

## ON A FUNCTIONAL EQUATION RELATED TO THE DETERMINANT OF SYMMETRIC TWO-BY-TWO MATRICES

KELLY B. HOUSTON AND PRASANNA K. SAHOO

ABSTRACT. The present work aims to find the solutions  $f, g, h, \ell, m : \mathbb{R}^2 \rightarrow \mathbb{R}$  of the functional equation  $f(ux - vy, uy - vx) = g(x, y) + h(u, v) + \ell(x, y)m(u, v)$  for all  $x, y, u, v \in \mathbb{R}$  without any regularity assumptions on the unknown functions. This equation is a generalization of a functional equation which arises from the characterization of the determinant of symmetric matrices.

### 1. INTRODUCTION

Let us define  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$f(x, y) = \det \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

for all  $x, y \in \mathbb{R}$ . Then, since

$$\det \begin{pmatrix} ux - vy & uy - vx \\ uy - vx & ux - vy \end{pmatrix} = \det \begin{pmatrix} x & y \\ y & x \end{pmatrix} \det \begin{pmatrix} u & v \\ v & u \end{pmatrix},$$

we have the functional equation

$$f(ux - vy, uy - vx) = f(x, y) f(u, v) \tag{1}$$

for all  $x, y, u, v \in \mathbb{R}$ . Obviously,  $f(x, y) = x^2 - y^2$  is a solution of the functional equation (1). A functional equation similar to (1) was studied in [2]. In the solved and unsolved problems column of the *News Letter* of the European Mathematical Society, the second author [6] posed the following problem: Determine the general solutions  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  of the functional equation

$$f(ux - vy, uy - vx) = f(x, y) + f(u, v) + f(x, y) f(u, v) \tag{2}$$

---

2000 *Mathematics Subject Classification*. Primary 39B22.

*Key words and phrases*. Determinant, functional equation, logarithmic function, multiplicative function.

for all  $x, y, u, v \in \mathbb{R}$ . In [4], among others we gave the general solution of the functional equation (2) without any regularity assumptions on  $f$ . A generalization of the functional equations (1) and (2) is the following:

$$f(ux - vy, uy - vx) = g(x, y) + h(u, v) + \ell(x, y) m(u, v) \quad (3)$$

for all  $x, y, u, v \in \mathbb{R}$ . Here  $f, g, h, \ell, m : \mathbb{R}^2 \rightarrow \mathbb{R}$  are unknown functions to be determined. For an account on the subject of functional equations the interested reader should refer to [1], [3], [5], [7], [8], and [9].

In this paper, we determine the general solution of the functional equation (3) without any regularity assumptions on the unknown functions  $f, g, h, \ell, m : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Our method of solution is elementary and direct.

## 2. PRELIMINARY RESULTS

A function  $M : \mathbb{R} \rightarrow \mathbb{R}$  is said to be a *multiplicative function* if and only if it satisfies  $M(xy) = M(x)M(y)$  for all  $x, y \in \mathbb{R}$ . An identically constant multiplicative function  $M$  is either  $M = 0$  or  $M = 1$ .

Let  $D$  be an interval in  $\mathbb{R}$  such that whenever  $x, y \in D$ , then  $xy \in D$ . A function  $L : D \rightarrow \mathbb{R}$  is said to be a *logarithmic function* if and only if  $L(xy) = L(x) + L(y)$  for all  $x, y \in D$ . Note that if  $0 \in D$ , then  $L = 0$ .

**Lemma 1.** *Let  $D \subseteq \mathbb{R}$  be an interval such that if  $x, y \in D$ , then  $xy \in D$ . The general solution  $f : D^2 \rightarrow \mathbb{R}$  of the functional equation*

$$f(x_1y_2, x_2y_1) = f(x_1, y_1) + f(y_2, x_2) \quad (4)$$

holding for all  $x_1, x_2, y_1, y_2 \in D$  is given by

$$f(x, y) = L_1(x) + L_2(y), \quad (5)$$

where  $L_1, L_2 : D \rightarrow \mathbb{R}$  are logarithmic functions. If  $D = \mathbb{R}$ , then  $f(x, y) \equiv 0$  is the only solution to functional equation (4).

