Endogenous Network Formation: Theory and Application

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Abstract

Economic networks have become a useful modeling tool. However most of the current literature on economic networks takes the networks themselves as given. This paper presents a model of endogenous network formation. Individual economic agents choose to form relationships with one another, thereby forming the links of a network. I describe an application of this model in the context of firms choosing input suppliers, forming a production network and analyze the outcome of a single firm losing its equilibrium input supplier. I show that when one of these firms loses its input supplier, aggregate output may actually increase. Simulations of the model indicate, on average, that when a firm loses its input supplier, the drop in output is smaller when: (1) the firm which loses its input supplier has a larger number of alternative suppliers to choose from and (2) the original equilibrium network is less connected. In the absence of this endogenous network formation, the answers to some economic questions, such as the effect of tariffs on international trade networks or the effect of regulation on air traffic networks, may not only be quantitatively incorrect but qualitatively incorrect.

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JEL Classifications: C67, D85, E23.

1. Introduction

When the US automobile industry was failing, the president of Ford supported the bailout of his competitors, General Motors and Chrysler. He did this because if GM and Chrysler failed, their upstream input suppliers would fail, and Ford would no longer have access to those suppliers. Recent economic literature has begun investigating how the interconnectedness of agents determine aggregate outcomes. Acemoglu et al. (2012) explore how the sector level input-output network leads to aggregate fluctuations. di Giovanni, Levchenko, and Mejean (2014) analyze what percentage of aggregate volatility can be attributed to network linkages between firms. Networks are being used as powerful modeling tools in economics. However, most of the existing literature on economic networks takes the networks themselves as given. When faced with an economic shock, economic agents adapt. They act to mitigate their losses or to improve their outcomes. Their decisions, and thus the links of the economic network, change. As a result, for some economic questions it is necessary to model the formation of the network. This paper describes a model that does this.

I present a model of endogenous network formation wherein a finite set of individual economic agents choose to form relationships with one another and thereby form the links of an equilibrium economic network. This model allows me to ask and answer new questions. The endogeneity of the network formation allows for individual agents to react to economic shocks and for these reactions to determine a new, resulting network. The finite set of agents allows me to investigate the decisions of large firms and their effect on the network. I apply this model to the context of individual firms choosing intermediate input suppliers and thereby forming an equilibrium production network. Then, I use it to find the effect on the production network when an individual firm loses its equilibrium input supplier. Furthermore, I analyze how the changes to the production network affect aggregate output. This effect will determine the change in aggregate output produced by all the firms in the network.

In the model, each agent chooses whether to form a relationship with a set of available other agents, thereby forming the links of an equilibrium network. I define three network allocations: a solution to the planner's problem, a pairwise-stable equilibrium, and a new refinement of the pairwise-stable equilibrium, a coordination-proof equilibrium. The planner considers all feasible networks and chooses the one that maximizes a measure of consumer welfare. The solution to the planner's problem is used to provide a baseline comparison for the efficiency of the equilibrium network definitions. The equilibrium definition used prominently in the literature, a pairwise-stable equilibrium, is a network in which no possible pair of potentially-related firms would be made better off by deviating to a network in which that relationship is chosen. This is a needlessly restrictive definition; it does not allow for the consideration of more than one agent deviating to a different potential relationship at a time. As such, I define a coordination-proof equilibrium as a network such that no *set* of potentially-related pairs can be made better off by a multi-lateral deviation to a different network. That is, considering all possible combinations of agents - of size 1, 2, up to the entire set of agents and each of their alternative relationships not in use - no set of potentially-related pairs would be made better off by switching to the network in which those relationships are in use.

In this paper, I describe the model in the context of firms choosing intermediate input suppliers. That is, the agents are firms and the relationships being chosen are the use of one firm's product by another firm in production. I analyze the effect of a firm losing its equilibrium input supplier on the network as a whole and on the aggregate output produced by the firms that make up the network. Note that the model is more general than this particular contextualization.

