International Journal of Mathematical Education in Science and Technology, Vol. 00, No. 00, 15 January 2008, 1–8

Classroom Notes

Simultaneous generalizations of the theorems of Ceva and Menelaus for field planes

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(*v4.1 released November 2007*)

In 1992, Klamkin and Liu proved a very general result in the Extended Euclidean Plane that contains the theorems of Ceva and Menelaus as special cases. In this paper we extend the Klamkin and Liu result to projective planes $PG(2, \mathbb{F})$ where $\mathbb{F}$ is a field.

Keywords: projective plane, field plane, homography

1. Introduction

The theorems of Ceva and Menelaus, two classical results in plane Euclidean geometry, are mentioned in many college geometry books [2,3,5,7]. The ancient Greek astronomer Menelaus of Alexandria formulated conditions under which three points are collinear and the seventeenth century Italian mathematician Giovanni Ceva formulated conditions under which three lines are concurrent. To state these results let $A_1A_2A_3$ be a triangle in the Euclidean plane and let B_1, B_2, B_3 be three points such that B_i lies on the side of $A_1A_2A_3$ opposite vertex A_i for $i = 1, 2, 3$. Ceva's Theorem states that the lines A_1B_1, A_2B_2, and A_3B_3 are concurrent if and only if

$$\frac{\overline{A_3B_1}}{\overline{B_1A_2}} \cdot \frac{\overline{A_1B_2}}{\overline{B_2A_3}} \cdot \frac{\overline{A_2B_3}}{\overline{B_3A_1}} = 1.$$

Menelaus's Theorem states that the points B_1, B_2, and B_3 are collinear if and only if

$$\frac{\overline{A_3B_1}}{\overline{B_1A_2}} \cdot \frac{\overline{A_1B_2}}{\overline{B_2A_3}} \cdot \frac{\overline{A_2B_3}}{\overline{B_3A_1}} = -1.$$

An expression of the form $\overline{PQ}$ refers to the directed or signed distance from P to Q. Figure 1 illustrates both results.

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ISSN: 1045-1129 print/ISSN 1029-0346 online

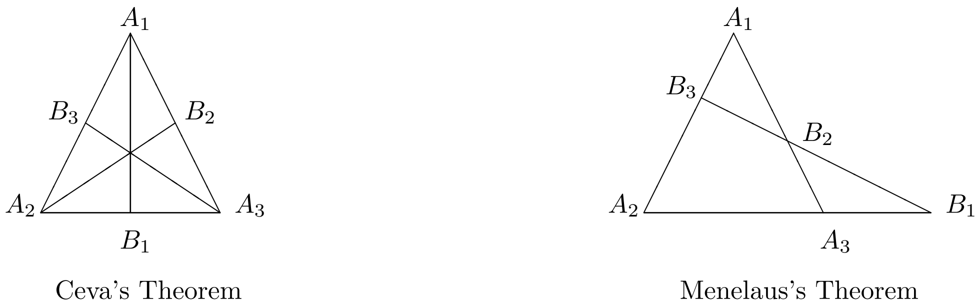

Figure 1.

Even though both theorems have similar looking equations, it is hard to imagine how to formulate a single statement that contains both of them as special cases. In 1992, Klamkin and Liu [8] stated and proved such a result. They worked in the Extended Euclidean plane with the convention that $\frac{\overline{A_iB}}{\overline{BA_j}} = -1$ if B is the ideal point of the line A_iA_j. By avoiding the notion of distance we are able to extend the Klamkin and Liu result to projective planes $PG(2, \mathbb{F})$ where $\mathbb{F}$ is a field. The theorems of Ceva and Menelaus can be considered dual to each other and so the approach given here provides a nice unifying framework.

The next section contains a small amount of background information on plane projective geometry and, in particular, on field planes. The final section includes a statement and proof of the main result.

2. Background

We start with some basic notation and terminology related to field planes. Our main reference is [1].