*Proof.* It is easy to check that the solution enumerated in (5) satisfies functional equation (4). Next, we show that (5) is the only solution of (4).

Suppose  $f$  is identically a constant, say  $f \equiv c$ . Then from (4) we have  $c = 0$ . Hence the identically constant solution of (4) is  $f(x, y) = 0$  for all  $x, y \in D$ , which is a solution included in (5).

From now on we assume that  $f$  is not identically constant. Let  $a \in D$  be a fixed element and  $f : D^2 \rightarrow \mathbb{R}$  be such that it satisfies (4). Then

$$\begin{aligned} f(x, y) &= f(x, y) + f(a, a) + f(a, a) - 2f(a, a) \\ &= f(xaa, aay) - 2f(a, a) \\ &= f((xa)a, (ya)a) - 2f(a, a) \\ &= f(xa, a) + f(a, ya) - 2f(a, a) \\ &= f(xa, a) - f(a, a) + f(a, ya) - f(a, a) \end{aligned}$$

$$= L_1(x) + L_2(y)$$

where

$$L_1(x) := f(xa, a) - f(a, a)$$

and

$$L_2(y) := f(a, ya) - f(a, a).$$

Now we show that  $L_1$  and  $L_2$  are logarithmic functions in  $D$ . Consider

$$\begin{aligned} L_1(xy) &= f(xya, a) - f(a, a) \\ &= f(xya, a) + f(a, a) - 2f(a, a) \\ &= f(xyaa, aa) - 2f(a, a) \\ &= f(xa, a) + f(ya, a) - 2f(a, a) \\ &= f(xa, a) - f(a, a) + f(ya, a) - f(a, a) \\ &= L_1(x) + L_1(y). \end{aligned}$$

Hence  $L_1$  is logarithmic. Similarly, one can show that

$$L_2(xy) = L_2(x) + L_2(y)$$

and hence  $L_2$  is logarithmic. Thus

$$f(x, y) = L_1(x) + L_2(y)$$

where  $L_1$  and  $L_2$  are logarithmic functions. □

**Lemma 2.** *Let  $D \subseteq \mathbb{R}$  be an interval such that if  $x, y \in D$ , then  $xy \in D$ . The general solution  $f : D^2 \rightarrow \mathbb{R}$  of the functional equation*

$$f(x_1y_2, x_2y_1) = f(x_1, y_1) f(y_2, x_2) \tag{6}$$

holding for all  $x_1, x_2, y_1, y_2 \in D$  is given by

$$f(x, y) = M_1(x) M_2(y), \tag{7}$$

where  $M_1, M_2 : D \rightarrow \mathbb{R}$  are multiplicative functions.

*Proof.* It is easy to check that the solution enumerated in (7) satisfies functional equation (6). Next, we will show that (7) is the only solution of (6).

Suppose  $f$  is identically a constant, say  $f \equiv c$ . Then from (6) we have  $c^2 - c = 0$  for all  $x, y \in D$ . Hence  $c = 1$  or  $c = 0$ . Thus the only identically constant solutions of (6) are  $f(x, y) = 1$  and  $f(x, y) = 0$  for all  $x, y \in D$ , which are solutions included in (7).

From now on we assume that  $f$  is not identically constant. Let  $a \in D$  be a fixed element and  $f : D^2 \rightarrow \mathbb{R}$  be such that it satisfies (6) with  $f(a, a) \neq 0$ . Then

$$\begin{aligned} f(x, y) &= f(x, y) f(a, a) f(a, a) f(a, a)^{-2} \\ &= f(xaa, aay) f(a, a)^{-2} \end{aligned}$$

$$\begin{aligned}
&= f((xa)a, (ya)a) f(a, a)^{-2} \\
&= f(xa, a) f(a, ya) f(a, a)^{-2} \\
&= f(xa, a) f(a, a)^{-1} f(a, ya) f(a, a)^{-1} \\
&= M_1(x) M_2(y)
\end{aligned}$$