We have evidence that firm choices are driven by the production network in which they are placed. As in the case of Ford, GM, and Chrysler, the links between a firm and its suppliers, as well as the links between other firms, play a role in the choices of firms. Furthermore, firms may lose input suppliers in many ways. It may be the result of a natural disaster, as in the case of American Toyota factories, when an earthquake in Japan prevented the factories from getting necessary parts for production. It may be the result of a cyberattack, in the case of many Ukranian businesses when the Ukranian shipping infrastructure was shut down due to a ransomware attack. It may even be policy driven. Protectionist trade policies may prevent the use of oil from Saudi Arabia or avocados from Mexico. Health and safety regulations lead to the loss of asbestos as a major construction input. Finally, these changes at the firm level do affect the macroeconomy. di Giovanni, Levchenko and Mejean find that a majority of aggregate volatility is driven by changes at the firm level and that this percentage is growing over time.

I show that, contrary to results of existing models, when a firm loses its equilibrium input supplier, aggregate output may actually increase. In the solution to the planner's problem, output will always decrease when a firm loses an input supplier. However, I show that there exist parameters of the model such that when an edge is deleted from a pairwise-stable equilibrium, it is possible for output to be greater in the new pairwise-stable equilibrium. In fact, this result survives in the coordination-proof equilibrium.

Output increases after an edge is deleted when that edge is the only edge preventing a particularly high-output network from being an equilibrium. In most cases, when an edge is deleted, the set of new equilibrium networks is a strict subset of the previous set of equilibrium networks and thus output is lower. However, there are situations in which the edge that is deleted was the only potentially-related pair preventing a new network from being an equilibrium. When that edge is deleted, the new network becomes an equilibrium network and if that new network has a higher output than the original equilibrium network then output will increase. Economically, this occurs when one buyer-supplier pair is being made better off at the expense of lower output in the economy as a whole. When this relationship is no longer possible, the higher-output network can be sustained as an equilibrium.

I simulate the model and the results of this simulation suggest network characteristics which lead to increased output. The level of connectivity in an economic network can affect the outcomes of that network. Acemoglu et al. (2015) analyzes the role of connectivity in the fragility of financial networks, for example. I measure the connectivity of a given equilibrium network using the average shortest path of the network. This is the average distance of the shortest directed path from each node of the network to every other node. The results of the simulations indicate that, on average, the less connected the original equilibrium network is, the higher output will be after a firm loses its input supplier. The intuition for this result is that the more connected a network is, the more directly each firm will be hit by the negative shock because there will be shorter distances between nodes. So the less connected the network is, the more isolated each firm will be from the shock.

In addition to investigating the role of the connectivity of the network as a whole, I also do this for the connectivity of the individual firm that loses its input supplier. Specifically, I ask how the number of alternative suppliers available to a firm affects the change in output when that firm loses its equilibrium input supplier. Intuitively, a larger number of alternative suppliers should be associated with a smaller drop in output because the firm has more suppliers to choose from to mitigate the loss of its equilibrium supplier. The results of the simulation suggest that this is indeed true on average; the more alternative suppliers available to the firm that loses its input supplier, the higher output will be after that firm loses its input suppliers.

2. Network Model

The input suppliers available to each firm define a *potential production network*. The potential network is made up of a finite set J of firms and directed edges between them. There is an edge pointing from firm a to firm b in the potential network if firm a's output *can* be used by firm b in production. When an equilibrium network is determined, the set of firms will remain the same, but the set of edges will be a subset of the edges in the potential network. An edge from a to b in an equilibrium network will mean that firm a's output *is* used in firm b's production.

Each firm $j \in J$ produces a single good. This good can be consumed in two ways: either as an input in another firm's production as described by the potential network or as a final consumption good by a representative consumer. That is, if y_j is the amount of good j which firm j produces, then y_j is partitioned in the following manner:

$$y_j = \sum_{e \in C_j} x(e) + y_j^0$$

where C_j is the set of edges pointing away from firm j to each of firm j's customers, $x(e)$ is the amount of good j used by each such customer as an input in their own production, and y_j^0 is the amount of good j consumed as a final good by the representative consumer.