Definition 2.1 Let $\mathbb{F}$ be a field. The **field plane** $PG(2, \mathbb{F})$ is a triple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ (the elements of $\mathcal{P}$ are called points, the elements of $\mathcal{L}$ are subsets of $\mathcal{P}$ called lines, and $\mathcal{I}$ is an incidence relation) such that
1. $\mathcal{P} = \{(x, y, z) | x, y, z \in \mathbb{F}, \text{ not all zero}\}$ with the restriction that for all $\rho \in \mathbb{F} \setminus \{0\}$, (x, y, z) and $\rho(x, y, z)$ refer to the same point.

2. $\mathcal{L} = \{[a, b, c] | a, b, c \in \mathbb{F}, \text{ not all zero}\}$ with the restriction that for all $\rho \in \mathbb{F} \setminus \{0\}$, $[a, b, c]$ and $\rho[a, b, c]$ refer to the same line.

3. $\mathcal{I}$ is the relation of incidence where the point $P = (x, y, z)$ is incident or lies on the line $[a, b, c]$ if $ax + by + cz = 0$. We refer to $ax + by + cz = 0$ as the equation of the line $[a, b, c]$.

If the field $\mathbb{F}$ equals the set $\mathbb{R}$ of real numbers, then the field plane $PG(2, \mathbb{F})$ is the real projective plane which is isomorphic to the Extended Euclidean Plane. In general, $PG(2, \mathbb{F})$ is a projective plane and so any two distinct points P and Q determine a unique line PQ. If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are two distinct points in $PG(2, \mathbb{F})$ then $R = (x_3, y_3, z_3)$ lies on the line PQ if and only if there exists $\alpha, \beta \in \mathbb{F}$ not both zero and $\rho \in \mathbb{F} \setminus \{0\}$ such that

$$\rho(x_3, y_3, z_3) = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2).$$

Since $\rho(x_3, y_3, z_3)$ refers to the point R we will simply write

$$R = \alpha P + \beta Q.$$

The equation that determines the line PQ can be expressed using a determinant:

$$\det \begin{pmatrix} a & b & c \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = 0.$$

So the three points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, and $R = (x_3, y_3, z_3)$ are collinear if and only if

$$\det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = 0.$$

Dually, the three lines $[a_1, b_1, c_1]$, $[a_2, b_2, c_2]$, and $[a_3, b_3, c_3]$ are concurrent if and only if

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0.$$

The next step is to introduce a special type of mapping of $PG(2, \mathbb{F})$.

Definition 2.2 Let A be an invertible 3×3 matrix with entries from the field $\mathbb{F}$. The mapping $T : PG(2, \mathbb{F}) \to PG(2, \mathbb{F})$ defined by

$$T(x, y, z) = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for all points $(x, y, z) \in PG(2, \mathbb{F})$ is called a **homography** of $PG(2, \mathbb{F})$.

So a homography T of $PG(2, \mathbb{F})$ can be thought of as a linear transformation from $\mathbb{F}^3$ to $\mathbb{F}^3$. Indeed, the linearity of T makes it a well defined mapping of $PG(2, \mathbb{F})$ since (x, y, z) and $\rho(x, y, z)$ represent the same point in $PG(2, \mathbb{F})$. Another consequence of the linearity is that any homography T of $PG(2, \mathbb{F})$ is a *collineation*, i.e., T maps collinear points to collinear points.

If the three points $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, and $R = (x_3, y_3, z_3)$ are non-collinear, then the matrix

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

is invertible. If A is the inverse of the previous matrix and T is the homography of $PG(2, \mathbb{F})$ based on the matrix A, then $T(P) = (1, 0, 0)$, $T(Q) = (0, 1, 0)$, and $T(R) = (0, 0, 1)$. The fact that there is a homography mapping three non-collinear points to the standard points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ will be useful when we prove our theorem.