where

$$M_1(x) := f(xa, a) f(a, a)^{-1}$$

and

$$M_2(y) := f(a, ya) f(a, a)^{-1}.$$

Now we show that  $M_1$  and  $M_2$  are multiplicative functions in  $D$ . Consider

$$\begin{aligned}
M_1(xy) &= f(xya, a) f(a, a)^{-1} \\
&= f(xya, a) f(a, a) f(a, a)^{-2} \\
&= f(xyaa, aa) f(a, a)^{-2} \\
&= f(xa, a) f(ya, a) f(a, a)^{-2} \\
&= f(xa, a) f(a, a)^{-1} f(ya, a) f(a, a)^{-1} \\
&= M_1(x) M_1(y).
\end{aligned}$$

Hence  $M_1$  is multiplicative. Similarly, one can show that

$$M_2(xy) = M_2(x) M_2(y)$$

and hence  $M_2$  is multiplicative. Thus

$$f(x, y) = M_1(x) M_2(y)$$

where  $M_1$  and  $M_2$  are multiplicative functions. □

**Remark 1.** Note that if we replace  $D$  by a commutative semigroup in Lemma 1 and Lemma 2, then both of the lemmas are still valid.

**Lemma 3.** *The general solution  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  of the functional equation*

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + g(x_1, y_1) g(x_2, y_2) \quad (8)$$

holding for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$  is given by

$$\begin{cases} f(x, y) = \delta^2 [M(xy) - 1] \\ g(x, y) = \delta [M(xy) - 1], \end{cases} \quad (9)$$

where  $M : \mathbb{R} \rightarrow \mathbb{R}$  is a multiplicative function and  $\delta$  is an arbitrary constant.

*Proof.* It is easy to check that the solution enumerated in (9) satisfies functional equation (8). Next, we will show that (9) is the only solution of (8).

Suppose  $g$  is identically constant, say  $g \equiv -\delta$ . Then from (8) we have that

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + \delta^2 \quad (10)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ . Now we define a function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$F(x, y) := f(x, y) + \delta^2 \quad (11)$$

for all  $x, y \in \mathbb{R}$ . Then (10) becomes

$$F(x_1 y_2, x_2 y_1) = F(x_1, y_1) + F(y_2, x_2) \quad (12)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ . From Lemma 1, we see that the solution of (12) is

$$F(x, y) = 0 \quad (13)$$

for all  $x, y \in \mathbb{R}$ . Hence

$$f(x, y) = -\delta^2 \quad (14)$$

for all  $x, y \in \mathbb{R}$ , which is a solution included in (9).

From now on we assume that  $g$  is not identically constant. Interchanging  $x_1$  with  $y_2$  and  $x_2$  with  $y_1$  in (8) and comparing the resulting equations with (8) we see that

$$g(x_1, y_1) g(x_2, y_2) = g(y_1, x_1) g(y_2, x_2) \quad (15)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ . Since  $g$  is non-constant, there exists  $x_0, y_0 \in \mathbb{R}$  such that  $g(y_0, x_0) \neq 0$ . Letting  $x_2 = x_0$  and  $y_2 = y_0$  in (15) we obtain

$$g(y, x) = \alpha g(x, y) \quad (16)$$

for all  $x, y \in \mathbb{R}$  where  $\alpha$  is a constant and  $\alpha \neq 0$  (since otherwise  $g \equiv 0$ ).

Setting  $x_2 = y_2 = 0$  in (8) we obtain

$$f(0, 0) = f(x_1, y_1) + f(0, 0) + g(x_1, y_1) g(0, 0) \quad (17)$$

for all  $x_1, y_1 \in \mathbb{R}$ . Thus,

$$f(x, y) = \alpha_0 g(x, y) \quad (18)$$

for all  $x, y \in \mathbb{R}$ . Again note that  $\alpha_0 \neq 0$  since otherwise  $f \equiv 0$  and hence  $g \equiv 0$  by (8). Therefore,

$$g(x, y) = k f(x, y) \quad (19)$$

for all  $x, y \in \mathbb{R}$  where the constant  $k \neq 0$ .