Let S_j denote the set of edges pointing to firm j; this describes the set of inputs available to firm j. Each such input, $e \in S_j$, defines a different production function:

$$y_j(e) = \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} z(e) x(e)^\alpha l_j^{1-\alpha}$$

where x(e) is the amount of the associated input good, l_j is the amount of labor used by firm j, and $z(e)$ is an edge-specific productivity parameter. The production parameter α is the same across all firms. I allow for the possibility of firm j using multiple intermediate input goods, however because the production functions exhibit constant returns to scale, only one of the intermediate inputs will be used in equilibrium.

The representative consumer has preferences over the products produced by the firms in J according to

$$U(y_1^0, y_2^0, \ldots, y_{|J|}^0) = \left(\sum_{j \in J} (y_j^0)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

and she supplies L units of labor, inelastically.

2.1. Possible Equilibrium Networks

The set of networks which could be equilibrium networks is the set of all subnetworks of the potential network such that each firm has exactly one edge pointing to it. Label this set of possible equilibrium networks, henceforth referred to as PENs for simplicity, $\mathcal{N}$. The size of this set - the number of PENs - for a given potential network is determined by the number of potential input suppliers available to each firm. Let $m_j = |S_j| =$ the in-degree of j for each $j \in J$.

Theorem 1. *The number of PENs for a given potential network is given by the product of the number of suppliers available to each firm. That is,* $|\mathcal{N}| = \prod_{j \in J} m_j$.

Proof. For each $N \in \mathcal{N}$, each firm has exactly one edge pointing to it. For each firm, j, the number of ways to pick one edge from the m_j available is m_j. The number of ways to pick one for every $j \in J$ is the number of ways to do it for the first firm, multiplied by the number of ways to do it for the second firm, multiplied by the number of ways to do it for the third firm, and so on for every firm. So the number of ways to pick one edge for each firm is $m_1 \cdot m_2 \cdot m_3 \cdot \cdots \cdot m_{|J|} = \prod_{j \in J} m_j$. $\square$

In order to define an equilibrium network from which agents do not wish to deviate, I first define how a deviation manifests. For a given network, $N \in \mathcal{N}$, an i-adjacent network is another network, $\widetilde{N} \in \mathcal{N}$ that differs by exactly i edges. See Figures 1 and 2. In the context of the model described in this paper, a firm switching from one supplier to another defines a 1-adjacent network, two firms switching from each of their suppliers to another defines a 2-adjacent network and so on. For a given network, N, let $\widetilde{N}^{F,S_F}$ denote the $|F|$-adjacent network to N associated with the firms in set F switching from the suppliers in use in N to the suppliers specified in S_F. Using the example in Figure 2, $F = \{1, 4\}$ and $S_F = \{4, 3\}$ and $\widetilde{N_3}^{F,S_F} = N_4$.

3. The Planner's Problem

Here I describe the planner's problem to both provide a basis for comparison for other equilibrium outcomes and to build intuition for the model. The planner considers all of the PENs and for each solves a standard consumer utility maximization problem.

$$\max_{\left\{\{y_j^0, x(e_j), l_j\}_{j\in J}\right\}_{N\in\mathcal{N}}} \Big(\sum_{j\in J}(y_j^0)^{\frac{\epsilon-1}{\epsilon}}\Big)^{\frac{\epsilon}{\epsilon-1}} \equiv Y^0$$

s.t.

$$y_j^0 + \sum_{e\in\hat{D}_j} x(e) \leq \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(e_j)x(e_j)^\alpha l_j^{1-\alpha} \qquad \forall j \in J$$

$$\sum_{j\in J} l_j = L$$

Each $N \in \mathcal{N}$ defines a different edge pointing to each firm j, labeled e_j, and a different set of edges pointing away from firm j to each of its customers, labeled $\hat{D}_j$. The first constraint is the technology constraint: the consumer and the customers of firm j cannot consume more than firm j produces using the input defined by N. The second constraint is the labor constraint: all of the firms in J use only the labor supplied by the representative consumer.

Each of the PENs in $\mathcal{N}$ defines a different maximization problem across $\{y_j^0, x(e_j), l_j\}_{j\in J}$, each of which the planner solves. The planner then selects the network and choice variables corresponding to the largest Y^0. The Lagrangian for each PEN is:

$$\mathcal{L} = Y^0 + \sum_{j\in J}\lambda_j\Big[\frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(e_j)x(e_j)^\alpha l_j^{1-\alpha} - y_j^0 - \sum_{e\in\hat{D}_j} x(e)\Big] + \mu\Big[L - \sum_{j\in J} l_j\Big].$$

Define the individual efficiency of each firm, $q_j \equiv \frac{\mu}{\lambda_j}$. This will be useful in characterizing individual and aggregate outcomes, as defined in the following theorems.

