BASIC PROBABILITY RULES

- Methods of Assigning Probability
- Multiplication Rule for Independent Events
- Addition Rule for Disjoint Events
- General Addition Rule
- Complement Rule
- Law of Large Numbers

University of Pittsburgh

TERMINOLOGY

- **Random phenomenon:** any process that leads to one of several potential results where the result cannot be predicted, but whose results have a regular distribution after many repetitions
  - ***Examples:*** *Rolling a die, flipping a coin, selecting a card from a full deck*
- **Trial:** an attempt of a random phenomenon
- **Outcome:** potential result of a random phenomenon
  - ***Examples:*** *Rolling a 4, flipping heads, selecting the 7 of spades*
- **Event:** any combination of outcomes, typically denoted by a capital letter or word
  - ***Examples:*** *Rolling an even number (2, 4, or 6), selecting any spade*

SAMPLE SPACE

- **Sample space:** set of all possible outcomes of a trial
  - Denoted by S
  - If there are a countable number of outcomes, use **set notation** (list outcomes in a set of braces)
  - If there are an infinite number of outcomes, use **interval notation** (denote upper and lower bounds inside parentheses/brackets)
- **Requirements of sample spaces:**
  1. **Exhaustive:** Includes all possible outcomes
  2. **Disjoint:** Two outcomes cannot occur simultaneously on the same trial

EXAMPLE: SAMPLE SPACE

- **Question:** What is the sample space in each of the experiments?
  1. Roll two dice and add the results
     - $S =$ ____________________________ or $S =$ _______________
  2. Measure the amount of time (in minutes) it takes a student to complete an exam during a MWF class at Pitt
     - $S =$ _________
  3. Flip two coins and look at the order of the results
     - $S =$ ____________________
  4. Flip two coins and count how many heads occurred
     - $S =$ ____________

PROBABILITY

- **Probability:** the chance that an event occurs
  - Denoted by $P(A)$, where A is the event being studied

- Every probability must satisfy two rules:
  - ***Rule #1:*** *A probability is a number between 0 and 1; that is, for any event A,* $0 \leq P(A) \leq 1$.
  - ***Rule #2:*** *The probability of the set of all possible outcomes in the sample space must be 1; that is,* $P(S) = 1$.

EXAMPLE: REQUIREMENTS OF PROBABILITIES

- **Scenario:** There are 50 marbles in a bag of four different colors: blue, red, green, and yellow. Select one marble from the bag.
- **Question:** What is the sample space?
- **Answer:** ___________________

- **Question:** What must be true about the blue, red, green, and yellow marbles?
- **Answer:**
  - Each color must have a probability of being selected __________________
  - ____________________________ $= 1$

Methods of Assigning Probability

- **Classical approach:** assigns same probability to each possible outcome
  - Used often in fair games of chance
- **Empirical approach:** long-run frequency with which an outcome occurs
  - Take total number of observations for an outcome and divide by total number of observations
- **Subjective approach:** degree of belief that we have in the occurrence of an event
  - Used when classical approach is unreasonable and no information exists to calculate proportions

Example: Methods of Assigning Probability

- **Task:** Identify the method used to assign probability in each of these scenarios.
  - **Scenario #1:** Probability of rolling 4 on a fair die is $\frac{1}{6}$.
    - **Method:** ________ approach: Fair die has __ sides, each outcome is ____________
  - **Scenario #2:** Probability that a basketball player makes his next free throw is 0.90 because he made 18 of his last 20 shots.
    - **Method:** _________ approach: ________________ tells us the proportion of shots made historically is _________
  - **Scenario #3:** The probability the Pitt wins the National Championship in football is 0.05.
    - **Method:** __________ approach: ________, but outcomes are not _____________

Union and Intersection

- **Union:** occurs when **either** event A or event B occurs
  - Look for the word "or"
  - Probabilities get added – larger value because we only need at least of two events to occur
  - Includes situations when both events occur
- **Intersection:** occurs when **both** event A and event B occur
  - Look for the word "and"
  - Probabilities usually get multiplied – smaller value because it is harder for two events to both occur
  - Probability of an intersection is called a **joint probability**