Now, using (19) in (8) we see that

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + k^2 f(x_1, y_1) f(x_2, y_2) \quad (20)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ . Using (16) and (19) we see that

$$k f(x, y) = g(x, y) = \alpha g(y, x) = \alpha k f(y, x)$$

which gives us

$$f(x, y) = \alpha f(y, x) \quad (21)$$

for all  $x, y \in \mathbb{R}$ .

Next, using (20) and (21) we have

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + \alpha k^2 f(x_1, y_1) f(y_2, x_2) \quad (22)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ . Define  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$F(x, y) := \alpha k^2 f(x, y) + 1 \quad (23)$$

for all  $x, y \in \mathbb{R}$ . Then using (22) and (23) we have

$$F(x_1 y_2, x_2 y_1) = F(x_1, y_1) F(y_2, x_2) \quad (24)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ .

From Lemma 2 we obtain

$$F(x, y) = M_1(x) M_2(y) \quad (25)$$

for all  $x, y \in \mathbb{R}$ . Thus from (19) and (23) we have

$$\begin{cases} f(x, y) = \frac{1}{\alpha k^2} [M_1(x) M_2(y) - 1] \\ g(x, y) = \frac{1}{\alpha k} [M_1(x) M_2(y) - 1] \end{cases} \quad (26)$$

for all  $x, y \in \mathbb{R}$ .

Substituting (26) back into (8) we see that  $\alpha = 1$  and  $M_1 = M_2 = M$  for all  $x \in \mathbb{R}$ . Thus if we write  $\delta = 1/k$ , (26) becomes

$$\begin{cases} f(x, y) = \delta^2 [M(xy) - 1] \\ g(x, y) = \delta [M(xy) - 1] \end{cases} \quad (27)$$

for all  $x, y \in \mathbb{R}$ , which is a solution to (8). This completes the proof of the lemma.  $\square$

**Lemma 4.** *The general solutions  $f, \ell, m : \mathbb{R}^2 \rightarrow \mathbb{R}$  of the functional equation*

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + \ell(x_1, y_1) m(x_2, y_2) \quad (28)$$

holding for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$  are given by

$$\begin{cases} f(x, y) \equiv 0 \\ \ell(x, y) m(u, v) = 0 \end{cases} \quad (29)$$

and

$$\begin{cases} f(x, y) = \frac{1}{k_1 k_2} [M_1(x) M_2(y) - 1] \\ \ell(x, y) = \frac{1}{k_2} [M_1(x) M_2(y) - 1] \\ m(x, y) = \frac{1}{k_1} [M_1(y) M_2(x) - 1], \end{cases} \quad (30)$$

where  $M_1, M_2 : \mathbb{R} \rightarrow \mathbb{R}$  are multiplicative functions and  $k_1$  and  $k_2$  are arbitrary nonzero constants.

*Proof.* It is easy to check that the solutions enumerated in (29)-(30) satisfy functional equation (28). Next, we will show that (29)-(30) are the only solutions of (28).

Suppose  $\ell(x, y) m(u, v) = 0$ . Then (28) becomes

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) \quad (31)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ . From (31) and Lemma 1 we see that

$$f(x, y) = 0 \quad (32)$$

for all  $x, y \in \mathbb{R}$ , which is a solution to (28).

Now, suppose  $\ell(x, y) = m(x, y)$  for all  $x, y \in \mathbb{R}$ . Then, (28) becomes

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + \ell(x_1, y_1) \ell(x_2, y_2) \quad (33)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ . From (33) and Lemma 3 we see that

$$\begin{cases} f(x, y) = \delta^2 [M(xy) - 1] \\ \ell(x, y) = \delta [M(xy) - 1] \\ m(x, y) = \delta [M(xy) - 1] \end{cases} \quad (34)$$

for all  $x, y \in \mathbb{R}$  where  $M : \mathbb{R} \rightarrow \mathbb{R}$  is a multiplicative function, which is a solution to (28) and a special case of (30).

