\ell_1,\ell_2}| < \eta$. To show that $\sigma \in R_\phi(\eta)$ we must show that there exists n such that $|\langle U_\sigma^n \phi | \phi \rangle| < \eta$.

Let $t \in \mathbb{Z}$. Then

$$\begin{aligned}
\left|\langle U_\sigma^t \phi|\phi\rangle\right| &= \left|\int_{[0,1)^2} \phi(\sigma^t(x))\overline{\phi(x)}d\mu\right| \\
&= \left|\int_{[0,1)^2} \phi(x_1+t\alpha_1, x_2+t\alpha_2)\overline{\phi(x_1,x_2)}dx_1dx_2\right| \\
&= \left|\int_{[0,1)^2} \left(\sum c_{\ell_1,\ell_2}e^{2\pi i(\ell_1x_1+\ell_1t\alpha_1+\ell_2x_2+\ell_2t\alpha_2)}\right)\left(\sum \overline{c}_{\ell_1',\ell_2'}e^{-2\pi i(\ell_1'x_1+\ell_2'x_2)}\right)dx_1dx_2\right| \\
&= \left|\sum c_{\ell_1,\ell_2}\overline{c}_{\ell_1',\ell_2'}\int_{[0,1)^2} e^{2\pi i(\ell_1x_1+\ell_1t\alpha_1+\ell_2x_2+\ell_2t\alpha_2)}e^{2\pi i(-\ell_1'x_1-\ell_2'x_2)}dx_1dx_2\right| \\
&= \left|\sum c_{\ell_1,\ell_2}\overline{c}_{\ell_1',\ell_2'}e^{2\pi i(\ell_1t\alpha_1+\ell_2t\alpha_2)}\int_{[0,1)^2} e^{2\pi i(\ell_1-\ell_1')x_1}e^{2\pi i(\ell_2-\ell_2')x_2}dx_1dx_2\right| \\
&= \left|\sum \left|c_{\ell_1,\ell_2}\right|^2 e^{2\pi i(\ell_1t\alpha_1+\ell_2t\alpha_2)}\right|
\end{aligned}$$

Let ν be the probability measure on $\mathbb{T}$ defined by

$$\nu = \sum \left|c_{\ell_1,\ell_2}\right|^2 \delta_{\ell_1\alpha_1+\ell_2\alpha_2}.$$

Observe the following:

$$\left|\langle U_\sigma^t\phi|\phi\rangle\right| = \left|\int_0^1 e^{2\pi itx}d\nu\right| = |\hat{\nu}(t)|.$$

By Wiener's theorem

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{2N+1}\sum_{t=-N}^{N}|\hat{\nu}(t)|^2 &= \sum_{x\in[0,1)}\nu^2(\{x\}) \\
&= \sum\left|c_{\ell_1,\ell_2}\right|^4 < \eta^2\sum\left|c_{\ell_1,\ell_2}\right|^2 \\
&= \eta^2.
\end{aligned}$$

Therefore there exists n with $|\hat{\nu}(n)|^2 < \eta^2$ and hence there exists n with $|\langle U_\sigma^n\phi|\phi\rangle| < \eta$. □

Lemma 4.4. *Let $G \in \mathcal{H}(\mathbb{T}^2)$ and $\phi \in C^0(\mathbb{T}^2)$. Then*

$$G^{-1}R_\phi(\eta)G = R_{\phi\circ G}(\eta).$$

Proof. Suppose $T \in R_{\phi\circ G}(\eta)$. Let n be such that $|\langle U_T^n(\phi\circ G)|\phi\circ G\rangle| < \eta$. Notice that

$$\begin{aligned}
\langle U_T^n(\phi\circ G)|\phi\circ G\rangle &= \int_{[0,1)^2}(\phi\circ G)(T^n(x))\overline{\phi\circ G(x)}d\mu \\
&= \int_{[0,1)^2}\phi(G\circ T^n\circ G^{-1}(x))\overline{\phi(x)}d\mu \\
&= \langle U^n_{G\circ T\circ G^{-1}}\phi|\phi\rangle.
\end{aligned}$$

Thus $G\circ T\circ G^{-1} \in R_\phi(\eta)$ and therefore $R_{\phi\circ G}(\eta) \subseteq G^{-1}R_\phi(\eta)G$. The other containment is proved in the same way. □

Lemma 4.5. *Let g be a continuous real-valued function on $\mathbb{T}$ which is twice differentiable and assume that g'' has only finitely many zeros. Then there exists $K \in \mathbb{N}$ such that for all M we have*

$$\left| \int_0^1 e^{2\pi i[Kg(x)+Mx]} dx \right| < 0.3.$$

Remark 4.6. *In the above lemma, K does not depend on M.*

Proof of Lemma 4.5. This follows from Van der Corput's Lemma for the second derivative (see [9] page 220). A similar lemma is proved in Section 5.

□

We are now prepared to prove Theorem 4.1.

Proof of Theorem 4.1 Let $\{\phi_i\}$ be a dense subset of $C^0(\mathbb{T}^2)$. We will show that

$$\mathcal{R}_1 = \bigcap_{i=1}^{\infty} R_{\phi_i}(0.99)$$

is our desired dense G_δ set. From Lemma 4.2 we see that $\mathcal{R}_1$ is the set of weakly mixing homeomorphisms of $\mathbb{T}^2$ inside $\mathcal{O}$. Let $\phi \in C^0(\mathbb{T}^2)$. It suffices to show that $R_\phi(0.99)$ is open and dense in $\mathcal{O}$.

To see that $R_\phi(0.99)$ is open in $\mathcal{O}$, notice that the set

$$\{T \in \mathcal{H}(\mathbb{T}^2) : \textit{there exists } n \textit{ with } |\langle U_T^n \phi | \phi \rangle| < 0.99\}$$

is open in $\mathcal{H}(\mathbb{T}^2)$. Therefore this set restricted to $\mathcal{O}$, which is $R_\phi(0.99)$, is open in $\mathcal{O}$.

The bulk of this proof is showing that $R_\phi(0.99)$ is dense in $\mathcal{O}$. Since $R_\phi(0.99) \subseteq \mathcal{O}$ and $\mathcal{O}$ is closed, it suffices to show that if $G_0 \in \mathcal{H}(\mathbb{T}^2)$ then $G_0 \circ \sigma \circ G_0^{-1} \in \overline{R_\phi(0.99)}$.

Suppose that (G_m) is a sequence in $\mathcal{H}(\mathbb{T}^2)$ such that $d_u(G_m \circ \sigma \circ G_m^{-1}, \sigma) \to 0$ as $m \to \infty$ and for all m, $G_m \circ \sigma \circ G_m^{-1} \in R_{\phi \circ G_0}(0.99)$. Since G_0, G_0^{-1} are continuous, $d_u(G_0 \circ G_m \circ \sigma \circ G_m^{-1} \circ G_0^{-1}, G_0 \circ \sigma \circ G_0^{-1}) \to 0$ as $m \to \infty$. Thus we can write

$$G_0 \circ \sigma \circ G_0^{-1} = \lim_{m \to \infty} (G_0 \circ G_m \circ \sigma \circ G_m^{-1} \circ G_0^{-1}) \in \overline{G_0 R_{\phi \circ G_0}(0.99) G_0^{-1}} = \overline{R_\phi(0.99)}.$$

Therefore we have reduced the rest of the proof of Theorem 4.1 to the following lemma:

Lemma 4.7. *Let $\epsilon > 0$ and $\phi \in C^0(\mathbb{T}^2)$. Then there exists $G \in \mathcal{H}(\mathbb{T}^2)$ such that the following two properties hold:*

(1) $d_u(G \circ \sigma \circ G^{-1}, \sigma) < \epsilon$

(2) $G \circ \sigma \circ G^{-1} \in R_\phi(0.99)$ *(ie. $\sigma \in R_{\phi \circ G}(0.99)$).*

Remark 4.8. *The ϕ that appears in the above lemma is $\phi \circ G_0$ from the proof of Theorem 4.1.*

Proof of Lemma 4.7. Let $\sum c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}$ be the Fourier expansion of $\phi \in C^0(\mathbb{T}^2)$. Recall that $\sum |c_{\ell_1,\ell_2}|^2 = 1$.

If for every index $(\ell_1, \ell_2) \in \mathbb{Z}^2$ we have $|c_{\ell_1,\ell_2}| < 0.99$ then by Lemma 4.3, $\sigma \in R_\phi(0.99)$. In this case if we take G to be the Id we have the lemma.

