Tree and Cycle Concentration and Covariance Graphical Markov Models are Faithful.

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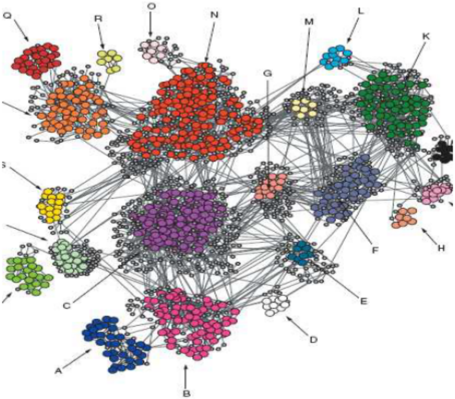
Motivation

yeast FOCI coexpression network. (*Magwene and Kim 2004*)



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Solution

A graph $G = (V, E)$ where V is the set of vertices and $E \subseteq V \times V$ is the set of edges and we define a separation criteria in G :

Separation statement $\Rightarrow$ Conditional independence statement.

It is the **global Markov property** of P with respect to G

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  - concentration graph : *natural* separation statement, ,*Whittaker 1990, Lauritzen 1996...*
  - covariance graph : $m-$separation statement *Cox and Wermuth 1996, Kauremann 1996...*

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- Mixed graphs (directed and undirected edges are present in G) : Chain graphs, Ancestral graphs *Anderson et al. 2001....*

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- Concentration graph : $G = (V, E(G))$

$$u \not\sim_G v \iff u \perp\!\!\!\perp v \mid V \setminus uv$$

where $u \perp\!\!\!\perp v \mid V \setminus uv$ is a shortcut of $X_u \perp\!\!\!\perp X_v \mid X_{V \setminus uv}$ and $X_{V \setminus uv} = (X_w, w \notin \{u, v\})$

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- Covariance graph : $H = (V, E(H))$

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Theorem (Lňenička and Matúš 2007, Malouche and Rajaratnam 2012)

*If P satisfies the the **pseudo graphoid** axiom : for any $(u, v, w) \in V^3$ and $S \subseteq V \setminus uvw$*

$$u \perp\!\!\!\perp v \mid Sw \text{ and } u \perp\!\!\!\perp w \mid Sv \;\Rightarrow\; u \perp\!\!\!\perp v \mid S \text{ and } u \perp\!\!\!\perp w \mid S$$

*then we can read from the **concentration** graph G and the **covariance** graph H the following two assertions : $\forall\,(A, B, S)$*

$$\text{If } A \perp_G B \mid S \Longrightarrow A \perp\!\!\!\perp B \mid S$$

and

$$\text{If } A \perp_H B \mid V \setminus (ABS) \Longrightarrow A \perp\!\!\!\perp B \mid S$$

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Question

When the faithfulness assumption can be satisfied ?

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Theorem (Becker et al. 2006,)

Assume that P satisfies the grahoid axiom (for example Gaussian and Elliptical distributions). If the concentration graph G is a ***tree*** *then P is faithful to G (represent all the CI statements) and H is complete (does not represent any CI statement).*

Theorem (Malouche and Rajaratnam 2010, Peña 2011,)

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Theorem (Malouche and Rajaratnam 2012)

*Assume that P satisfies the grahoid axiom. If the concentration G or covariance graph H is a **cycle** then all the CI statements of P are represented by G and H together.*

Further reading

[1] Pearl, J. Probabilistic Reasoning in Intelligent Systems, Morgan-Kaufmann, 1988

[2] Whittaker, J. Graphical models in Applied Multivariate Statistics. Wiley, 1990

[3] Lauritzen, S. L. Graphical Models New York : Oxford University Press., 1996

[4] Kauermann, G. On a dualization of graphical Gaussian models Scand. J. Statist., 1996, 23, 105-116

[5] Cox, D. R. & Wermuth, N. Multivariate Depencies : Models, Analysis and Interpretations Chapman and Hall, 1996

[6] Anderson, S. A., Madigan, D. and Perlman, M. D. 2001. Alternative Markov properties for chain graphs. Scand. J. Statist. 28 33–86.

Further reading...

[7] P. Magwene and J. Kim. Estimating genomic coexpression networks using first-order conditional independence. Genom Biol., 5(12), 2004.

[8] Lňenička, R., Matúš, F.,, 2007. On gaussian conditional independence structures. Kybernetika 43, 327–342.

[9] Peña, 2011, J. Reading Dependencies from covariance grahs, preprint in arXiv.

[10] Malouche, D. and Rajaratnam B. 2011, Covariance Gaussian Faithful Trees, Journal of Probability and Statistics,

[11] Malouche, D. and Rajaratnam B. Pseudographoids satisfy also the global Markov property wrt covariance graphs, communication submitted to JDS 2012.