Finally, suppose  $\ell(x, y) \neq m(x, y)$  and neither is identically zero. Setting  $y_2 = x_2 = 0$  in (28) we see that

$$f(0, 0) = f(x_1, y_1) + f(0, 0) + \ell(x_1, y_1) m(0, 0) \quad (35)$$

for all  $x_1, y_1 \in \mathbb{R}$ . Thus,

$$\ell(x, y) = k_1 f(x, y) \quad (36)$$

for all  $x, y \in \mathbb{R}$  where  $k_1 \neq 0$  (since otherwise  $\ell \equiv 0$ ).

Similarly, setting  $y_1 = x_1 = 0$  in (28) we see that

$$f(0, 0) = f(0, 0) + f(y_2, x_2) + \ell(0, 0) m(x_2, y_2) \quad (37)$$

for all  $x_2, y_2 \in \mathbb{R}$ . Thus,

$$m(x, y) = k_2 f(y, x) \quad (38)$$

for all  $x, y \in \mathbb{R}$  where  $k_2 \neq 0$  (since otherwise  $m \equiv 0$ ).

Now using (28), (36), and (38) we have

$$f(x_1 y_2, x_2 y_1) = f(x_1, y_1) + f(y_2, x_2) + k_1 k_2 f(x_1, y_1) f(y_2, x_2) \quad (39)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ . Define a function  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$H(x, y) := k_1 k_2 f(x, y) + 1 \quad (40)$$

for all  $x, y \in \mathbb{R}$ . Then (39) becomes

$$H(x_1 y_2, x_2 y_1) = H(x_1, y_1) H(y_2, x_2) \quad (41)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ .

From Lemma 2 we get

$$H(x, y) = M_1(x) M_2(y) \quad (42)$$

for all  $x, y \in \mathbb{R}$ . Thus using (36), (38), (40), and (42) we see that

$$\begin{cases} f(x, y) = \frac{1}{k_1 k_2} [M_1(x) M_2(y) - 1] \\ \ell(x, y) = \frac{1}{k_2} [M_1(x) M_2(y) - 1] \\ m(x, y) = \frac{1}{k_1} [M_1(y) M_2(x) - 1], \end{cases} \quad (43)$$

for all  $x, y \in \mathbb{R}$  where  $M_1, M_2 : \mathbb{R} \rightarrow \mathbb{R}$  are multiplicative functions, which is a solution to (28).  $\square$

### 3. MAIN RESULT

Now we are ready to prove our main result.

**Theorem 1.** *The general solutions  $f, g, h, \ell, m : \mathbb{R}^2 \rightarrow \mathbb{R}$  of the functional equation*

$$f(ux - vy, uy - vx) = g(x, y) + h(u, v) + \ell(x, y) m(u, v) \quad (44)$$

holding for all  $x, y, u, v \in \mathbb{R}$  are given by

$$\begin{cases} f(x, y) \equiv \alpha_1 \alpha_2 - \beta_1 - \beta_2 \\ g(x, y) = \alpha_1 \ell(x, y) - \beta_2 + \alpha_1 \alpha_2 \\ h(x, y) = -\beta_1 \\ \ell(x, y) \text{ is arbitrary} \\ m(x, y) \equiv -\alpha_1 \end{cases} \quad (45)$$

$$\begin{cases} f(x, y) \equiv \alpha_1 \alpha_2 - \beta_1 - \beta_2 \\ g(x, y) \equiv -\beta_2 \\ h(x, y) = \alpha_2 m(x, y) - \beta_1 + \alpha_1 \alpha_2 \\ \ell(x, y) \equiv -\alpha_2 \\ m(x, y) \text{ is arbitrary} \end{cases} \quad (46)$$

