Representing Conditional independencies and dependencies using undirected graphs Covariance and concentration graphs

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Outline

Motivation

Global Markov Properties

Relations

Pseudographoids

Reading conditional dependencies

Multivariate Gaussian distribution.

V a finite set, $X = (X_v,\ v \in V)' \sim P = \mathcal{N}_{|V|}(\mu, \Sigma)$ with density function

$$f(x) = \frac{\sqrt{|K|}}{(2\pi)^{d/2}} \exp\left\{-\frac{1}{2}(x-\mu)' K (x-\mu)\right\} \quad \forall x \in \mathbb{R}^{|V|}$$

where

- $\Sigma = (\sigma_{uv})_{(u,v)\in V\times V}$ is the $|V| \times |V|$ covariance matrix
- $K = \Sigma^{-1} = (k_{uv})_{(u,v)\in V\times V}$ is the $|V| \times |V|$ precision matrix

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Theorem

Let $X = (X_v,\ v \in V)' \sim P = \mathcal{N}_{|V|}(\mu, \Sigma)$ and let A, B and S be a triplet of disjoint subsets of V (A and B are non empty) :

$$X_A \perp\!\!\!\perp X_B \mid X_S \iff \forall (u,v) \in A \times B \quad |\Sigma_{uS,vS}| = 0$$

Covariance and Concentration graphs.

- A graph $G = (V, E)$ is a pair of sets : V is a set of vertices and $E \subseteq V \times V$ a set of edges.

$$G \text{ is undirected} \iff \forall (u, v) \in V \times V \;\; (u, v) \in E \iff (v, u) \in E$$

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  - $H = (V, E(H))$ is the Concentration graph :

$$u \not\sim_H v \iff X_u \perp\!\!\!\perp X_v \mid X_{V \setminus uv} \underbrace{\iff k_{uv} = 0}_{\text{If Gaussian}}$$

Questions

1. Can these graphs be used to read many other relationships between the variables of X ?
2. What happens in a more general cases (other than Gaussian distributions) ?

Examples (Covariance graph)

$$\Sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 & \sigma_{41} & 0 \\ 0 & \sigma_{22} & 0 & \sigma_{42} & \sigma_{5,2} \\ 0 & 0 & \sigma_{33} & \sigma_{43} & 0 \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{54} \\ 0 & \sigma_{52} & 0 & \sigma_{54} & \sigma_{55} \end{bmatrix}$$

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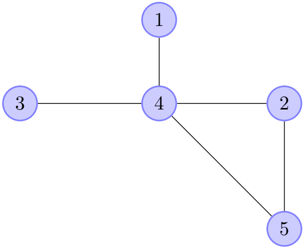

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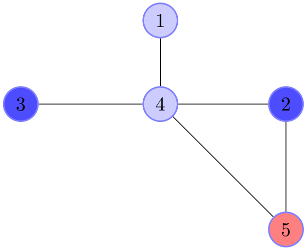

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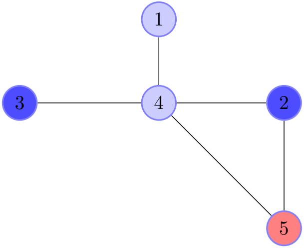

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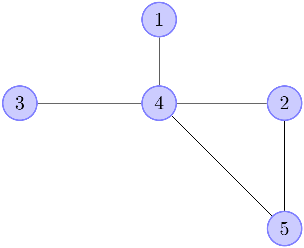

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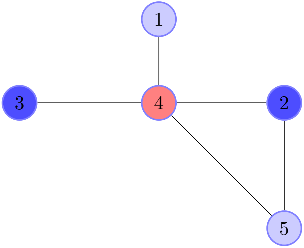

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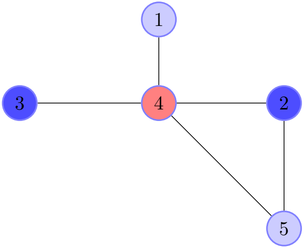

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- Let A and $B \subseteq V \setminus S$, $A \cap B = \emptyset$ and A and B are non empty. A and B are separated by S in G, i.e., $A \perp_G B \mid S$ if $\forall\, (u, v) \in A \times B$ we have $u \perp_G v S$.