Otherwise there is exactly one index, call it $c_0 = c_{N_1,N_2}$, such that $|c_0| \geq 0.99$. Let $\phi_1 = c_0 e^{2\pi i(N_1x_1+N_2x_2)}$ and $\phi_2 = \phi - \phi_1$. Then $1 = \|\phi\|_2^2 = \|\phi_1\|_2^2 + \|\phi_2\|_2^2$. Since $\|\phi_1\|_2^2 = |c_0|^2 \geq 0.9801$ we have $\|\phi_2\|_2^2 < 0.02$ and therefore $\|\phi_2\|_2 < 0.2$. For all n and $G \in \mathcal{H}(\mathbb{T}^2)$ we have the following:

$$\begin{aligned}
|\langle U_\sigma^n(\phi\circ G)|\phi\circ G\rangle| &= |\langle U_\sigma^n((\phi_1+\phi_2)\circ G)|(\phi_1+\phi_2)\circ G\rangle| \\
&= |\langle U_\sigma^n(\phi_1\circ G) + U_\sigma^n(\phi_2\circ G)|(\phi_1\circ G)+(\phi_2\circ G)\rangle| \\
&\leq |\langle U_\sigma^n(\phi_1\circ G)|\phi_1\circ G\rangle| + |\langle U_\sigma^n(\phi_1\circ G)|\phi_2\circ G\rangle| \\
&+ |\langle U_\sigma^n(\phi_2\circ G)|\phi_1\circ G\rangle| + |\langle U_\sigma^n(\phi_2\circ G)|\phi_2\circ G\rangle| \\
&\leq |\langle U_\sigma^n(\phi_1\circ G)|\phi_1\circ G\rangle| + \|U_\sigma^n(\phi_1\circ G)\|_2 \|\phi_2\circ G\|_2 \\
&+ \|U_\sigma^n(\phi_2\circ G)\|_2 \|\phi_1\circ G\|_2 + \|U_\sigma^n(\phi_2\circ G)\|_2 \|\phi_2\circ G\|_2 \\
&< |\langle U_\sigma^n(\phi_1\circ G)|\phi_1\circ G\rangle| + 3\|\phi_2\|_2 \\
&< |\langle U_\sigma^n(\phi_1\circ G)|\phi_1\circ G\rangle| + 0.6.
\end{aligned}$$

From the above we see that it suffices to find n and $G \in \mathcal{H}(\mathbb{T}^2)$ such that

$$|\langle U_\sigma^n(\phi_1\circ G)|\phi_1\circ G\rangle| < 0.3.$$

Let $\phi_0 = \frac{\phi_1}{c_0}$. Since $|c_0| \leq 1$ it suffices to find n and $G \in \mathcal{H}(\mathbb{T}^2)$ such that $|\langle U_\sigma^n(\phi_0\circ G)|\phi_0\circ G\rangle| < 0.3$ (ie. $\sigma \in R_{\phi_0\circ G}(0.3)$).

Let $\psi = \phi_0 \circ G$. Our goal is to show that there exists $G \in \mathcal{H}(\mathbb{T}^2)$ such that $\sigma \in R_\psi(0.3)$. By Lemma 4.3 it suffices to show that for every index $(\ell_1,\ell_2) \in \mathbb{Z}^2$, the absolute value of the coefficients in the Fourier expansion of ψ are less than 0.3. Hence we need to show that for every $(\ell_1,\ell_2) \in \mathbb{Z}^2$ we have

$$\left|\langle\psi|e^{2\pi i(\ell_1x_1+\ell_2x_2)}\rangle\right| = \left|\langle\phi_0\circ G|e^{2\pi i(\ell_1x_1+\ell_2x_2)}\rangle\right| < 0.3.$$

Recall that $\phi_0 = e^{2\pi i(N_1x_1+N_2x_2)}$ where N_1 and N_2 cannot both be equal to zero since $\phi \in C^0(\mathbb{T}^2)$.

Suppose that $N_1 \neq 0$. In this case let $G \in \mathcal{H}(\mathbb{T}^2)$ have the form

$$G(x_1,x_2) = (x_1 + f(x_2), x_2)$$

where f is a continuous function. The inverse of G is easily defined as well and is equal to $G^{-1}(x_1,x_2) = (x_1 - f(x_2), x_2)$. For all $(\ell_1,\ell_2) \in \mathbb{Z}^2$ we have

$$\begin{aligned}
\left|\langle\phi_0\circ G|e^{2\pi i(\ell_1x_1+\ell_2x_2)}\rangle\right| &= \left|\langle e^{2\pi i(N_1x_1+N_1f(x_2)+N_2x_2)}|e^{2\pi i(\ell_1x_1+\ell_2x_2)}\rangle\right| \\
&= \left|\int_{[0,1)^2} e^{2\pi i[N_1x_1+N_1f(x_2)+N_2x_2]}e^{-2\pi i[\ell_1x_1+\ell_2x_2]}dx_1dx_2\right| \\
&= \left|\int_{[0,1)^2} e^{2\pi i(N_1-\ell_1)x_1}e^{2\pi i[N_1f(x_2)+(N_2-\ell_2)x_2]}dx_1dx_2\right| \\
&= \delta_0\left|\int_0^1 e^{2\pi i[N_1f(x_2)+(N_2-\ell_2)x_2]}dx_2\right|
\end{aligned}$$

where δ_0 is 0 when $N_1 \neq \ell_1$ and 1 otherwise.

Let $g : \mathbb{T} \to \mathbb{R}$ be continuous, twice differentiable, and such that g'' has finitely many zeros. The same properties hold for $N_1 g$ and thus by Lemma 4.5 there exists $K \in \mathbb{N}$ such that for all M we have

$$\left| \int_0^1 e^{2\pi i[KN_1 g(x)+Mx]} dx \right| < 0.3.$$

Hence for any $\ell_2 \in \mathbb{Z}$

$$\delta_0 \left| \int_0^1 e^{2\pi i[KN_1 g(x_2)+(N_2-\ell_2)x_2]} dx_2 \right| < 0.3.$$

Any G of the above form where $f(x_2) = Kg(x_2)$ will satisfy (2). However, we need to be a little more careful since we also need G to satisfy (1). To that end, let $\delta > 0$ be such that if $|x - x'| < \delta$ then $|Kg(x) - Kg(x')| < \frac{\epsilon}{2}$. Let $q \in \mathbb{Z} \setminus \{0\}$ be such that $\left|\alpha_2 - \frac{p}{q}\right| < \frac{1}{q^2}$ and $\frac{1}{q} < \delta$ where $p \in \mathbb{Z}$.

Now we will define the formula for G explicitly. Let $f(x_2) = Kg(qx_2)$ on $\mathbb{T}$. Then $G(x_1, x_2) = (x_1 + Kg(qx_2), x_2)$. In order to estimate the uniform distance between $G \circ \sigma \circ G^{-1}$ and σ observe the following:

$$G \circ \sigma \circ G^{-1}(x_1, x_2) = (x_1 + \alpha_1 + f(x_2 + \alpha_2) - f(x_2), x_2 + \alpha_2)$$

and

$$G \circ \sigma^{-1} \circ G^{-1}(x_1, x_2) = (x_1 - \alpha_1 + f(x_2 - \alpha_2) - f(x_2), x_2 - \alpha_2).$$

For our estimate we will need the fact that $|q\alpha_2| < \delta$. Now

$$|f(x_2 + \alpha_2) - f(x_2)| = |Kg(qx_2 + q\alpha_2) - Kg(qx_2)| < \frac{\epsilon}{2}$$

and

$$|f(x_2 - \alpha_2) - f(x_2)| = |Kg(qx_2 - q\alpha_2) - Kg(qx_2)| < \frac{\epsilon}{2}.$$

Therefore $d_u(G \circ \sigma \circ G^{-1}, \sigma) = \overline{d}(G \circ \sigma \circ G^{-1}, \sigma) + \overline{d}(G \circ \sigma^{-1} \circ G^{-1}, \sigma^{-1}) < \epsilon$ and (1) is proved.

To verify (2), notice that

$$\int_0^1 e^{2\pi i[KN_1 g(x_2)+(N_2-\ell_2)x_2]} dx_2$$

is a Fourier coefficient of the function $e^{2\pi i[KN_1 g(x_2)]}$. Since the set of nonzero Fourier coefficients of $e^{2\pi i[KN_1 g(x_2)]}$ is the same as that of $e^{2\pi i[N_1 Kg(qx_2)]} = e^{2\pi i N_1 f(x_2)}$ we have

$$\delta_0 \left| \int_0^1 e^{2\pi i[N_1 f(x_2)+(N_2-\ell_2)x_2]} dx_2 \right| < 0.3$$

proving (2).

In the case that $N_2 \neq 0$ we follow a very similar argument except we skew G in the second variable. If $s \in \mathbb{Z} \setminus \{0\}$ is chosen so that $\left|\alpha_1 - \frac{r}{s}\right| < \frac{1}{s^2}$ and $\frac{1}{s} < \delta$ where $r \in \mathbb{Z}$ then let $G(x_1, x_2) = (x_1, x_2 + Kg(sx_1))$. It is easy to see that this G satisfies (1) and (2). $\square$

4.2. **Genericity of Uniformly Rigid Homeomorphisms.** In this subsection we will show that uniform rigidity is generic in the set $\mathcal{O}$. Specifically we will prove the following theorem:

Theorem 4.9. *There exists a dense G_δ subset $\mathcal{R}_2$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}_2$, $(\mathbb{T}^2, T)$ is uniformly rigid.*

Proof. Recall that $\{n\alpha : n \in \mathbb{Z}\}$ where $\alpha = (\alpha_1, \alpha_2)$, is dense in $\mathbb{T}^2$. If $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ is defined by $\sigma(x_1, x_2) = (x_1 + \alpha_1, x_2 + \alpha_2)$ then σ is uniformly rigid. This follows easily from the above statement regarding the density of $\{n\alpha : n \in \mathbb{Z}\}$. Let (n_m) be the uniform rigidity sequence for σ.