Theorem 2. *In the solution to the planner's problem the efficiency of a given firm is a function of the efficiency of the input supplier of that firm. That is, $q_j = z(e_j)q_{s(e_j)}^\alpha$, where $s(e_j)$ is the identity of the input supplier used by j.*

Proof. The technology constraint gives

$$\lambda_j = \frac{1}{z(e_j)}\lambda_{s(e_j)}^\alpha \mu^{1-\alpha}.$$

Using the definition of q_j and then rearranging,

$$\frac{\mu}{q_j} = \frac{1}{z(e_j)}\Big(\frac{\mu}{q_{s(e_j)}}\Big)^{\alpha}\mu^{1-\alpha}$$

$$q_j = \mu\Big(z(e_j)\Big(\frac{q_{s(e_j)}}{\mu}\Big)^{\alpha}\frac{1}{\mu^{1-\alpha}}\Big)$$

$$q_j = z(e_j)q_{s(e_j)}^{\alpha}$$

□

Theorem 3. *The measure of aggregate output, Y^0, is a function of the efficiencies of all of the individual firm efficiencies.*

$$Y^0 = \Big(\sum_{j\in J} q_j^{\epsilon-1}\Big)^{\frac{1}{\epsilon-1}} \cdot L$$

See Appendix A for proof.

4. Equilibrium Definitions

Here I define a pairwise-stable equilibrium network and a refinement of it, a coordination-proof equilibrium network. The definition of these require a list of payoffs for each firm for each PEN, $\big\{\{\pi\}_{j\in J}\big\}_{N\in\mathcal{N}}$. The optimal derivation of these will be described in the following section. A pairwise-stable network is a network, $N \in \mathcal{N}$, such that no firm j, along with any potential supplier of j, would be made better off by moving to the 1-adjacent network defined by j and the potential supplier. Formally, it is a network $\big\{N \in \mathcal{N} : \forall j, \forall k \in S_j \setminus \{i_j^N\}, \neg\exists(j,k) \text{ s.t. } \pi_j^{\widetilde{N}^{j,k}} > \pi_j^N \text{ and } \pi_k^{\widetilde{N}^{j,k}} > \pi_k^N\big\}$, where i_j^N is the supplier of firm j in N. Note that this restricts the potential firm deviations considered. It does not allow for the individual firms to consider the possibility of the other firms simultaneously deviating in their decision. A pairwise-stable equilibrium network merely needs to be better than a 1-adjacent network for one pair of firms at a time. Next I define an equilibrium network that needs to be better than i-adjacent network for i firms at a time.

A coordination-proof equilibrium network is a network, $N \in \mathcal{N}$, such that not only no *one* pair of firm and alternate supplier would be made better off by moving to the 1-adjacent network defined by the pair, but no two pairs would be made better off, no three pairs, and so on up to the number of firms. Let C_J^i be the set of all combinations of size i of the firms in J. E.g., if $J = \{1, 2, 3, 4\}$, then

$C_J^2 = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$. Let $C_J = \{C_J^i\}_{i=1}^{|J|}$. Formally, a coordination-proof network is a network $\{N \in \mathcal{N} : \forall C_J^i \in C_J, \forall j \in C_J^i, \forall k \in S_j \setminus \{i_j^N\}, \neg\exists(j,k) \text{ s.t. } \pi_j^{\widetilde{N}^{j,k}} > \pi_j^N \text{ and } \pi_k^{\widetilde{N}^{j,k}} > \pi_k^N\}$. Any coordination-proof network is also pairwise-stable but not necessarily vice-versa. Therefore, the number of coordination-proof networks will be less than or equal to the number of pairwise-stable networks.