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- Concentration Global Markov property : P is Global Markov to the concentration graph H if $\forall (A, B, S)$ pairwise disjoint subsets of V then

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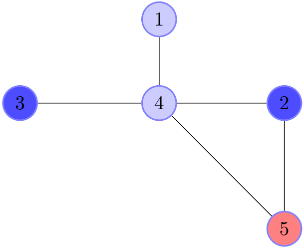

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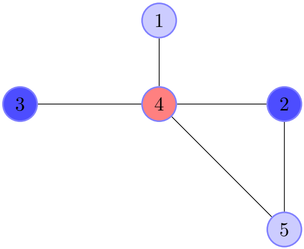

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Sufficient conditions for GMP

Let A, B and C be any triplet of pairwise disjoint subsets of V.

- Lauritzen (1996)

$$A \perp\!\!\!\perp B \mid C \cup D \text{ and } A \perp\!\!\!\perp C \mid B \cup D \text{ then } A \perp\!\!\!\perp B \cup C \mid D \quad (1)$$

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Theorem

- *(Pearl and Paz 1987)*
  If P satisfies (1) then the concentration GMP is satisfied.
- *(Kauermann 1996, Banarjee and Richardson 2003)*
  If P satisfies (2) then the covariance GMP is satisfied.

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- We give new sufficient conditions to get GMP : pseudographoid axioms.
- We show perfect duality between covariance and concentration graphs

Relations

- $$\begin{aligned}\mathcal{T}(V) &= \{(A,B,S),\ \text{where } A \text{ and } B \neq \emptyset,\\ &\quad A,\ B \text{ and } S \subseteq V \text{ and pairwise disjoint}\}\end{aligned}$$
- A relation L is a subset of $\mathcal{T}(V)$.
- We associate to $L \mapsto \tau(L)$ another relation called the *dual* of L such that

$$\tau(L) = \{(A,B,S) \in \mathcal{T}(V) \text{ such that } (A,B,V \setminus ABS) \in L\}\,.$$

Example of relations

- Probabilistic relations : $X = (X_v,\ v \in V)' \sim P$ and

$$L = L[P] = \{(A, B, S) \in \mathcal{T}(V) \text{ such that } A \perp\!\!\!\perp B \mid S\}$$

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- Graphical relations : $G = (V, E)$ is an undirected graph and

$$L = L[G] = \{(A, B, S) \in \mathcal{T}(V) \text{ such that } A \perp_G B \mid S\}$$

We denote $S(G) = L = L[G]$.

A set of relations

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Definition

$L \in \Phi(V)$ if $\forall (A,B,S) \in \mathcal{T}(V)$, $(A,B,S) \in L \iff$
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Theorem (Matúš, 1992)

If L is a probabilistic relation, i.e., $L = L[P]$, then $L \in \Phi(V)$.

Other subsets of relations : Pseudographoids

- **pseudographoids** (Lňenička and Matúš 2007) : $L \in \Psi(V)^{\rightarrow}$ if and only if $\forall\, S \subseteq V$ and u, v and $w \in V \setminus S$.

$$\{(u, v, Sw), (u, w, Sv)\} \subseteq L \Rightarrow \{(u, v, S), (u, w, S)\} \subseteq L$$

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- **symmetric-pseudographoids** $\psi(V) = \Psi(V)^{\rightarrow} \cap \Psi(V)^{\leftarrow}$. Then $L \in \psi(V)$ if and only if $\forall S \subseteq V$ and u, v and $w \in V \setminus S$.

$$\{(u, v, Sw), (u, w, Sv)\} \subseteq L \iff \{(u, v, S), (u, w, S)\} \subseteq L$$

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If P has a positive density with respect to a measure μ, then $L[P]$ is a pseudographoid, i.e., $L[P] \in \Psi(V)^{\rightarrow}$.

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- *If $G = (V, E)$ is an undirected graph then $L[G]$ is a symmetric-pseudographoid, i.e., $L[G] \in \psi(V)$*

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Lemma

$$\tau(\Psi(V)^{\rightarrow}) = \Psi(V)^{\leftarrow} \quad \textit{and} \quad \tau(\Psi(V)^{\leftarrow}) = \Psi(V)^{\rightarrow}$$

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Covariance-concentration global Markov property

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Let $L \subseteq \mathcal{T}(V)$ be a relation in $\Phi(V)$.