Let

$$R_{i,\epsilon} = \{T \in \mathcal{O} \ : \ \textit{there exists } n_m \geq i \textit{ with } d_u(T^{n_m}, Id) < \epsilon\}.$$

We will show that

$$\mathcal{R}_2 = \bigcap_{i=1}^{\infty} R_{i,\frac{1}{i}}$$

is our desired dense G_δ set. By the definition of uniform rigidity, the set $\mathcal{R}_2$ is the set of uniformly rigid homeomorphisms in $\mathcal{O}$ with respect to a subsequence of (n_m). Since $R_{i,\epsilon}$ is clearly open in $\mathcal{O}$, it suffices to show that $R_{i,\epsilon}$ is dense in $\mathcal{O}$.

To show that $R_{i,\epsilon}$ is dense in $\mathcal{O}$ we will show that $O(\sigma) \subseteq R_{i,\epsilon}$. To do this we will rely on

$$d_u(\sigma^{n_m}, Id) \to 0$$

as $m \to \infty$. Let $G \in \mathcal{H}(\mathbb{T}^2)$. Our goal is to show that there exists $n_m \geq i$ such that $d_u(G \circ \sigma^{n_m} \circ G^{-1}, Id) < \epsilon$. Recall that we can write

$$d_u(G \circ \sigma^{n_m} \circ G^{-1}, Id) = \overline{d}(G \circ \sigma^{n_m} \circ G^{-1}, Id) + \overline{d}(G \circ \sigma^{-n_m} \circ G^{-1}, Id)$$

and so we need to find $n_m \geq i$ such that $\overline{d}(G(\sigma^{n_m}), G(Id)) < \frac{\epsilon}{2}$ and $\overline{d}(G(\sigma^{-n_m}), G(Id)) < \frac{\epsilon}{2}$. Since $d_u(\sigma^{n_m}, Id) \to 0$ we can find a large enough n_m for our purpose. Therefore $G^{-1} \circ \sigma \circ G \in R_{i,\epsilon}$.

□

Remark 4.10. *If (n_m) is the uniform rigidity sequence for σ then for every $T \in \mathcal{R}_2$, $(\mathbb{T}^2, T)$ is uniformly rigid with respect to a subsequence of (n_m).*

4.3. **Genericity of Strictly Ergodic Homeomorphisms.** The goal of this subsection is to prove the following theorem:

Theorem 4.11. *There exists a dense G_δ subset $\mathcal{R}_3$ of $\mathcal{O}$ such that for every $T \in \mathcal{R}_3$, $(\mathbb{T}^2, T)$ is strictly ergodic.*

For more information regarding strict ergodicity see [4].

Proof. First notice that σ is strictly ergodic. If $G \in \mathcal{H}(\mathbb{T}^2)$ then $G \circ \sigma \circ G^{-1}$ is also strictly ergodic. Hence $O(\sigma)$ is a subset of the strictly ergodic homeomorphisms. Let $\phi \in C(\mathbb{T}^2)$

and $\epsilon > 0$. Consider the open set $R_{\phi,\epsilon}$ defined by $T \in \mathcal{O}$ such that there exists N and c with $\left|\frac{1}{N}\sum_{n=0}^{N-1}\phi(T^n(x)) - c\right| < \epsilon$ for all $x \in \mathbb{T}^2$. Let $\{\phi_i\}$ be dense in $C(\mathbb{T}^2)$. We will show that

$$\mathcal{R}_3 = \bigcap_{i,j=1}^{\infty} R_{\phi_i,\frac{1}{j}}$$

is our desired dense G_δ set.

By the definition of unique ergodicity, the set $\mathcal{R}_3$ is the set of uniquely ergodic homeomorphisms in $\mathcal{O}$. The unique ergodic measure is a product of Lebesgue measures and therefore has full topological support. Thus, $\mathcal{R}_3$ is the set of strictly ergodic homeomorphisms in $\mathcal{O}$ by Remark 2.10. Clearly, $R_{\phi_i,\frac{1}{i}}$ is open and dense in $\mathcal{O}$.

□

4.4. **Main Result.** In this subsection we put all of the previous theorems together to give a positive answer to the question stated at the beginning of this section, proving Theorem 1.2.

Proof of Theorem 1.2. Let

$$\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3.$$

□

5. Uniform Rigidity Sequences

In this section we are interested in sequences that are uniform rigidity sequences for weakly mixing (ergodic) homeomorphisms of $\mathbb{T}^2$. We will show that if a sequence satisfies a certain growth rate, then we can construct a weakly mixing (ergodic) homeomorphism of $\mathbb{T}^2$ for which the given sequence is a uniform rigidity sequence. This gives a result in the direction of Question 1.3.

The goal of this section is then to prove Theorems 1.4 and 1.5. We will use the structure involved in the previous category argument to construct the desired homeomorphisms. The first step in the construction is to choose an irrational rotation that we will then conjugate. In [3] Eggleston shows that if an increasing sequence of natural numbers (n_m) is such that $\lim_{m\to\infty}\frac{n_{m+1}}{n_m} = \infty$ then $\lim_{m\to\infty}|n_m x| = 0$ holds for an uncountable set of x values. In the following lemma we use a similar argument.

Lemma 5.1. *Let $\psi(x) = x^{x^3}$ and suppose (n_m) is an increasing sequence of natural numbers satisfying $\frac{n_{m+1}}{n_m} \geq \psi(n_m)$. Then there exists $\alpha = (\alpha_1, \alpha_2)$ such that $\{n\alpha : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$ and*

$$\frac{1}{4(n_m)^2} < |n_m\alpha_i| < \frac{1}{2(n_m)^2}$$

for $i = 1, 2$.

Proof. Our goal is to build a Cantor set using some of the n_m-th roots of unity. From this Cantor set we will be able to select α_1, α_2 irrational and rationally independent such that the desired bounds hold.

Let $h_m = 2n_m^2$. In this case $\frac{1/h_m}{n_m/n_{m+1}} \to \infty$ as $m \to \infty$. Let M be large enough so that for all $m \geq M$ we have $n_{m+1} \geq 10 n_m h_m$.

Now we will build our Cantor set inductively. Suppose $m \geq M$. As part of the construction put two intervals close to some of the n_m-th roots of unity (determined as part of the induction) such that any point in either of the intervals is at most $\frac{1}{n_m h_m}$ away from the n_m-th root of unity and at least $\frac{1}{2n_m h_m}$ away. In this stage of the construction note that each n_m-th root of unity that appears above has two symmetric intervals close to it, one on either side, each of length $\frac{1}{2n_m h_m}$. Call this collection of intervals C_m.

Since $\frac{1}{n_{m+1}} << \frac{1}{2n_m h_m}$, there are many points of the form $\frac{j}{n_{m+1}}$ in each interval of C_m. Now select in C_m pairs of symmetric intervals, each of size $\frac{1}{2n_{m+1}h_{m+1}}$, close to some of the n_{m+1}-th roots of unity in the same way as above. Call this collection of intervals C_{m+1}.

Continue on in this manner and let the Cantor set C be defined as

$$C = \bigcap_{m=M}^{\infty} C_m.$$

For each point $x \in C$ we have that $n_m x$ is at most $\frac{1}{h_m}$ away from an integer and at least $\frac{1}{2h_m}$ away from an integer. That is,

$$\frac{1}{2h_m} < |n_m x| < \frac{1}{h_m}.$$

Hence, if $x \in C$ then $|n_m x| \to 0$ as $m \to \infty$.

Note that C is uncountable. Thus there exists $\alpha_1 \in C$ that is irrational. Since the set of all irrational numbers in C that are rationally dependent to α_1 is countable, there exists $\alpha_2 \in C$ that is irrational and rationally independent with respect to α_1. Hence, $\alpha = (\alpha_1, \alpha_2)$ has the desired properties.

□

We can now prove Theorem 1.4.

Proof of Theorem 1.4. First use Eggleston's result in [3] to obtain $\alpha = (\alpha_1, \alpha_2)$ such that $|n_m \alpha_i| \to 0$ as $m \to \infty$ for $i = 1, 2$. Let $\sigma(x_1, x_2) = (x_1 + \alpha_1, x_2 + \alpha_2)$ be an irrational rotation of $\mathbb{T}^2$. Then $\sigma^{n_m}(x_1, x_2) = (x_1 + n_m\alpha_1, x_2 + n_m\alpha_2)$ and $d_u(\sigma^{n_m}, Id) \to 0$ as $m \to \infty$. Thus σ is an ergodic homeomorphism of $\mathbb{T}^2$ that is uniformly rigid with respect to (n_m). □

We will also need the following lemma regarding an oscillatory integral estimate in our proof of the main theorem of this section.

Lemma 5.2. *Suppose N is a given constant, $g(y) = \sin(2\pi y)$, and $\frac{1}{4} < A < \frac{3}{4}$. Then there exists $K \in \mathbb{N}$, which only depends on N, such that for all M we have*

$$\left| \int_0^1 e^{2\pi i [K(g(y+A) - g(y)) + My]} dy \right| < \frac{0.3}{N^3}.$$

Moreover, if $N \geq \pi$, then any integer larger than $260N^9$ will suffice for K.