5. Prices and Profit Maximization

When exactly one edge is pointing to each firm, there are only two possible network shapes that can make up each connected component of the entire network. These are cycles and branches. A cycle is a set of nodes such that the in-degree and out-degree of each node is exactly one. A branch is a set of nodes such that the in-degree of each node is one but the out-degree is unrestricted. See Figure 3. Any connected component must contain exactly one cycle and any branch in that connected component must have its root on the cycle.[1] These two shapes are critical in calculating the prices.

Firms set prices for both the portion of their output consumed by the representative consumer and the portion consumed by each of their network customers - the other firms which use their good as an input. Label the price of y_j^0 as p_j^0. For each network customer of firm j, j charges a two-part tariff. That is, j sets $\{p(e), \tau(e)\}_{e\in\hat{D}_j}$, for each $\hat{D}_j$ defined by each $N \in \mathcal{N}$. Here $\tau(e)$ is fixed fee and $p(e)$ is a price per unit of product j.

Theorem 4. *The per-unit price, $p(e)$, that firm j charges the customer to which edge e points is firm j's marginal cost of production.*

See Appendix C for proof. As a result of this, the per-unit price firm j charges is a function of the marginal cost of the input supplier used by firm j, $p(e) = MC_j = \frac{1}{z(e_j)} MC_{s(e_j)} w^{1-\alpha}$, where w is the price of labor to all firms. Because the price charged by each firm can be written in terms of the supplier's marginal cost, all of these prices can be calculated using only the network structure and $z(e_j)$'s. The price charged by any firm on a cycle can be traced back through each supplier until it is expressed in terms of itself, thus there is a closed form solution for any price on a cycle. The price charged by any firm on a branch can be traced up to the root node of the cycle, which

[1]If it had no cycle then there would need to exist one node with no supplier and if there was more than one cycle then there would exist some node with more that one supplier.

must be on a cycle, thus any such price can be calculated. See Appendix C for formal derivation.

The profit maximization problem each firm j solves is

$$\max_{p_j^0, y_j^0, x(e_j), l_j} p_j^0 y_j^0 + \sum_{e \in \hat{D}_j} [p(e)x(e) + \tau(e)] - [p(e_j)x(e_j) + \tau(e_j)] - w l_j$$

s.t.

$$y_j^0 + \sum_{e \in \hat{D}_j} x(e) \leq \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} z(e_j) x(e_j)^\alpha l_j^{1-\alpha}$$

and all firms are jointly subject to the labor constraint, $\sum_{j \in J} l_j = L$.

Each $N \in \mathcal{N}$ defines a different profit maximization problem for each firm and the solutions to these produce a set of payoffs for each firm for each PEN. These payoffs are what determine the pairwise-stable and the coordination-proof equilibria, as defined in the previous section.

Just as the efficiency of an individual firm in the planner's problem is defined as the ratio of the two shadow costs μ and λ_j, define the efficiency of the individual firms in this case as, $\widetilde{q}_j \equiv \frac{w}{MC_j}$. As in the case of the planner's problem, this can be written in terms of the efficiency of firm j's input supplier.

Theorem 5. *The efficiency of a given firm is a function of the efficiency of the input supplier of that firm, $\widetilde{q}_j = z(e_j)\widetilde{q}^{\alpha}_{s(e_j)}$, where $s(e_j)$ is the identity of the input supplier used by j.*

Proof. Cost minimization and the result above,

$$MC_j = \frac{1}{z(e_j)} MC^{\alpha}_{s(e_j)} w^{1-\alpha}.$$

The definition of $\widetilde{q}_j$ gives,

$$\frac{w}{\widetilde{q}_j} = \frac{1}{z(e_j)} \Big(\frac{w}{\widetilde{q}_{s(e_j)}}\Big)^{\alpha} w^{1-\alpha}$$

$$\widetilde{q}_j = z(e_j)\widetilde{q}_{s(e_j)}.$$

□

6. Input Removal

I compare the aggregate output generated by an equilibrium network before and after an edge of the network is removed. Let e^* be the deleted edge and j^* be the identity of the firm to which e^* points. That is, j^* uses e^* in its production. When this edge is deleted, it creates a new potential network, and thus a new set of PENs. This new set of PENs is a strict subset of the original set. The new equilibrium network is determined from among this new set of PENs.