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3. *If $L \in \psi(V)$ then $\tau(S(G)) \cap S(H) \subseteq L$.*

Proof

- Since $L \in \Psi(V)^{\rightarrow}$ and by induction on $|S|$ we show $u \perp_H v \mid S \Rightarrow (u, v, S) \in L : s(H) \subseteq \mathcal{L}$ where

$$s(H) = \{(u, v, S) \in \mathcal{T}(V) \text{ such that } u \perp_H v \mid S\}$$

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- Then $S(G) \subseteq \tau(L)$.

Next...

- We find a graphical criteria to read conditional independence statements
- New graphical criteria in order to read conditional dependencies ?

Semigraphoid relations

Definition (Lňenička and Matúš 2007)

$L \in \Pi(V)$ if and only if $\forall\, u,\ v,\ w \in V$ and $S \subseteq V \setminus uvw$

$$\{(u, w, S), (u, v, Sw)\} \subseteq L \implies \{(u, v, S), (u, w, Sv)\} \subseteq L.$$

Any relation L in $\Pi(V)$ is a called a **semipseudographoid**.

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1st graphical criteria

Theorem

Let $L \subseteq \mathcal{T}(V) \in \Phi(V) \cap \Pi(V)$.

Let $L \mapsto \prec G, H \succ$ *be the covariance-concentration graphs associated with* L.

Let $u, v \in V$ *and* $S \subseteq V \setminus uv$ *and* $S \neq \emptyset$. *Then*

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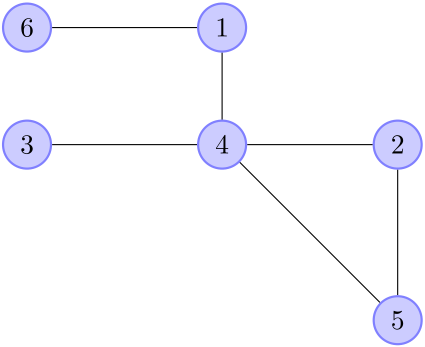
2. *Assume that $L \in \Psi(V)^{\rightarrow}$, $u \sim_H v$ and that $\forall\, w \in V \setminus Suv$, $\forall p \in \mathcal{P}(w, v, H)$, $p \ni u$ then $(u, v, S) \notin L$.*

Examples (Covariance graph)



$4 \perp\!\!\!\perp 5 \mid 1, 6$?

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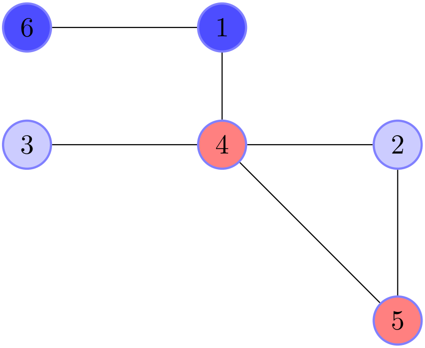

$$\Sigma_{416,516} = \begin{pmatrix} \sigma_{45} & \sigma_{41} & 0 \\ 0 & \sigma_{11} & \sigma_{16} \\ 0 & \sigma_{16} & \sigma_{66} \end{pmatrix}$$

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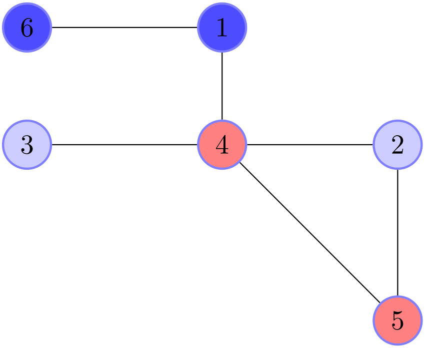

$$4 \not\perp\!\!\!\perp 5 \mid 1, 6$$

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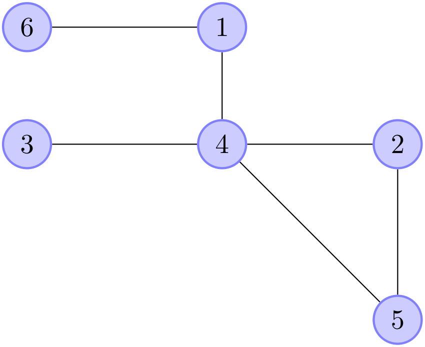

$4 \perp\!\!\!\perp 5 \mid 2$?