Proof. To obtain this estimate we will use Van der Corput's lemma (see [9] page 220) with respect to the second derivative.

Let

$$\psi(y) = 2\pi[Kg(y+A) - Kg(y) + My] = 2\pi[K\sin(2\pi(y+A)) - K\sin(2\pi y) + My].$$

Then,

$$\psi''(y) = -8\pi^3 K[\sin(2\pi(y+A)) - \sin(2\pi y)].$$

The proof will now be split into cases depending on the location of A inside the interval $(\frac{1}{4}, \frac{3}{4})$.

For our first case, suppose that $\frac{1}{4} < A < \frac{1}{2}$. Then the zeros of ψ'' are $a_1 = \frac{1}{4} - \frac{A}{2}$ and $a_2 = \frac{3}{4} - \frac{A}{2}$. This can be seen algebraically or graphically. Also note that $\psi''(0)$ is negative. Let $\epsilon = \frac{0.01}{N^3}$.

Suppose that $2\epsilon < a_1$ and $|\psi''(y)| \geq \lambda > 0$ on $[\epsilon, a_1 - \epsilon]$. Then

$$\begin{aligned}\left|\int_0^1 e^{i\psi(y)}dy\right| &\leq 6\epsilon + \left|\int_\epsilon^{a_1-\epsilon} e^{i\psi(y)}dy\right| + \left|\int_{a_1+\epsilon}^{a_2-\epsilon} e^{i\psi(y)}dy\right| + \left|\int_{a_2+\epsilon}^{1-\epsilon} e^{i\psi(y)}dy\right| \\ &\leq 6\epsilon + 3\left(\frac{6}{\lambda^{\frac{1}{2}}}\right).\end{aligned}$$

Upon inspecting the graph we see that we can take $\lambda = \psi''(a_1+\epsilon) > 0$ where

$$\psi''(a_1+\epsilon) = -8\pi^3 K[\sin(2\pi(a_1+\epsilon+A)) - \sin(2\pi(a_1+\epsilon))].$$

Thus we need to find $K \in \mathbb{N}$ such that

$$6\epsilon + 3\left(\frac{6}{(-8\pi^3 K[\sin(2\pi(a_1+\epsilon+A)) - \sin(2\pi(a_1+\epsilon))])^{\frac{1}{2}}}\right) < \frac{0.3}{N^3}.$$

In particular we need to find $K \in \mathbb{N}$ such that

$$K > \frac{5,625N^6}{-8\pi^3[\sin(2\pi(a_1+\epsilon+A)) - \sin(2\pi(a_1+\epsilon))]}.$$

After using some standard estimations we see that as long as K is an integer greater than $\frac{703.125N^6}{\sqrt{2}\pi^3\sin(\frac{\pi}{50N^3})}$ we have the claim.

If we use the underestimate

$$\sin\left(\frac{\pi}{50N^3}\right) \approx \left(\frac{\pi}{50N^3}\right) - \frac{1}{6}\left(\frac{\pi}{50N^3}\right)^3$$

and suppose that $N \geq \pi$, we can take $K = 260N^9$.

Suppose $2\epsilon \geq a_1$ and $|\psi''(y)| \geq \lambda > 0$ on $[a_1+\epsilon, a_2-\epsilon]$. Then

$$\begin{aligned}\left|\int_0^1 e^{i\psi(y)}dy\right| &\leq 6\epsilon + \left|\int_{a_1+\epsilon}^{a_2-\epsilon} e^{i\psi(y)}dy\right| + \left|\int_{a_2+\epsilon}^{1-\epsilon} e^{i\psi(y)}dy\right| \\ &\leq 6\epsilon + 2\left(\frac{6}{\lambda^{\frac{1}{2}}}\right).\end{aligned}$$

In this case, λ is the same as before and the same value for K is sufficient.

For our second case, suppose that $\frac{1}{2} < A < \frac{3}{4}$. Then the zeros of ψ'' are $a_1 = \frac{3}{4} - \frac{A}{2}$ and $a_2 = \frac{5}{4} - \frac{A}{2}$. Also note that $\psi''(0)$ is positive. This case is similar to the above (here $\lambda = \psi''(a_1 - \epsilon)$) and yields the same K value.

The final case is when $A = \frac{1}{2}$. In this case,

$$\psi''(y) = -8\pi^3 K[\sin(2\pi(y + \frac{1}{2})) - \sin(2\pi y)] = 16\pi^3 K \sin(2\pi y)$$

and the zeros of ψ'' are 0 and $\frac{1}{2}$. This case is also very similar to the first one and yields a K value of $181N^9$.

After we put all of the cases together we see that the K value from case 1 is sufficient for all cases.

□

Before we give the proof of Theorem 1.5 we need to discuss a quantitative aspect of estimating a function in $L_2(\mathbb{T}^2, \mu)$ by it's partial Fourier sum. Let $S_N f$ be the N-th partial Fourier sum of some function $f \in L_2(\mathbb{T}^2, \mu)$. Let $F_N(x_1, x_2)$ be the Fejér kernel defined on $\mathbb{T}^2$ by

$$F_N(x_1, x_2) = \sum_{|\ell_i| \le N} \left(1 - \frac{|\ell_1|}{N}\right)\left(1 - \frac{|\ell_2|}{N}\right) e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)} = F_N(x_1) F_N(x_2)$$

where $F_N(x_i)$ is the usual Fejér kernel on $\mathbb{T}$. When discussing the Fejér kernel, it is easier to view $\mathbb{T}^2$ as $[\frac{-1}{2}, \frac{1}{2}]^2$ under addition modulo one in each coordinate.

Recall that F_N is an approximate identity and that $\int_{[\frac{-1}{2},\frac{1}{2}]^2} F_N(x_1, x_2) dx_1 dx_2 = 1$. Another important fact that we will need about F_N is the following:

$$F_N(x_1, x_2) \le \min\left(\frac{1}{(Nx_1^2)(Nx_2^2)}, (N+1)^2\right).$$

This follows from the representation of F_N where

$$F_{N-1}(x_1, x_2) = \left(\frac{1}{N}\right)^2 \left(\frac{\sin(N\pi x_1)}{2\pi x_1}\right)^2 \left(\frac{\sin(N\pi x_2)}{2\pi x_2}\right)^2.$$

Lemma 5.3. *Suppose f is a function in $C^0(\mathbb{T}^2)$ with modulus of continuity $\omega_f(\delta) \le M\delta$ for some constant M. If $N \ge 2^{20} M$ then*

$$\|S_N f - f\|_2 < 0.01.$$

Proof. Let $\delta = \frac{1}{1{,}000M}$ and suppose $N \ge 2^{20} M$. Our goal is to show that $\|S_N f - f\|_2 < 0.01$. The N-th partial Fourier sum of f is the best L_2-approximation of f by trigonometric polynomials of degree N. Since $f * F_N$ is a trigonometric polynomial of degree N, $\|S_N f - f\|_2 \le \|f * F_N - f\|_2$. Thus it suffices to show that $\|f * F_N - f\|_2 < 0.01$. Notice that $N > \frac{1}{\delta}$. To this end, observe

$$\|f * F_N - f\|_2 = \left\| \int_{[\frac{-1}{2},\frac{1}{2}]^2} f(x_1 - y_1, x_2 - y_2) F_N(y_1, y_2) dy_1 dy_2 - f(x_1, x_2) \right\|_2$$

$$
\begin{aligned}
&= \left\| \int_{[\frac{-1}{2},\frac{1}{2}]^2} (f(x_1-y_1,x_2-y_2)-f(x_1,x_2))F_N(y_1,y_2)dy_1dy_2 \right\|_2 \\
&\le \sup_{|y_i|<\delta} (\|f(x_1-y_1,x_2-y_2)-f(x_1,x_2)\|_2) \\
&+ 4\,\|f(x_1,x_2)\|_2 \int_{\substack{|y_1|<\delta\\|y_2|>\delta}} |F_N(y_1,y_2)|\,dy_1dy_2 \\
&+ 2\,\|f(x_1,x_2)\|_2 \int_{|y_i|>\delta} |F_N(y_1,y_2)|\,dy_1dy_2 \\
&< M\delta + 4\int_{|y_2|>\delta} |F_N(y_2)|\,dy_2 + \frac{2(2+\frac{1}{\delta})^2}{N^2} \\
&\le M\delta + \frac{4(2+\frac{1}{\delta})}{N} + \frac{2(2+\frac{1}{\delta})^2}{N^2} \\
&\le 0.001 + 0.004 + 0.004 \\
&< 0.01.
\end{aligned}
$$

□

Remark 5.4. *In the previous proof we used the Fejér kernel to obtain an estimate for* $\|S_N f - f\|_2$*. Alternately, you could use the uniform estimate found on the bottom of page 115 of* [12].