and

$$\begin{cases} f(x, y) = \frac{1}{k_1 k_2} [M_1(x+y)M_2(x-y) - 1] + \alpha_1 \alpha_2 - \beta_1 - \beta_2 \\ g(x, y) = \frac{1}{k_1 k_2} [M_1(x+y)M_2(x-y) - 1] \\ \quad + \frac{\alpha_1}{k_2} [M_1(x+y)M_2(x-y) - 1] - \beta_2 \\ h(x, y) = \frac{1}{k_1 k_2} [M_1(x+y)M_2(x-y) - 1] \\ \quad + \frac{\alpha_2}{k_1} [M_1(x-y)M_2(x+y) - 1] - \beta_1 \\ \ell(x, y) = \frac{1}{k_2} [M_1(x+y)M_2(x-y) - 1] - \alpha_2 \\ m(x, y) = \frac{1}{k_1} [M_1(x-y)M_2(x+y) - 1] - \alpha_1, \end{cases} \quad (47)$$

where  $M_1, M_2 : \mathbb{R} \rightarrow \mathbb{R}$  are multiplicative functions,  $\alpha_1, \alpha_2, \beta_2$  and  $\beta_1$  are arbitrary constants, and  $k_1$  and  $k_2$  are nonzero arbitrary constants.



*Proof.* It is easy to check that the solutions enumerated in (45)-(47) satisfy functional equation (44). Next, we will show that (45)-(47) are the only solutions of (44).

First, we define functions  $F, G, H, L, N : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$\begin{cases} F(x, y) := f\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ G(x, y) := g\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ H(x, y) := h\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ L(x, y) := \ell\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \\ N(x, y) := m\left(\frac{x+y}{2}, \frac{x-y}{2}\right) \end{cases} \quad (48)$$

for all  $x, y \in \mathbb{R}$ . Using (48) in (44) we see that

$$\begin{aligned} & F((x+y)(u-v), (x-y)(u+v)) \\ &= G(x+y, x-y) + H(u+v, u-v) + L(x+y, x-y) N(u+v, u-v) \end{aligned} \quad (49)$$

for all  $x, y, u, v \in \mathbb{R}$ . Letting  $x_1 = x+y$ ,  $y_1 = x-y$ ,  $x_2 = u+v$ , and  $y_2 = u-v$  we have

$$F(x_1 y_2, x_2 y_1) = G(x_1, y_1) + H(x_2, y_2) + L(x_1, y_1) N(x_2, y_2) \quad (50)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ .

Setting  $x_2 = y_2 = 1$  in (50) we have

$$F(x_1, y_1) = G(x_1, y_1) + H(1, 1) + L(x_1, y_1) N(1, 1) \quad (51)$$

for all  $x_1, y_1 \in \mathbb{R}$ . Simplifying we get

$$F(x_1, y_1) = G(x_1, y_1) - \alpha_1 L(x_1, y_1) - \beta_1 \quad (52)$$

for all  $x_1, y_1 \in \mathbb{R}$  where  $\alpha_1$  and  $\beta_1$  are arbitrary constants.

Similarly, setting  $x_1 = y_1 = 1$  in (50) we have

$$F(y_2, x_2) = G(1, 1) + H(x_2, y_2) + L(1, 1) N(x_2, y_2) \quad (53)$$

for all  $x_2, y_2 \in \mathbb{R}$ . Simplifying we get

$$F(y_2, x_2) = H(x_2, y_2) - \alpha_2 N(x_2, y_2) - \beta_2 \quad (54)$$

for all  $x_2, y_2 \in \mathbb{R}$  where  $\alpha_2$  and  $\beta_2$  are arbitrary constants.

Then using (50), (52), and (54) we have

$$\begin{aligned} F(x_1 y_2, x_2 y_1) - \beta &= F(x_1, y_1) + F(y_2, x_2) + \alpha_1 L(x_1, y_1) \\ &\quad + \alpha_2 N(x_2, y_2) + L(x_1, y_1) N(x_2, y_2) \end{aligned} \quad (55)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$  where  $\beta = \beta_1 + \beta_2$ . Next, we define functions  $F_1, L_1, N_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$\begin{cases} F_1(x, y) := F(x, y) + \beta - \alpha_1 \alpha_2 \\ L_1(x, y) := L(x, y) + \alpha_2 \\ N_1(x, y) := N(x, y) + \alpha_1 \end{cases} \quad (56)$$

for all  $x, y \in \mathbb{R}$ . Thus using (55) and (56) we see that

$$F_1(x_1 y_2, x_2 y_1) = F_1(x_1, y_1) + F_1(y_2, x_2) + L_1(x_1, y_1) N_1(x_2, y_2) \quad (57)$$

for all  $x_1, x_2, y_1, y_2 \in \mathbb{R}$ .