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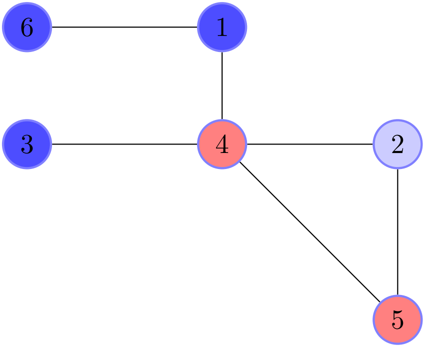

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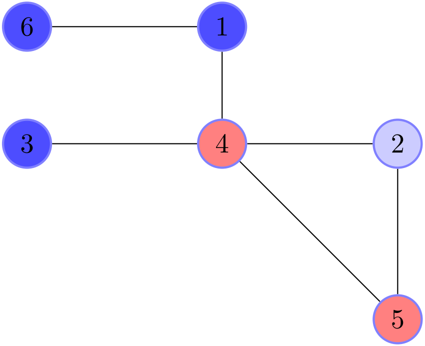

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1-connection

Definition (Peña 2010)

Let u, $v \in V$ and $S \subseteq V \setminus uv$. We say that u and v are 1−connected given S if $\mathcal{P}(u, v, G_{Suv}) = \{p\}$, i.e., $u \sim^1_G v \mid S$.

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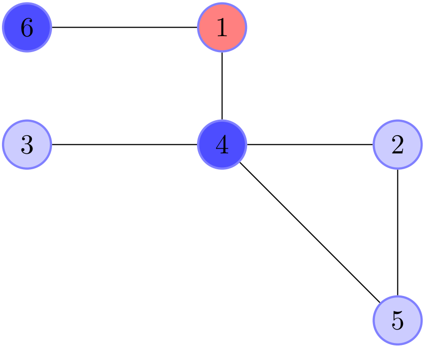

$$4 \sim^1 6 \mid 1$$

Weak Transitive

Definition

$L \in Delta(V)$ if and only if $\forall\, u$, v and w in V and $S \subseteq V \setminus uvw$

$$\{(u, v, S), (uv, Sw)\} \subseteq \mathcal{L} \implies (u, w, S) \in \mathcal{L} \text{ or } (v, w, S) \in \mathcal{L}$$

When $L \in \Delta(V)$ we say that L satisfies the **weak transitive axiom.**

Lemma

if $L = L[P]$ *where* P *is Gaussian then* $L \in \Delta(V)$

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Lemma

$L \in \Delta(V) \cap \Pi(V) \iff \tau(L) \in \Delta(V) \cap \Pi(V)$

2nd Graphical Criteria

Theorem

Let $L \subseteq \mathcal{T}(V) \in \Phi(V) \cap \Pi(V) \cap \Delta(V)$.

Let $L \mapsto \prec G, H \succ = \{V, E(G), E(H)\}$ *be the covariance-concentration graphs associated with* L.

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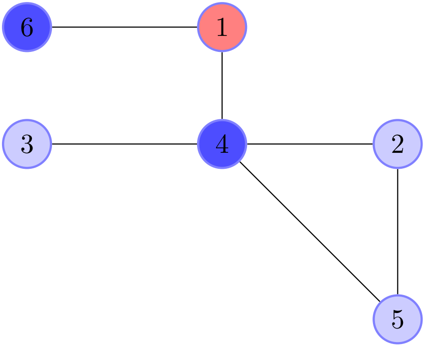

$$4 \sim^1 6 \mid 1$$

If covariance graph : $4 \not\perp\!\!\!\perp 6 \mid 1$

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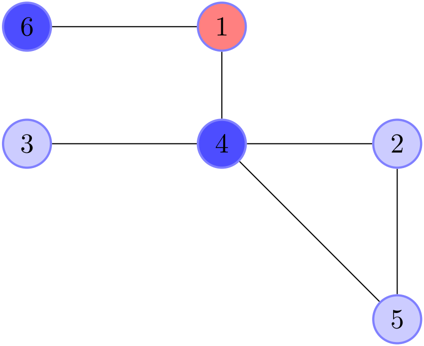

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If concentration graph :

$$4 \not\perp\!\!\!\perp 6 \mid 2, 3, 5$$