Example: Union and Intersection

- **Scenario:** Flip two coins and record if they land on heads or tails
- **Question:** Which of the following events are unions? Which are intersections?
  I. Both flips are heads
  II. At least one flip is heads
  III. Neither flip is heads

- **Answer:**
  - **Union:** ____________
  - **Intersection:** ____________
    - For I, need ______________ to be heads
    - For II, need ______________________ to be heads
    - For III, need _____________ to be tails

Independent Events

- **Independent events:** two events are independent if the outcome of one trial does not influence or impact the outcome of another

- **Question:** Which of these pairs of events are independent?
  I. Selecting a card from a deck, not replacing it, and drawing a 2nd card
  II. Flipping a coin two times in a row
  III. The grade you receive in Calc 1 and Calc 2

- **Answer:** _________
  I. Not replacing the card makes all other cards _____________________
  II. Coins have __________: First coin flip has ___________ on the second
  III. Your grade in Calc 1 is ______________________ of your Calc 2 grade

Multiplication Rule for Independent Events

- **Multiplication Rule for Independent Events:** If A and B are independent, then $P(A \text{ and } B) = P(A) \times P(B)$.

- **Scenario:** In the United States, 44% have type O blood, 42% of people have Type A blood, 11% have Type B blood, and 3% have Type AB blood. Suppose two people are selected randomly.
- **Question:** What is the probability both people have Type A blood?
- **Answer:**
  - Blood types are ______________
  - $P(A \text{ and } A) =$ _____________ = ___________ = ________

Multiplication Rule for Independent Events

- **Scenario:** In the United States, 44% have type O blood, 42% of people have Type A blood, 11% have Type B blood, and 3% have Type AB blood. Suppose two people are selected randomly.
- **Question:** What is the probability both people have the same blood type?
- **Answer:**
  - ______ different ways event can be satisfied
  - Blood types are ______________
  - $P(\text{Same}) = P(O \text{ and } O) + P(A \text{ and } A) + P(B \text{ and } B) + P(AB \text{ and } AB)$
    $= P(O)\times P(O) + P(A)\times P(A) + P(B)\times P(B) + P(AB)\times P(AB)$
    $=$ ____
    $=$ ________

Disjoint Events

- **Disjoint events:** events that cannot both occur at the same time
  - For two events A and B, $P(A \text{ and } B) = 0$
  - A occurring prevents B from occurring and vice versa
- **Question:** Which of these events are disjoint?
  I. Rolling an odd number and an even number
  II. Rolling an odd number and rolling a number less than 4
  III. Rolling a number greater than 4 and a number less than 4
- **Answer:** I and III
  I. Event A can be __________, Event B can be __________ → __________
  II. Event A can be __________, Event B can be __________ → __________
  III. Event A can be __________, Event B can be __________ → __________

Addition Rule for Disjoint Events

- **Addition Rule for Disjoint Events:** If A and B are disjoint, then:

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Scenario:** There are 50 marbles in a bag: 23 blue, 15 red, 8 yellow, and 4 green.
- **Question:** What is the probability of picking a red or blue marble?
- **Answer:**
  - Red and blue are __________
  - Want _____________ so we can _________________________
  - $P(R \text{ or } B) =$ ____________ $=$ ____________ $=$ _____ $=$ _____

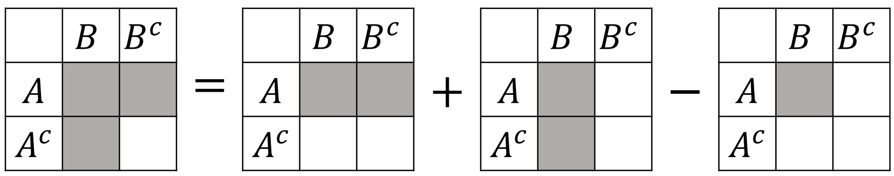
Motivation: General Addition Rule

- **Scenario:** In a deck of cards, there are 13 cards in each of 4 suits. Each number from 2-10 as well as jacks, queens, kings, and aces appear 4 times each. You draw one card from the deck.
- **Question:** What is the probability of picking a king?
- **Answer:** ____________________
- **Question:** What is the probability of picking a club?
- **Answer:** ____________________
- **Question:** Is the probability of drawing either a king or a club $\frac{17}{52}$?
- **Answer:** ________
  - Counted _______________: once with the _____ and once with the _____

