ON TWO FUNCTIONAL EQUATIONS AND THEIR SOLUTIONS

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Abstract. The present work aims to determine the solution $f : \mathbb{R}^2 \to \mathbb{R}$ of the equation $f(ux - vy,\, uy - vx) = f(x, y) + f(u, v) + f(x, y)\, f(u, v)$ for all $x, y, u, v \in \mathbb{R}$ without any regularity assumption. The solution of the functional equation $f(ux + vy,\, uy - vx) = f(x, y) + f(u, v) + f(x, y)\, f(u, v)$ is also determined. The methods of solution of these equations are simple and elementary. These two equations arise in connection with the characterizations of determinant and permanant of two-by-two symmetric matrices, respectively.

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1. Introduction

In the solved and unsolved problems column of the *News Letter* of the European Mathematical Society, the second author [5] posed the following problem: Determine the general solutions $f : \mathbb{R}^2 \to \mathbb{R}$ of the functional equation

$$f(ux - vy,\, uy - vx) = f(x, y) + f(u, v) + f(x, y)\, f(u, v) \tag{1}$$

for all $x, y, u, v \in \mathbb{R}$ (see [4]). Another similar functional equation is the following:

$$f(ux + vy,\, uy - vx) = f(x, y) + f(u, v) + f(x, y)\, f(u, v) \tag{2}$$

for all $x, y, u, v \in \mathbb{R}$. These two equations are connected with the characterizations of determinant and permanant of two-by-two symmetric matrices, respectively. The interested reader should refer to [1–4, 6–8] for an in-depth account on the subject of functional equations.

In this paper, we determine the general solution of the functional equation (1) as well as the functional equation (2) without any regularity assumption on the unknown function $f : \mathbb{R}^2 \to \mathbb{R}$. Our method of solution is simple, direct and interesting.

2. Solution of functional equation (1)

A function $M : \mathbb{R} \to \mathbb{R}$ is said to be a *multiplicative function* if and only if it satisfies $M(xy) = M(x)\, M(y)$ for all $x, y \in \mathbb{R}$. An identically constant multiplicative function M is either $M = 0$ or $M = 1$.

Theorem 1. *The general solution of the functional equation* (1) *is given by*

$$f(x,y) = M(x^2 - y^2) - 1, \tag{3}$$

where $M : \mathbb{R} \to \mathbb{R}$ *is a multiplicative function.*

Proof. It is easy to check that the solution enumerated in (3) satisfies the functional equation (1). Next, we show that (3) is the only solution of (1).

Suppose f is identically a constant, say $f \equiv c$. Then from (1), we have $c^2 + c = 0$ which implies $c = 0$ or $c = -1$. Hence the identically constant solutions of (1) are $f(x,y) = 0$ and $f(x,y) = -1$ for all $x, y \in \mathbb{R}$. These solutions are included in the solution (3).

From now on we assume that f is not identically constant, that is $f \not\equiv c$, where c is a constant. We define a function $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x,y) = f\left(\frac{x+y}{2}, \frac{x-y}{2}\right) + 1 \tag{4}$$

for all $x, y \in \mathbb{R}$. Next, using (4) in (1), we obtain

$$F((x+y)(u-v),\, (x-y)(u+v)) = F(x+y,\, x-y)\, F(u+v,\, u-v) \tag{5}$$

for all $x, y, u, v \in \mathbb{R}$. Substituting $x_1 = x+y,\ y_1 = x-y,\ x_2 = u+v$ and $y_2 = u-v$ in (5), we have

$$F(x_1 y_2, x_2 y_1) = F(x_1, y_1)\, F(x_2, y_2) \tag{6}$$

for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Setting $y_1 = x_2 = 1$ in (6), we see that

$$F(x_1 y_2, 1) = F(x_1, 1)\, F(1, y_2) \tag{7}$$

for all $x_1, y_2 \in \mathbb{R}$. Interchanging x_1 with y_2 in (7) and comparing the resulting equation with (7) we have

$$F(x_1, 1)\, F(1, y_2) = F(y_2, 1)\, F(1, x_1) \tag{8}$$

for all $x_1, y_2 \in \mathbb{R}$. Since f is non-constant, there exists a $x_0 \in \mathbb{R}$ such that $F(x_0, 1) \neq 0$ and letting $x_1 = x_0$ in (8), we obtain

$$F(1, y_2) = \alpha\, F(y_2, 1) \tag{9}$$

where α is an arbitrary constant. We claim that α is nonzero. If $\alpha = 0$, then letting $x_1 = x_2 = 1$ in (6) and using (9) we get f to be identically constant contrary to the assumption that f is not identically constant. Hence $\alpha \neq 0$. Using (9) in (7), we get

$$F(x_1 y_2, 1) = \alpha\, F(x_1, 1)\, F(y_2, 1) \tag{10}$$

for all $x_1, y_2 \in \mathbb{R}$. Defining $M : \mathbb{R} \to \mathbb{R}$ by

$$M(x) = \alpha\, F(x, 1) \tag{11}$$

for all $x \in \mathbb{R}$, we see that (10) reduces to

$$M(x_1 y_2) = M(x_1)\, M(y_2) \tag{12}$$

for all $x_1, y_2 \in \mathbb{R}$. Hence $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative map which is not identically constant.