3. The Main Result

To state our result, let $A_1A_2A_3$ be a triangle in $PG(2, \mathbb{F})$. We introduce six points, two on each side of the triangle $A_1A_2A_3$. Specifically, let B_1, C_1, B_2, C_2, and B_3, C_3 be three pairs of points where each pair lie on the line A_2A_3, A_3A_1, and A_1A_2, respectively. So we get the following expressions:

$$B_1 = \alpha_1 A_2 + \beta_1 A_3, \quad B_2 = \alpha_2 A_1 + \beta_2 A_3, \quad B_3 = \alpha_3 A_1 + \beta_3 A_2$$

and

$$C_1 = \gamma_1 A_2 + \delta_1 A_3, \quad C_2 = \gamma_2 A_1 + \delta_2 A_3, \quad C_3 = \gamma_3 A_1 + \delta_3 A_2.$$

We only require that $B_1 \neq A_2$, $B_2 \neq A_3$, $B_3 \neq A_1$, $C_1 \neq A_3$, $C_2 \neq A_1$, and $C_3 \neq A_2$ which is equivalent to $\beta_1\alpha_2\beta_3\gamma_1\delta_2\gamma_3 \neq 0$. Thus our main result makes sense even for the Fano plane $PG(2, \mathbb{Z}_2)$.

Theorem 3.1 In $PG(2, \mathbb{F})$, the lines C_1B_2, C_2B_3, and C_3B_1 are concurrent if and only if

$$b_1b_2b_3 + c_1c_2c_3 + b_1c_1 + b_2c_2 + b_3c_3 = 1 \tag{1}$$

where $b_1 = \alpha_1/\beta_1$, $b_2 = \beta_2/\alpha_2$, $b_3 = \alpha_3/\beta_3$, $c_1 = \delta_1/\gamma_1$, $c_2 = \gamma_2/\delta_2$, and $c_3 = \delta_3/\gamma_3$.

Before we prove the theorem we will give an example in $PG(2, \mathbb{Z}_5)$ to illustrate the result.

Example 3.2

For this example, the vertices of the initial triangle are $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, and $A_3 = (0, 0, 1)$. Consider the following points in $PG(2, \mathbb{Z}_5)$:

$$B_1 = (0, 1, 2), \quad B_2 = (1, 0, 3), \quad B_3 = (1, 2, 0)$$

and

$$C_1 = (0, 1, 3), \quad C_2 = (1, 0, 2), \quad C_3 = (1, 3, 0)$$

Note that $B_1, C_1 \in A_2A_3(x = 0)$, $B_2, C_2 \in A_3A_1(y = 0)$, and $B_3, C_3 \in A_1A_2(z = 0)$. Also, notice that

$$C_1B_2 : 3x + 3y + 4z = 0, \quad C_2B_3 : x + 2y + 2z = 0, \quad C_3B_1 : x + 3y + z = 0.$$

and that

$$\det \begin{pmatrix} 3 & 3 & 4 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix} = 3(2 - 6) - 3(1 - 2) + 4(3 - 2) =$$

$$= 3(-4) - 3(-1) + 4(1) = -12 + 3 + 4 = -5 = 0 (\text{mod } 5).$$

Thus the lines C_1B_2, C_2B_3, C_3B_1 are concurrent in $PG(2, \mathbb{Z}_5)$. We now check that equation (1) given in Theorem 1 is satisfied. Observe that

$$B_1 = A_2 + 2A_3 \Rightarrow b_1 = \frac{1}{2} = 1 \cdot 2^{-1} = 1 \cdot 3 = 3.$$

In a similar manner, $b_2 = 3$, $b_3 = 3$, $c_1 = 3$, $c_2 = 3$, and $c_3 = 3$. If we substitute these values into equation (1), then we obtain