General Addition Rule

- **General Addition Rule:** If A and B are any two events, then:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$



  - **Rationale:** Subtracting off the probability of the intersection accounts for the fact that it was originally included both times when we added the individual probabilities together

Example: General Addition Rule

- **Scenario:** In a deck of cards, there are 13 cards in each of 4 suits. Each number from 2-10 as well as jacks, queens, kings, and aces appear 4 times each. You draw one card from the deck.
- **Question:** What is the probability of drawing either a king or a club?
- **Answer:**
  - $P(K) =$ _____ and $P(C) =$ _____
  - $P(K \text{ and } C) =$ _____
  - $P(K \text{ or } C) =$ _____________ $=$ ____

Cards	Club	Not Club	Total
King	1	3	**4**
Not King	12	36	**48**
Total	**13**	**39**	**52**

Complement

- **Complement:** set of all outcomes not included in an original event A, and denoted by A^c

- **Question:** What is the complement of each of these events?
  1. Rolling a sum of less than 8 on two fair dice
     - $A^c =$ ______________
  2. Taking at least 40 minutes to complete an exam in a MWF class
     - $B^c =$ _______
  3. Flipping heads more than once on two fair coins
     - $C^c =$ _______

Complement Rule

- **Complement Rule:** The probability that an event occurs is 1 minus the probability that it does not occur; that is, $P(A) = 1 - P(A^c)$

- **Scenario:** A hotel asks its customers to rate the cleanliness of its rooms on a 5-point scale. The probabilities are displayed below.

Rating	Poor	Fair	Average	Good	Excellent
Probability	0.08	0.12	0.18	0.25	0.37

- **Question:** What is the probability a rating was at least "fair"?
- **Answer:**
  - "At least fair" is the same as ___________________
  - $P(\text{At least "fair"}) =$ _________ $=$ _______ $=$ ____

Example: Complement Rule

- **Scenario:** Flip four coins and record each result. 16 possible combinations, each with probability of $\frac{1}{16}$ of occurring.

HHHH	HHHT	HHTH	HTHH	THHH	HHTT	HTHT	HTTH
TTHH	THTH	THHT	TTTH	TTHT	THTT	HTTT	TTTT

- **Question:** What is the probability of flipping heads at least once?
- **Answer #1:** Find all combinations with _________________
  - **1 head:** ___ combinations
  - **2 heads:** ___ combinations
  - **3 heads:** ___ combinations
  - **4 heads:** ___ combination
  - **Final answer:** $P(\text{At least one head}) =$ ______________ $=$ ____

EXAMPLE: COMPLEMENT RULE

- **Scenario:** Flip four coins and record each result. 16 possible combinations, each with probability of $\frac{1}{16}$ of occurring.

HHHH	HHHT	HHTH	HTHH	THHH	HHTT	HTHT	HTTH
TTHH	THTH	THHT	TTTH	TTHT	THTT	HTTT	TTTT

- **Question:** What is the probability of flipping heads at least once?
- **Answer #2:** Using the Complement Rule
  - Find all combinations with _____ heads: ______
  - **Final answer:** $P(\text{At least one head}) =$ _______________ $=$ ______ $=$ __

EXAMPLE: COMPLEMENT RULE

- **Scenario:** In one lottery game, three digits from 0-9 are selected at random and combined to make a three-digit number. Players win some amount of money if they match at least one digit.
- **Question:** How can a person win?
- **Answer:** Let M be a _______ and X be a _____
  - Match ______________ (______)
  - Match ____ of the three digits (________________)
  - Match ____ of the three digits (________________)
- **Question:** Why is this probability complicated to calculate?
- **Answer:** Requires ___ separate calculations

EXAMPLE: COMPLEMENT RULE

- **Scenario:** In one lottery game, three digits from 0-9 are selected at random and combined to make a three-digit number. Players win some amount of money if they match at least one digit.
- **Question:** What is another way of approaching this problem?
- **Answer:** Find probability of ________ and use the _______________
  - **Only way of losing:** Miss _______________ (_____)
  - Each digit is _____________ of the other two digits
  - **Probability of winning:** $P(\text{Win}) =$ ___________

    $=$ ___________________

    $=$ ___________________

    $=$ ___________

    $=$ ______