Lemma 5.5. *Let* $\phi \in C^0(\mathbb{T}^2)$*,* $G \in \mathcal{H}(\mathbb{T}^2)$*, and* σ *be a rotation of* $\mathbb{T}^2$*. Suppose that* $\phi_0 \in C^0(\mathbb{T}^2)$ *is such that* $\|\phi - \phi_0\|_2 < 0.01$*. If* $\sigma \in R_{\phi_0 \circ G}(0.9)$ *then* $\sigma \in R_{\phi\circ G}(0.93)$*.*

Proof.
Let $t \in \mathbb{N}$. Now it suffices to observe the following:

$$
\begin{aligned}
\left|\langle \phi\circ G(\sigma^t)|\phi\circ G\rangle\right| &= \left|\langle (\phi-\phi_0)\circ G(\sigma^t) + \phi_0\circ G(\sigma^t)|(\phi-\phi_0)\circ G + \phi_0\circ G\rangle\right| \\
&\le \left|\langle \phi_0\circ G(\sigma^t)|\phi_0\circ G\rangle\right| + \left|\langle (\phi-\phi_0)\circ G(\sigma^t)|(\phi-\phi_0)\circ G\rangle\right| \\
&+ \left|\langle (\phi-\phi_0)\circ G(\sigma^t)|\phi_0\circ G\rangle\right| + \left|\langle \phi_0\circ G(\sigma^t)|(\phi-\phi_0)\circ G\rangle\right| \\
&\le \left|\langle \phi_0\circ G(\sigma^t)|\phi_0\circ G\rangle\right| + \|\phi-\phi_0\|_2^2 + 2\,\|\phi-\phi_0\|_2\,\|\phi_0\|_2 \\
&< \left|\langle \phi_0\circ G(\sigma^t)|\phi_0\circ G\rangle\right| + 3(0.01).
\end{aligned}
$$

□

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let (n_m) be a sequence of natural numbers satisfying

$$n_{m+1} \ge \psi(n_m)n_m$$

where $\psi(x) = x^{x^3}$. From Lemma 5.1 we obtain irrationals α_1, α_2 such that

$$\frac{1}{4(n_m)^2} < |n_m\alpha_i| < \frac{1}{2(n_m)^2}$$

for $i = 1, 2$. We will need both of these bounds later in the proof. Let $\alpha = (\alpha_1, \alpha_2)$ and $\sigma : \mathbb{T}^2 \to \mathbb{T}^2$ be a rotation defined by $\sigma(x_2, x_2) = (x_1 + \alpha_1, x_2 + \alpha_2)$. By the nature of our choice of α, (n_m) is a uniform rigidity sequence for σ.

Let $\{\phi_i\}$ be a sequence of trigonometric polynomials that are a countable dense subset of $C^0(\mathbb{T}^2)$. Let $M_i > 0$ be the modulus of continuity of ϕ_i, that is

$$\omega_{\phi_i}(\delta) = \sup_{d(x,x')<\delta} |\phi_i(x) - \phi_i(x')| \leq M_i \delta.$$

Suppose that the ϕ_i are ordered so that the corresponding sequence, (M_i), is nondecreasing.

Recall that the set of weakly mixing homeomorphisms in $\mathcal{O} = \overline{O(\sigma)}$ is a dense G_δ set and is equal to

$$\bigcap_{i=1}^{\infty} R_{\phi_i}(0.99)$$

where $R_{\phi_i}(0.99)$ is the set of $T \in \mathcal{O}$ such that there exists $t \in \mathbb{N}$ with the property that

$$\left|\langle U_T^t(\phi_i)|\phi_i\rangle\right| < 0.99.$$

We are going to show that successive conjugations of σ converge to a weakly mixing homeomorphism that is uniformly rigid with respect to (n_m). We will form a nested sequence of closed balls B_i such that each $B_i \subseteq R_{\phi_i}(0.99)$. Then $\bigcap_{i=1}^{\infty} B_i$ will contain a homeomorphism T_0 that is weakly mixing. The center of each B_i will be a conjugation of σ and will be chosen carefully so that in the end, T_0 will be the uniform limit of these conjugations and (n_m) will be a uniform rigidity sequence for T_0. We will use Lemma 5.2 to help us form this nested sequence of closed balls. This will be an inductive construction.

To begin let $0 < \epsilon_1 < 1$. The first step is to find $G_1 \in \mathcal{H}(\mathbb{T}^2)$ such that

(1) $d_u(G_1 \circ \sigma \circ G_1^{-1}, \sigma) < \epsilon_1$

(2) $G_1 \circ \sigma \circ G_1^{-1} \in R_{\phi_1}(0.93)$ (ie. $\sigma \in R_{\phi_1 \circ G_1}(0.93) \subseteq R_{\phi_1 \circ G_1}(0.99)$)

The homeomorphism G_1 will have a similar form as the homeomorphism G in the generic argument in Section 4.1. However, this construction is more technical because we need explicit constants in order to use the given growth rate to form our first closed ball B_1. In constructing G_1 we will use Lemma 5.2 to ensure that our oscillatory integral is small. To that end, let $g(y) = \sin(2\pi y)$ be defined on $\mathbb{T}$. The Lipschitz constant for g is 2π and from here on will be denoted by C (ie. $C = 2\pi$). Therefore the modulus of continuity of g is

$$\omega_g(\delta) = \sup_{|x-x'|<\delta} |g(x) - g(x')| \leq C\delta.$$

Suppose the trigonometric polynomial ϕ_1 is of the form

$$\phi_1(x_1, x_2) = \sum_{|\ell_i| \leq N_1} c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}.$$

Recall from the category argument that the homeomorphism G that we define in Section 4.1 is skewed in one variable. Similarly, the homeomorphism G_1 that we construct here will be skewed in one variable. To decide which variable to skew, notice that at least one of the following must occur:

(1) $\sum_{|\ell_2|\leq N_1} |c_{0,\ell_2}|^2 < 0.6$

(2) $\sum_{|\ell_1|\leq N_1} |c_{\ell_1,0}|^2 < 0.6$

WLOG suppose $\sum_{|\ell_2|\leq N_1} |c_{0,\ell_2}|^2 < 0.6$. This means that if we skew G_1 in the first variable we can use the oscillation to ensure that ϕ_1 is not an eigenfunction.

Let $K_1 = 260N_1^9$ (note that this is the K value from Lemma 5.2), $C_1 = 2 + K_1C$, and $\delta_1 = \frac{\epsilon_1}{K_1C}$. If $|x - x'| < \delta_1$, then $|K_1g(x) - K_1g(x')| < \epsilon_1$. Since (n_m) is an increasing sequence, there exists M such that $n_M > \max(\frac{1}{\delta_1}, C_1, (17334\pi)2^{180}M_2^9)$. WLOG suppose that $n_1 > \max(\frac{1}{\delta_1}, C_1, (17334\pi)2^{180}M_2^9)$. Let $G_1 : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by

$$G_1(x_1, x_2) = (x_1 + f_1(x_2), x_2)$$

where

$$f_1(x_2) = K_1g(n_1x_2) = 260N_1^9 \sin(2\pi n_1x_2).$$

The modulus of continuity of G_1 satisfies $\omega_{G_1}(\delta) \leq C_1n_1\delta$.

Now we need to show that $G_1 \in \mathcal{H}(\mathbb{T}^2)$ satisfies (1) (ie. $d_u(G_1\circ\sigma\circ G_1^{-1}, \sigma) < \epsilon_1$). Consider,

$$G_1 \circ \sigma \circ G_1^{-1}(x_1, x_2) - \sigma(x_1, x_2) = (f_1(x_2 + \alpha_2) - f_1(x_2), 0)$$

and

$$G_1 \circ \sigma^{-1} \circ G_1^{-1}(x_1, x_2) - \sigma^{-1}(x_1, x_2) = (f_1(x_2 - \alpha_2) - f_1(x_2), 0).$$

If we use the two statements above, we see that

$$\begin{aligned} d_u(G_1 \circ \sigma \circ G_1^{-1}, \sigma) &= \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, \sigma) + \overline{d}(G_1 \circ \sigma^{-1} \circ G_1^{-1}, \sigma^{-1}) \\ &= |f_1(x_2 + \alpha_2) - f_1(x_2)| + |f_1(x_2 - \alpha_2) - f_1(x_2)| \\ &= |K_1g(n_1x_2 + n_1\alpha_2) - K_1g(n_1x_2)| + |K_1g(n_1x_2 - n_1\alpha_2) - K_1g(n_1x_2)|. \end{aligned}$$

Recall that α_2 was chosen to satisfy $|n_1\alpha_2| < \frac{1}{2(n_1)^2} < \frac{1}{2n_1} < \frac{\delta_1}{2}$. Thus $d_u(G_1 \circ \sigma \circ G_1^{-1}, \sigma) < \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1$ and we have (1).

At this point our first conjugation of σ remains close to σ. Now we need to check that the conjugation of σ we chose belongs to $R_{\phi_1}(0.93)$, that is $\sigma \in R_{\phi_1\circ G_1}(0.93)$. Our goal is to find $t_1 \in \mathbb{N}$ such that $|\langle U_\sigma^{t_1}(\phi_1 \circ G_1)|\phi_1 \circ G_1\rangle| < 0.93$. Recall that in our generic argument in Section 4.1, we only showed the existence of such a t_1 using Wiener's Lemma. For this proof we need to explicitly calculate t_1. This is where the upper and lower bounds on $|n_m\alpha_2|$ come into play.