By Lemma 4, (48), (52), (54), and (56) we see that

$$\begin{cases} f(x, y) \equiv \alpha_1 \alpha_2 - \beta \\ g(x, y) = \alpha_1 \ell(x, y) + \beta_1 + \alpha_1 \alpha_2 - \beta \\ h(x, y) \equiv \beta_2 - \beta \\ \ell(x, y) \text{ is arbitrary} \\ m(x, y) \equiv -\alpha_1 \end{cases} \quad (58)$$

$$\begin{cases} f(x, y) \equiv \alpha_1 \alpha_2 - \beta \\ g(x, y) \equiv \beta_1 - \beta \\ h(x, y) = \alpha_2 m(x, y) - \beta + \beta_2 + \alpha_1 \alpha_2 \\ \ell(x, y) \equiv -\alpha_2 \\ m(x, y) \text{ is arbitrary} \end{cases} \quad (59)$$

and

$$\begin{cases} f(x, y) = \frac{1}{k_1 k_2} [M_1(x+y) M_2(x-y) - 1] + \alpha_1 \alpha_2 - \beta \\ g(x, y) = \frac{1}{k_1 k_2} [M_1(x+y) M_2(x-y) - 1] \\ \quad + \frac{\alpha_1}{k_2} [M_1(x+y) M_2(x-y) - 1] + \beta_1 - \beta \\ h(x, y) = \frac{1}{k_1 k_2} [M_1(x+y) M_2(x-y) - 1] \\ \quad + \frac{\alpha_2}{k_1} [M_1(x-y) M_2(x+y) - 1] - \beta + \beta_2 \\ \ell(x, y) = \frac{1}{k_2} [M_1(x+y) M_2(x-y) - 1] - \alpha_2 \\ m(x, y) = \frac{1}{k_1} [M_1(x-y) M_2(x+y) - 1] - \alpha_1 \end{cases} \quad (60)$$

for all  $x, y \in \mathbb{R}$  where  $M_1, M_2 : \mathbb{R} \rightarrow \mathbb{R}$  are multiplicative functions. Note that  $\beta = \beta_1 + \beta_2$ . Hence (58) - (60) yield the asserted solutions (45) - (47). The proof of the theorem is now complete.  $\square$

**Acknowledgment.** The authors are very thankful to the referee for valuable suggestions. This work was partially supported by a SROP grant from the Graduate School and an IRIG grant from the Office of the VP for Research, University of Louisville.

## REFERENCES

- [1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, 1989.
- [2] J. K. Chung and P. K. Sahoo, *General solution of some functional equations related to the determinant of symmetric matrices*, Demonstr. Math., 35 (2002), 539-544.
- [3] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, Singapore 2002.
- [4] K. B. Houston and P. K. Sahoo, *On two functional equations and their solutions*, submitted to Aequationes Math., 2006.
- [5] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Prace Nauk. Uniw. Sla 489, Polish Scientific Publishers, Warsaw-Cracow-Katowice, 1985.
- [6] P. K. Sahoo, *Solved and Unsolved Problems, Problem 2*, News Letter of the European Mathematical Society, 58 (2005), 43-44.
- [7] P. K. Sahoo and T. R. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific Publishing Co., NJ, 1998.
- [8] J. Smital, *On Functions and Functional Equations*, Adam Hilger, Bristol-Philadelphia, 1988.
- [9] L. Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups*. World Scientific, Singapore 1991.

(Received: September 8, 2006)

(Revised: January 5, 2007)

K. B. Houston and P. K. Sahoo  
Department of Mathematics  
University of Louisville  
Louisville, KY, 40292 USA  
E-mail: kbhous01@louisville.edu  
E-mail: sahoos@louisville.edu