$$(3)(3)(3) + (3)(3)(3) + (3)(3) + (3)(3) + (3)(3) = 81 = 1 (\text{mod } 5).$$

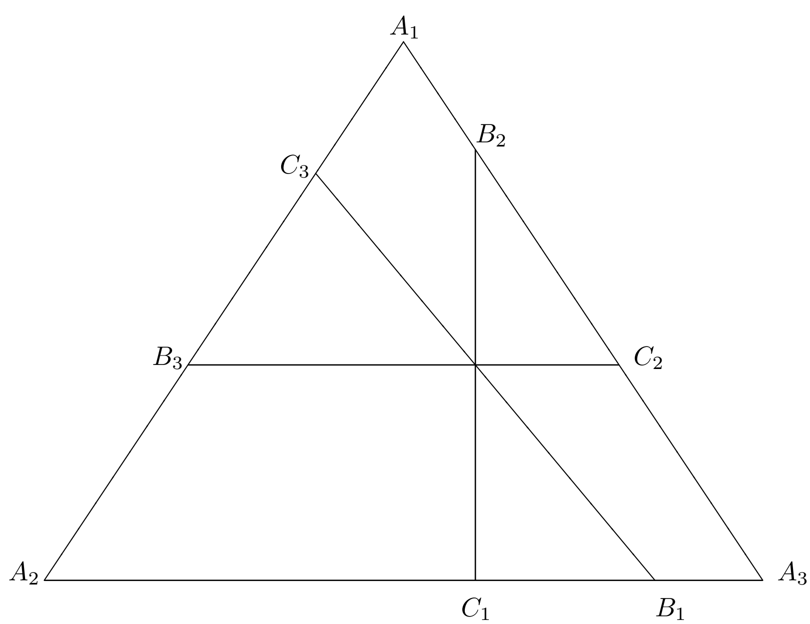

Figure 2. C_1B_2, C_2B_3, C_3B_1 are concurrent

Proof:

As mentioned above, there exists a homography T of $PG(2,\mathbb{F})$ such that the three non-collinear points A_1, A_2, and A_3 can be mapped to the standard points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Therefore, we may assume that $A_1=(1,0,0)$, $A_2=(0,1,0)$, and $A_3=(0,0,1)$. Note that the equations of the lines A_1A_2, A_2A_3, and A_3A_1 are $z=0$, $x=0$, and $y=0$, respectively. Also notice that b_1, b_2, b_3, c_1, c_2, and c_3 are preserved under the homography T by the linearity of T. Thus $B_1=(0,\alpha_1,\beta_1)$, $B_2=(\alpha_2,0,\beta_2)$, $B_3=(\alpha_3,\beta_3,0)$, $C_1=(0,\gamma_1,\delta_1)$, $C_2=(\gamma_2,0,\delta_2)$, and $C_3=(\gamma_3,\delta_3,0)$.

Next we will compute the equations of the lines that we want to show are concurrent. Notice that the line C_1B_2 has the following equation:

$$\det\begin{pmatrix} x & y & z \\ 0 & \gamma_1 & \delta_1 \\ \alpha_2 & 0 & \beta_2 \end{pmatrix} = \beta_2\gamma_1 x + \alpha_2\delta_1 y - \alpha_2\gamma_1 z = 0.$$

In a similar manner the line C_2B_3 has equation

$$-\beta_3\delta_2 x + \alpha_3\delta_2 y + \beta_3\gamma_2 z = 0$$

and the line C_3B_1 has equation

$$\beta_1\delta_3 x - \beta_1\gamma_3 y + \alpha_1\gamma_3 z = 0.$$

Therefore, the lines C_1B_2, C_2B_3, and C_3B_1 are concurrent if and only if

$$\det\begin{pmatrix} \beta_2\gamma_1 & \alpha_2\delta_1 & -\alpha_2\gamma_1 \\ -\beta_3\delta_2 & \alpha_3\delta_2 & \beta_3\gamma_2 \\ \beta_1\delta_3 & -\beta_1\gamma_3 & \alpha_1\gamma_3 \end{pmatrix} = 0.$$

After calculating and simplifying the previous determinant one obtains equation (1) and the proof of the theorem is complete.

□

It should be immediately noted that the Klamkin and Liu result is Theorem 3.1 with $\mathbb{F}=\mathbb{R}$.

There are two important consequences of Theorem 1. First, consider the situation where $C_1=A_2$, $C_2=A_3$, and $C_3=A_1$. This means that $\delta_1=0$, $\gamma_2=0$, and $\delta_3=0$ and so $c_1=c_2=c_3=0$. Therefore, equation (1) reduces to $b_1b_2b_3=1$. Moreover, we get $C_1B_2=A_2B_2$, $C_2B_3=A_3B_3$, and $C_3B_1=A_1B_1$.