Consider $t_1 = n_1^2$. Then $\frac{1}{4} < |t_1n_1\alpha_2| < \frac{1}{2}$ and $\frac{1}{4} < \|t_1n_1\alpha_2\| < \frac{3}{4}$ where $\|\cdot\|$ denotes fractional part.

We can now use Lemma 5.2 with $N = N_1$ and $A = \|t_1n_1\alpha_2\|$ to show that

$$\left|\langle U_\sigma^{t_1}(\phi_1 \circ G_1)|\phi_1 \circ G_1\rangle\right| < 0.93.$$

First consider

$$
\begin{aligned}
\left|\langle U_\sigma^{t_1}(\phi_1\circ G_1)|\phi_1\circ G_1\rangle\right| &= \left|\langle(\phi_1\circ G_1)(\sigma^{t_1})|\phi_1\circ G_1\rangle\right| \\
&= \left|\int_{[0,1)^2}\phi_1\circ G_1(x_1+t_1\alpha_1, x_2+t_1\alpha_2)\overline{\phi_1\circ G_1(x_1,x_2)}dx_1dx_2\right| \\
&\le \sum_{|\ell_i|\le N_1,\left|\ell_i'\right|\le N_1}\left|c_{\ell_1,\ell_2}\overline{c}_{\ell_1',\ell_2'}e^{2\pi i(\ell_1 t_1\alpha_1+\ell_2 t_1\alpha_2)}\right| \\
&\cdot\left|\int_{[0,1)^2} e^{2\pi i[(\ell_1-\ell_1')x_1+\ell_1 f_1(x_2+t_1\alpha_2)-\ell_1' f_1(x_2)+(\ell_2-\ell_2')x_2]}dx_1dx_2\right| \\
&\le \sum_{|\ell_i|\le N_1,\left|\ell_2'\right|\le N_1}\left|c_{\ell_1,\ell_2}\right|\left|\overline{c}_{\ell_1,\ell_2'}\right| \\
&\cdot\left|\int_0^1 e^{2\pi i[\ell_1K_1(g(n_1x_2+A)-g(n_1x_2))+(\ell_2-\ell_2')x_2]}dx_2\right| \\
&\le \sum_{\ell_1\neq 0,|\ell_i|\le N_1,\left|\ell_2'\right|\le N_1}\left|\int_0^1 e^{2\pi i[\ell_1K_1(g(n_1x_2+A)-g(n_1x_2))+(\ell_2-\ell_2')x_2]}dx_2\right| \\
&+\sum_{|\ell_2|\le N_1}\left|c_{0,\ell_2}\right|^2 \\
&< \sum_{\ell_1\neq 0,|\ell_i|\le N_1,\left|\ell_2'\right|\le N_1}\left|\int_0^1 e^{2\pi i[\ell_1K_1(g(n_1x_2+A)-g(n_1x_2))+(\ell_2-\ell_2')x_2]}dx_2\right| \\
&+0.6.
\end{aligned}
$$

Notice that

$$\int_0^1 e^{2\pi i[\ell_1K_1(g(n_1x_2+A)-g(n_1x_2))+(\ell_2-\ell_2')x_2]}dx_2$$

is a Fourier coefficient of the function $e^{2\pi i[\ell_1K_1(g(n_1x_2+A)-g(n_1x_2))]}$. Since the set of nonzero Fourier coefficients of $e^{2\pi i[\ell_1K_1(g(n_1x_2+A)-g(n_1x_2))]}$ is the same as that of $e^{2\pi i[\ell_1K_1(g(x_2+A)-g(x_2))]}$ and by Lemma 5.2 $\left|\int_{[0,1)^2} e^{2\pi i[\ell_1K_1(g(x_2+A)-g(x_2))+(\ell_2-\ell_2')x_2]}dx_2\right| < \frac{0.3}{N_1^3}$ we have

$$\left|\langle U_\sigma^{t_1}(\phi_1\circ G_1)|\phi_1\circ G_1\rangle\right| < N_1^3\left(\frac{0.3}{N_1^3}\right)+0.6 < 0.93.$$

Thus (2) is satisfied (ie. $G_1\circ\sigma\circ G_1^{-1}\in R_{\phi_1}(0.93)$).

Now that we have $G_1\circ\sigma\circ G_1^{-1}\in R_{\phi_1}(0.93)$ we proceed with finding a closed ball, which we will call B_1, centered at $G_1\circ\sigma\circ G_1^{-1}$ such that $B_1\subseteq R_{\phi_1}(0.99)$. We need to explicitly calculate the radius of B_1 to ensure that $B_1\subseteq R_{\phi_1}(0.99)$. Let

$$\kappa_1 = \frac{0.06}{2n_1(C_1n_1)^{2n_1^2-1}}$$

and

$$B_1 = \left\{T\in\mathcal{O} : d_u(G_1\circ\sigma\circ G_1^{-1}, T)\le\kappa_1\right\}.$$

Notice that for any $n \in \mathbb{N}$ and $T \in \mathcal{O}$ we have $d_u(T^n, G_1 \circ \sigma^n \circ G_1^{-1}) = \overline{d}(T^n, G_1 \circ \sigma^n \circ G_1^{-1}) + \overline{d}(T^{-n}, G_1 \circ \sigma^{-n} \circ G_1^{-1})$. Consider the following:

$$\begin{aligned}
\overline{d}(T^n, G_1 \circ \sigma^n \circ G_1^{-1}) &= \overline{d}((G_1 \circ \sigma \circ G_1^{-1})(G_1 \circ \sigma^{n-1} \circ G_1^{-1}), T(T^{n-1})) \\
&\le \overline{d}((G_1 \circ \sigma \circ G_1^{-1})(G_1 \circ \sigma^{n-1} \circ G_1^{-1}), (G_1 \circ \sigma \circ G_1^{-1})(T^{n-1})) \\
&+ \overline{d}((G_1 \circ \sigma \circ G_1^{-1})(T^{n-1}), T(T^{n-1})) \\
&\le \omega_{G_1 \circ \sigma \circ G_1^{-1}}(\overline{d}(G_1 \circ \sigma^{n-1} \circ G_1^{-1}, T^{n-1})) + \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T) \\
&\le \sum_{i=0}^{n-1} \omega^i_{G_1 \circ \sigma \circ G_1^{-1}}(\overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T)) \\
&\le \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T) \sum_{i=0}^{n-1} [(C_1 n_1)^2]^i \\
&= \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T) \frac{(C_1 n_1)^{2n} - 1}{(C_1 n_1)^2 - 1} \\
&\le \overline{d}(G_1 \circ \sigma \circ G_1^{-1}, T)(C_1 n_1)^{2n-1}
\end{aligned}$$

where $\omega^0 = Id$. A similar calculation can be carried out to yield $\overline{d}(T^{-n}, G_1 \circ \sigma^{-n} \circ G_1^{-1}) \le \overline{d}(G_1 \circ \sigma^{-1} \circ G_1^{-1}, T^{-1})(C_1 n_1)^{2n-1}$. Thus

$$d_u(T^n, G_1 \circ \sigma^n \circ G_1^{-1}) \le d_u(G_1 \circ \sigma \circ G_1^{-1}, T)(C_1 n_1)^{2n-1}.$$

We will show that $B_1 \subseteq R_{\phi_1}(0.99)$. Let $T \in B_1$. In this case

$$\begin{aligned}
\left|\langle U_T^{t_1}(\phi_1)|\phi_1\rangle\right| &= \left|\langle[\phi_1(T^{t_1}) - \phi_1(G_1 \circ \sigma^{t_1} \circ G_1^{-1})] + \phi_1(G_1 \circ \sigma^{t_1} \circ G_1^{-1})|\phi_1\rangle\right| \\
&\le \left\|\phi_1(T^{t_1}) - \phi_1(G_1 \circ \sigma^{t_1} \circ G_1^{-1})\right\|_2 \|\phi_1\|_2 + \left|\langle U_\sigma^{t_1}(\phi_1 \circ G_1)|\phi_1 \circ G_1\rangle\right| \\
&< M_1 d_u(G_1 \circ \sigma \circ G_1^{-1}, T)(C_1 n_1)^{2t_1 - 1} + 0.93 \\
&\le n_1 \kappa_1 (C_1 n_1)^{2n_1^2 - 1} + 0.93 \\
&< 0.99.
\end{aligned}$$

Hence we have the desired result (ie. $B_1 \subseteq R_{\phi_1}(0.99)$).