Now letting $y_1 = 1 = y_2$ in (6), we obtain

$$F(x_1, x_2) = F(x_1, 1)\, F(x_2, 1) \tag{13}$$

for all $x_1, x_2 \in \mathbb{R}$ which by (11) yields

$$F(x_1, x_2) = k\, M(x_1)\, M(x_2) \tag{14}$$

where $k = \frac{1}{\alpha^2}$. Using (14) in (6), we see that $k = 1$ (since $k = 0$ yields a constant function f). Thus from (14), (4) and the fact that $k = 1$, we have

$$f(x, y) = F(x + y, x - y) - 1 = M(x + y)\, M(x - y) - 1 = M(x^2 - y^2) - 1 \tag{15}$$

for all $x, y \in \mathbb{R}$, which is the solution (3). □

Corollary 1. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous solution of the functional equation* (1) *if and only if either $f = 0$, or $f = -1$, or f has one of the following forms*

$$f(x,\, y) = |x^2 - y^2|^{\alpha} - 1, \tag{16}$$

$$f(x,\, y) = |x^2 - y^2|^{\alpha}\, sgn(x^2 - y^2) - 1 \tag{17}$$

where α is an arbitrary positive real constant and

$$sgn\,(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases} \tag{18}$$

Proof. The proof follows from Theorem 6 on page 311 of [4] and Theorem 1. □

3. Solution of functional equation (2)

Theorem 2. *The general solution of the functional equation* (2) *is given by*

$$f(x, y) = M(x^2 + y^2) - 1, \tag{19}$$

where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function.

Proof. We define a function $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x, y) = f(x, y) + 1 \tag{20}$$

for all $x, y \in \mathbb{R}$. Then using (20) in the functional equation (2) we have

$$g(ux + vy,\, uy - vx) = g(x, y)\, g(u, v) \tag{21}$$

for all $x, y, u, v \in \mathbb{R}$.

Interchanging x with u and y with v in (21) we obtain

$$g(ux + vy,\, vx - uy) = g(u,\, v)\, g(x,\, y) \tag{22}$$

for all $x, y, u, v \in \mathbb{R}$. Comparing (21) and (22) we have

$$g(ux + vy,\, uy - vx) = g(ux + vy,\, vx - uy) \tag{23}$$

for all $x, y, u, v \in \mathbb{R}$. Hence (23) yields

$$g(x,\, y) = g(x,\, -y) \tag{24}$$

for all $x, y \in \mathbb{R}$.

Next, letting $v = 0 = y$ in (21) we obtain

$$g(ux,\, 0) = g(u,\, 0)\, g(x,\, 0) \tag{25}$$

for all $u, x \in \mathbb{R}$. Defining a function $m : \mathbb{R} \to \mathbb{R}$ by

$$m(x) = g(x,\, 0) \tag{26}$$

for all $x \in \mathbb{R}$, and from (26) we see that m satisfies $m(ux) = m(u)\, m(x)$ for all $u, x \in \mathbb{R}$. Thus m is a multiplicative function on $\mathbb{R}$.

From (24) and (21) we see that

$$\begin{aligned} g(ux + vy,\, uy - vx) &= g(u,\, v)\, g(x,\, y) \\ &= g(x,\, y)\, g(u,\, -v) \\ &= g(ux - vy,\, uy + vx) \end{aligned} \tag{27}$$

for all $x, y, u, v \in \mathbb{R}$. Let $x_1 = ux+vy$, $x_2 = uy-vx$, $y_1 = ux-vy$, and $y_2 = uy+vx$. Then it is easy to see that $x_1^2 + x_2^2 = (u^2 + v^2)(x^2 + y^2) = y_1^2 + y_2^2$. Hence (27) becomes

$$g(x_1,\, x_2) = g(y_1,\, y_2) \tag{28}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that

$$x_1^2 + x_2^2 = y_1^2 + y_2^2. \tag{29}$$

Letting $y_2 = 0$ in (28) and using (29) and (26), we obtain

$$g(x,\, y) = g(\sqrt{x^2 + y^2},\, 0) = m(\sqrt{x^2 + y^2}\,) \tag{30}$$

where $m : \mathbb{R} \to \mathbb{R}$ is a multiplicative function. Since $m(\sqrt{x}) = \sqrt{m(x)}$ for $x \geq 0$, therefore (30) can be rewritten as $g(x,\, y) = M(x^2 + y^2)$, where $M : \mathbb{R} \to \mathbb{R}$ is a multiplicative function. Using this $g(x,\, y)$ in (20) we obtain the asserted solution (19). □

Using Theorem 6 on page 311 of [4] and Theorem 2, we have the following corollary.

Corollary 2. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous solution of the functional equation* (2) *if and only if either $f = 0$, or $f = -1$, or f has the form*

$$f(x,\, y) = (x^2 + y^2)^\alpha - 1, \tag{31}$$

where α is an arbitrary positive real constant.

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