Corollary 3.3 In $PG(2,\mathbb{F})$, the lines A_2B_2, A_3B_3, and A_1B_1 are concurrent if and only if

$$b_1b_2b_3=1$$

where $b_1=\alpha_1/\beta_1$, $b_2=\beta_2/\alpha_2$, $b_3=\alpha_3/\beta_3$.

The second situation is where $B_1 = C_1$, $B_2 = C_2$, and $B_3 = C_3$. Then $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ for $i = 1, 2, 3$. In this case, equation (1) reduces to $b_1b_2b_3 = -1$. Notice that $C_1B_2 = B_1B_2$, $C_2B_3 = B_2B_3$, and $C_3B_1 = B_3B_1$. Thus these three lines are concurrent if and only if the points B_1, B_2, and B_3 are collinear.

Corollary 3.4 In $PG(2,\mathbb{F})$, the points B_1, B_2, and B_3 are collinear if and only if

$$b_1b_2b_3 = -1$$

where $b_1 = \alpha_1/\beta_1$, $b_2 = \beta_2/\alpha_2$, $b_3 = \alpha_3/\beta_3$.

These two corollaries are versions of Ceva and Menelaus's theorems for field planes. For the classical versions of these theorems let $\mathbb{F}$ be the field of real numbers $\mathbb{R}$ and assume that the points A_1, A_2, A_3, B_1, B_2, B_3 are ordinary points in the xy-plane. We now show that $b_1 = \frac{\overline{A_3B_1}}{\overline{B_1A_2}}$, which was motivated by [4]. Since B_1, A_2, A_3 are collinear it follows that the vectors $\overrightarrow{A_3B_1}$ and $\overrightarrow{B_1A_2}$ are parallel. So there exists a real number k such that

$$(B_1 - A_3) = k(A_2 - B_1).$$

Using the Euclidean distance formula and keeping track of signs we can solve for k to get

$$k = \frac{\overline{A_3B_1}}{\overline{B_1A_2}}.$$

Therefore,

$$\overline{B_1A_2}(B_1 - A_3) = \overline{A_3B_1}(A_2 - B_1).$$

Since $\overline{A_3B_1} + \overline{B_1A_2} = \overline{A_3A_2}$ it follows that

$$\overline{A_3A_2}\ B_1 = \overline{A_3B_1}\ A_2 + \overline{B_1A_2}\ A_3$$

and so

$$B_1 = \frac{\overline{A_3B_1}}{\overline{A_3A_2}}\ A_2 + \frac{\overline{B_1A_2}}{\overline{A_3A_2}}\ A_3.$$

In $PG(2,\mathbb{R})$, $B_1 = \alpha_1A_2 + \beta_1A_3$ and so there exists a nonzero real number ρ such that

$$\rho(\alpha_1A_2 + \beta_1A_3) = \frac{\overline{A_3B_1}}{\overline{A_3A_2}}\ A_2 + \frac{\overline{B_1A_2}}{\overline{A_3A_2}}\ A_3.$$

Now A_2 and A_3 are not scalar multiples of each other since they are two distinct points in $PG(2,\mathbb{R})$. Therefore, $\rho\alpha_1 = \frac{\overline{A_3B_1}}{\overline{A_3A_2}}$ and $\rho\beta_1 = \frac{\overline{B_1A_2}}{\overline{A_3A_2}}$. Hence $b_1 = \frac{\alpha_1}{\beta_1} = \frac{\overline{A_3B_1}}{\overline{B_1A_2}}$. In a similar way we see that $b_2 = \frac{\overline{A_1B_2}}{\overline{B_2A_3}}$ and $b_3 = \frac{\overline{A_2B_3}}{\overline{B_3A_1}}$. In sum, for points in the Euclidean plane, the product $b_1b_2b_3$ is the product of ratios of directed distances and we get the standard conclusions of Ceva and Menelaus's theorems.

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