Thus far we have constructed the closed ball B_1 centered at $G_1 \circ \sigma \circ G_1^{-1}$ such that $B_1 \subseteq R_{\phi_1}(0.99)$. The next step in our inductive procedure is to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $G_1 \circ G_2 \circ \sigma \circ G_2^{-1} \circ G_1^{-1} \in R_{\phi_2}(0.99) \cap B_1$ and then construct the closed ball B_2 centered at $G_1 \circ G_2 \circ \sigma \circ G_2^{-1} \circ G_1^{-1}$ such that $B_2 \subseteq R_{\phi_2}(0.99) \cap B_1$. To that end, let $\epsilon_2 = \frac{\kappa_1}{2C_1 n_1} < \epsilon_1$. Now similar to before we want to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that

(1) $d_u(G_2 \circ \sigma \circ G_2^{-1}, \sigma) < \epsilon_2$

(2) $G_2 \circ \sigma \circ G_2^{-1} \in R_{\phi_2 \circ G_1}(0.93)$ (ie. $\sigma \in R_{\phi_2 \circ G_1 \circ G_2}(0.93) \subseteq R_{\phi_2 \circ G_1 \circ G_2}(0.99)$)

To proceed as before, let $\phi_2' = \phi_2 \circ G_1$ and suppose that the Fourier expansion has the form:

$$\phi_2'(x_1, x_2) = \sum c_{\ell_1, \ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}.$$

Now we need to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $d_u(G_2 \circ \sigma \circ G_2^{-1}, \sigma) < \epsilon_2$ and $\sigma \in R_{\phi_2' \circ G_2}(0.93)$. The first step is to use an approximation of ϕ_2' in our calculations. We will use Lemma 5.3 and Lemma 5.5 to aid us in this step. Let $N_2 = 2^{20} M_2 C_1 n_1$ (this is the N value from Lemma 5.3) and

$$\widetilde{\phi_2'}(x_1, x_2) = S_{N_2}\phi_2'(x_1, x_2) = \sum_{|\ell_i| \le N_2} c_{\ell_1,\ell_2} e^{2\pi i(\ell_1 x_1 + \ell_2 x_2)}.$$

By Lemma 5.3 $\left\|\widetilde{\phi_2'} - \phi_2'\right\|_2 < 0.01$ and therefore it suffices to find $G_2 \in \mathcal{H}(\mathbb{T}^2)$ such that $\sigma \in R_{\widetilde{\phi_2'} \circ G_2}(0.9)$ by Lemma 5.5.

The next step is to decide which variable to skew. As before, notice that at least one of the following must occur:

(1) $\sum_{|\ell_2| \le N_2} |c_{0,\ell_2}|^2 < 0.6$

(2) $\sum_{|\ell_1| \le N_2} |c_{\ell_1,0}|^2 < 0.6$

WLOG suppose $\sum_{|\ell_1| \le N_2} |c_{\ell_1,0}|^2 < 0.6$. This means that if we skew G_2 in the second variable we can use the oscillation to ensure that ϕ_2 is not an eigenfunction.

Let $K_2 = 260 N_2^9$ (note that this is the K value from Lemma 5.2), $C_2 = 2 + K_2 C$, and $\delta_2 = \frac{\epsilon_2}{K_2 C}$. If $|x - x'| < \delta_2$, then $|K_2 g(x) - K_2 g(x')| < \epsilon_2$. If $n_2 \ge \max(C_2, (17334\pi)2^{180} M_3^9)$ then we proceed similar to before by letting $G_2(x_1, x_2) = (x_1, x_2 + f_2(x_1))$ where $f_2(x_1) = K_2 g(n_2 x_1) = 260 N_2^9 \sin(2\pi n_2 x_1)$. The modulus of continuity of G_2 satisfies $\omega_{G_2}(\delta) \le C_2 n_2 \delta$. If this is not the case then we use ϕ_1 in place of ϕ_i at each stage of our induction until a term of the sequence (n_m) exceeds $\max(C_2, (17334\pi)2^{180} M_3^9)$.

We now proceed with the induction under the assumption that $n_2 \ge \max(C_2, (17334\pi) 2^{180} M_3^9)$. Similar to before, to show that $d_u(G_2 \circ \sigma \circ G_2^{-1}, \sigma) < \epsilon_2$ we need to check that $\frac{1}{2n_2} < \frac{\delta_2}{2}$ or $n_2 > \frac{1}{\delta_2}$. Observe

$$\begin{aligned}
\frac{1}{\delta_2} &= \frac{260 N_2^9 C(2C_1 n_1)}{\kappa_1} \\
&\le ((17334\pi)2^{180} M_2^9)(C_1 n_1)^{10}(2n_1)(C_1 n_1)^{2n_1^2 - 1} \\
&\le n_1(2n_1)(C_1 n_1)^{2n_1^2 + 9} \\
&< n_1^{n_1^3} \cdot n_1 \\
&= \psi(n_1) n_1 \\
&\le n_2.
\end{aligned}$$

Therefore $d_u(G_2 \circ \sigma \circ G_2^{-1}, \sigma) < \epsilon_2$.

Next we need to show that $\sigma \in R_{\widetilde{\phi_2'} \circ G_2}(0.9)$. Our goal is now to find $t_2 \in \mathbb{N}$ such that $\left|\langle U_\sigma^{t_2}(\widetilde{\phi_2'} \circ G_2) | \widetilde{\phi_2'} \circ G_2 \rangle\right| < 0.9$. Let $t_2 = n_2^2$. It follows that $\frac{1}{4} < |t_2 n_2 \alpha_1| < \frac{1}{2}$ and $\frac{1}{4} < \|t_2 n_2 \alpha_1\| < \frac{3}{4}$. We can now use Lemma 5.2 with $N = N_2$ and $A = \|t_2 n_2 \alpha_1\|$. Consider

$$
\begin{aligned}
\left|\langle U_\sigma^{t_2}(\widetilde{\phi_2'}\circ G_2)|\widetilde{\phi_2'}\circ G_2\rangle\right| &= \left|\langle(\widetilde{\phi_2'}\circ G_2)(\sigma^{t_2})|\widetilde{\phi_2'}\circ G_2\rangle\right| \\
&= \left|\int_{[0,1)^2}\widetilde{\phi_2'}\circ G_2(x_1+t_2\alpha_1,x_2+t_2\alpha_2)\overline{\widetilde{\phi_2'}\circ G_2(x_1,x_2)}dx_1dx_2\right| \\
&\leq \sum_{|\ell_i|\leq N_2,\left|\ell_i'\right|\leq N_2}\left|c_{\ell_1,\ell_2}\overline{c}_{\ell_1',\ell_2'}e^{2\pi i(\ell_1t_2\alpha_1+\ell_2t_2\alpha_2)}\right| \\
&\cdot\left|\int_{[0,1)^2}e^{2\pi i[(\ell_1-\ell_1')x_1+\ell_2f_2(x_1+t_2\alpha_1)-\ell_2'f_2(x_1)+(\ell_2-\ell_2')x_2]}dx_1dx_2\right| \\
&\leq \sum_{|\ell_i|\leq N_2,\left|\ell_1'\right|\leq N_2}\left|c_{\ell_1,\ell_2}\right|\left|\overline{c}_{\ell_1',\ell_2}\right| \\
&\cdot\left|\int_0^1 e^{2\pi i[\ell_2K_2(g(n_2x_1+A)-g(n_2x_1))+(\ell_1-\ell_1')x_1]}dx_1\right| \\
&\leq \sum_{\ell_2\neq0,|\ell_i|\leq N_2,\left|\ell_1'\right|\leq N_2}\left|\int_0^1 e^{2\pi i[\ell_2K_2(g(n_2x_1+A)-g(n_2x_1))+(\ell_1-\ell_1')x_1]}dx_1\right| \\
&+\sum_{|\ell_1|\leq N_2}\left|c_{\ell_1,0}\right|^2 \\
&< \sum_{\ell_2\neq0,|\ell_i|\leq N_2,\left|\ell_1'\right|\leq N_2}\left|\int_0^1 e^{2\pi i[\ell_2K_2(g(n_2x_1+A)-g(n_2x_1))+(\ell_1-\ell_1')x_1]}dx_1\right| \\
&+0.6.
\end{aligned}
$$

Similar to before using Lemma 5.2

$$\left|\langle U_\sigma^{t_2}(\widetilde{\phi_2'}\circ G_2)|\widetilde{\phi_2'}\circ G_2\rangle\right| < N_2^3\left(\frac{0.3}{N_2^3}\right)+0.6=0.9.$$

Therefore $\sigma\in R_{\widetilde{\phi_2'}\circ G_2}(0.9)$ which implies that $\sigma\in R_{\phi_2\circ G_1\circ G_2}(0.93)$. Thus (2) is satisfied (ie. $G_2\circ\sigma\circ G_2^{-1}\in R_{\phi_2\circ G_1}(0.93)$).

Recall that our goal for the second step in the inductive procedure is to find $G_2\in\mathcal{H}(\mathbb{T}^2)$ such that $G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1}\in R_{\phi_2}(0.99)\cap B_1$. Thus far we have constructed $G_2\in\mathcal{H}(\mathbb{T}^2)$ such that $G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1}\in R_{\phi_2}(0.99)$. We now need to check that $G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1}\in B_1$. To that end, observe

$$
\begin{aligned}
\overline{d}(G_1\circ G_2\circ\sigma\circ G_2^{-1}\circ G_1^{-1},G_1\circ\sigma\circ G_1^{-1}) &= \overline{d}(G_1(G_2\circ\sigma\circ G_2^{-1}),G_1(\sigma)) \\
&\leq C_1n_1\left(\frac{\epsilon_2}{2}\right) \\
&= C_1n_1\left(\frac{\kappa_1}{4C_1n_1}\right) \\
&= \frac{\kappa_1}{4}.
\end{aligned}
$$

Similarly $\overline{d}(G_1 \circ G_2 \circ \sigma^{-1} \circ G_2^{-1} \circ G_1^{-1}, G_1 \circ \sigma^{-1} \circ G_1^{-1}) \leq \frac{\kappa_1}{4}$ and $d_u(G_1 \circ G_2 \circ \sigma \circ G_2^{-1} \circ G_1^{-1}, G_1 \circ \sigma \circ G_1^{-1}) \leq \frac{\kappa_1}{2}$ which implies that $G_1 \circ G_2 \circ \sigma \circ G_2^{-1} \circ G_1^{-1} \in B_1 \subseteq R_{\phi_1}(0.99)$.

Let $\overline{G_2} := G_1 \circ G_2$. With this new notation we have shown that $\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1} \in R_{\phi_2}(0.99) \cap B_1$. Now we need to find a closed ball, call it B_2, centered at $\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}$ that is a subset of $R_{\phi_2}(0.99) \cap B_1$. Let

$$\kappa_2 = \frac{0.06}{2^2 n_2 (C_1 C_2 n_1 n_2)^{2n_2^2-1}}$$

and

$$B_2 = \left\{T \in \mathcal{O} : d_u(\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}, T) \leq \kappa_2\right\}.$$

We will first show that $B_2 \subseteq R_{\phi_2}(0.99)$. Let $T \in B_2$ and consider

$$\begin{aligned}
\left|\langle U_T^{t_2}(\phi_2)|\phi_2\rangle\right| &= \left|\langle[\phi_2(T^{t_2}) - \phi_2(\overline{G_2} \circ \sigma^{t_2} \circ (\overline{G_2})^{-1})] + \phi_2(\overline{G_2} \circ \sigma^{t_2} \circ (\overline{G_2})^{-1})|\phi_2\rangle\right| \\
&\leq \left\|\phi_2(T^{t_2}) - \phi_2(\overline{G_2} \circ \sigma^{t_2} \circ (\overline{G_2})^{-1})\right\|_2 \|\phi_2\|_2 + \left|\langle U_\sigma^{t_2}(\phi_2 \circ \overline{G_2})|\phi_2 \circ \overline{G_2}\rangle\right| \\
&< M_2 d_u(\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}, T)(C_1 C_2 n_1 n_2)^{2t_2-1} + 0.93 \\
&\leq n_2 \kappa_2 (C_1 C_2 n_1 n_2)^{2n_2^2-1} + 0.93 \\
&< 0.99.
\end{aligned}$$

Hence we have the desired result, that is $B_2 \subseteq R_{\phi_2}(0.99)$.

Next we will show that $B_2 \subseteq B_1$. Let $T \in B_2$ and consider,

$$\begin{aligned}
d_u(T, G_1 \circ \sigma \circ G_1^{-1}) &\leq d_u(T, \overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}) \\
&\quad + d_u(\overline{G_2} \circ \sigma \circ (\overline{G_2})^{-1}, G_1 \circ \sigma \circ G_1^{-1}) \\
&\leq \kappa_2 + \frac{\kappa_1}{2} \\
&< \frac{\kappa_1}{2} + \frac{\kappa_1}{2} \\
&= \kappa_1.
\end{aligned}$$

Therefore $B_2 \subseteq R_{\phi_2}(0.99) \cap B_1$.

Thus far in our inductive procedure we have constructed two closed nested balls $B_1 \supseteq B_2$ centered at conjugations of σ such that $B_1 \subseteq R_{\phi_1}(0.99)$ and $B_2 \subseteq R_{\phi_2}(0.99)$. The general inductive step can be carried out in the same way and for brevity we don't include it here.

In the end, this inductive procedure produces a nested sequence of closed balls (B_m) and a sequence $(\overline{G_m})$ of homeomorphisms where each G_m is of the form

$$G_m(x_1, x_2) = (x_1 + f_m(x_2), x_2)$$

or

$$G_m(x_1, x_2) = (x_1, x_2 + f_m(x_1).$$

After the m-th stage of the construction has been completed we have a homeomorphism G_m in one of the above forms that satisfies:

(1) $d_u(G_m \circ \sigma \circ G_m^{-1}, \sigma) < \epsilon_m$ where $\epsilon_m = \frac{\kappa_{m-1}}{2C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}}$

(2) $G_m \circ \sigma \circ G_m^{-1} \in R_{\phi_m \circ G_1 \circ \cdots \circ G_{m-1}}(0.93)$ or $\overline{G_m} \circ \sigma \circ \left(\overline{G_m}\right)^{-1} \in R_{\phi_m}(0.93)$.

At the end of this stage we also have a closed ball B_m centered at $\overline{G_m} \circ \sigma \circ \left(\overline{G_m}\right)^{-1}$ with radius

$$\kappa_m = \frac{0.06}{2^m n_m \left(C_1 \cdots C_m n_1 \cdots n_m\right)^{2n_m^2 - 1}}$$

such that $B_m \subseteq R_{\phi_m}(0.99)$. Recall that we are working in a complete metric space. Let $T_0 = \bigcap_{m=1}^{\infty} B_m$. Therefore T_0 is weakly mixing. Also, $\overline{G_m} \circ \sigma \circ \left(\overline{G_m}\right)^{-1}$ converges uniformly to T_0 since $\overline{G_m} \circ \sigma \circ \left(\overline{G_m}\right)^{-1}$ is the center of B_m.

Now that we have T_0 which is weakly mixing, we need to show that it is uniformly rigid with respect to (n_m). To do this, we need to make a preliminary estimate. First notice that, depending on which variable is skewed, we have either

$$G_m \circ \sigma^{n_m} \circ G_m^{-1}(x_1, x_2) - \sigma^{n_m}(x_1, x_2) = (K_m g(n_m x_2 + n_m^2 \alpha_2) - K_m g(n_m x_2), 0)$$

or

$$G_m \circ \sigma^{n_m} \circ G_m^{-1}(x_1, x_2) - \sigma^{n_m}(x_1, x_2) = (0, K_m g(n_m x_1 + n_m^2 \alpha_1) - K_m g(n_m x_1)).$$

In either case

$$\left|n_m^2 \alpha_i\right| < \frac{1}{2n_m} < \frac{\delta_m}{2}$$

and we can conclude that $d_u(G_m \circ \sigma^{n_m} \circ G_m^{-1}, \sigma^{n_m}) < \epsilon_m$. Now observe the following:

$$\begin{aligned}
\overline{d}(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) &= \overline{d}(\overline{G_{m-1}}(G_m \circ \sigma^{n_m} \circ G_m^{-1}), \overline{G_{m-1}}(\sigma^{n_m})) \\
&\leq (C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}) \left(\frac{\epsilon_m}{2}\right) \\
&= \frac{\kappa_{m-1}}{4}.
\end{aligned}$$

Hence $d_u(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) \leq \frac{\kappa_{m-1}}{2}$.

The final estimate will show that T_0 is uniformly rigid with respect to (n_m). Observe

$$\begin{aligned}
d_u(T_0^{n_m}, Id) &\leq d_u(T_0^{n_m}, \overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}) \\
&\quad + d_u(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) + d_u(\overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}, Id) \\
&= d_u(T_0^{n_m}, \overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}) + d_u(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) \\
&\quad + \overline{d}(\overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}, Id) + \overline{d}(\overline{G_{m-1}} \circ \sigma^{-n_m} \circ (\overline{G_{m-1}})^{-1}, Id) \\
&= d_u(T_0^{n_m}, \overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}) + d_u(\overline{G_m} \circ \sigma^{n_m} \circ (\overline{G_m})^{-1}, \overline{G_{m-1}} \circ \sigma^{n_m} \circ (\overline{G_{m-1}})^{-1}) \\
&\quad + \overline{d}(\overline{G_{m-1}}(\sigma^{n_m}), \overline{G_{m-1}}(Id)) + \overline{d}(\overline{G_{m-1}}(\sigma^{-n_m}), \overline{G_{m-1}}(Id)) \\
&\leq \kappa_m (C_1 \cdots C_m n_1 \cdots n_m)^{2n_m - 1} + \frac{\kappa_{m-1}}{2} + 2\left(\frac{C_1 \cdots C_{m-1} n_1 \cdots n_{m-1}}{n_m^2}\right) \\
&\leq \left(\frac{0.06}{2^m n_m (C_1 \cdots C_m n_1 \cdots n_m)^{2n_m^2 - 1}}\right) (C_1 \cdots C_m n_1 \cdots n_m)^{2n_m - 1} + \frac{\kappa_{m-1}}{2} \\
&\quad + 2\left(\frac{n_1 n_2 \cdots n_{m-1}}{n_m}\right)^2
\end{aligned}$$

$$\leq \frac{1}{2^m} + \frac{\kappa_{m-1}}{2} + 2\left(\frac{\psi(n_{m-1})}{n_m}\right)^2$$
$$\leq \frac{1}{2^m} + \frac{\kappa_{m-1}}{2} + 2\left(\frac{1}{n_{m-1}}\right)^2.$$

Thus $d_u(T_0^{n_m}, Id) \to 0$ as $m \to \infty$ and T_0 is uniformly rigid with respect to (n_m). Therefore we have constructed a weakly mixing homeomorphism that is uniformly rigid with respect to (n_m).

□

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