An intuitive analysis of the result from this edge deletion may suggest a path-dependent cascade of the drop in j^*'s efficiency along all of the edges emanating from j^* and its customers, holding the other equilibrium network links fixed. However, this is not an equilibrium. See Figure 4 for an example. Figure 4(a) shows the potential network and Figure 4(b) shows the original coordination-proof equilibrium. Note that this is also then a pairwise-stable equilibrium. If the edge from firm 2 to firm 1 is deleted and the other edges are held fixed, while firm 1 chooses the lowest marginal cost supplier available to it - firm 4 - then the network will be as shown in Figure 4(c). However, this network is neither pairwise-stable nor coordination-proof. The coordination-proof equilibrium that results from deleting the edge from firm 2 to firm 1 is shown in Figure 4(d).

While in the case of a solution to the planner's problem, the new output will be lower than the original output, this is not necessarily true in the case of a new pairwise-stable equilibrium, and, in fact, this result survives the equilibrium refinement of the coordination-proof equilibrium.

Result 1. *There exist parameters of the model such that the output produced by a pairwise-stable equilibrium or by a coordination-proof equilibrium increases when an edge is deleted and a new equilibrium of the same type is determined.*

This will occur when the edge that is deleted is the only edge preventing a higher-output network from being pairwise stable or coordination proof. If, when an edge is deleted the set of new equilibrium networks is a strict subset of the original set of equilibrium networks, then certainly output will decrease. However, this need not be the case. There are situations when deleting an edge makes it possible for a new network to be pairwise stable or coordination proof. If the output produced by such a new network is higher than the output in the original network, then output will increase. See Figure 5 for an example. There are three coordination-proof (and pairwise-stable) networks corresponding to the potential network shown in Figure 5(a). Of those, the one that offers the highest output, 0.1430, is depicted in Figure 5(b). When the edge from firm 4 to firm 1 is deleted, the new set of coordination-proof equilibria consists of five networks. From those, the highest possible output is now 0.1904. The network that produces this output is depicted in Figure 5(c).

6.1. Network Connectivity and Firm Centrality

To avoid overly cumbersome phrasing, I restrict my focus to the situation in which output falls when an edge is deleted. These results hold in the case that output rises, as well. The connectivity of the equilibrium network as a whole plays a roll in how far aggregate output falls after an edge is deleted. The intuitive relationship may be that the more connected an equilibrium production network is, the harder each firm will be hit when j^* loses its input supplier and has to choose a different one, and this aggregate output will drop by more when a network is more connected. While the simulations described in the next section show that on average this is the case, the opposite may also occur.

Result 2. *There exist parameters of the model such that higher connectivity in an original equilibrium network can lead to a smaller decrease in aggregate output.*

See Figures 6 and 7 for an example. In the first potential network, shown in Figure 6(a), firm 2 has two available suppliers and in the second potential network, Figure 6(b), firm 2 has three available suppliers. Figure 6(c) shows the coordination-proof equilibrium they have in common, in which output is 0.1430. When the edge from firm 4 to firm 2 is deleted from both of them, the resulting new coordination-proof equilibria are shown in Figure 7(a) and Figure 7(b), respectively. The output in the first one is 0.0371, while the output in the second one is 0.0356.

Just as the connectivity of the network as a whole affects the drop in output, the level of connectedness, or centrality, of j^* in the potential network, as measured by the number of alternative suppliers available to j^*, plays a roll as well. As one might expect, the average outcome is that if j^* has more alternative suppliers to choose from when it looses its equilibrium input supplier, the drop in aggregate output will be smaller than if j^* had fewer alternative suppliers. However, the opposite effect is possible.

Result 3. *There exist parameters of the model such that a higher number of alternative suppliers for j^* can lead to a larger drop in aggregate output.*

See Figures 8 and 9 for an example. The network in Figure 8(a) is less connected than the network in Figure 8(b); however when the edge from firm 4 to firm 2 is deleted from both of them, the output in the less connected network decreases more.

7. Simulation Results

I simulate this model by generating potential production networks and then finding a solution to the planner's problem, a pairwise-stable equilibrium, and a coordination proof equilibrium. I create the potential network by drawing a number of possible input suppliers for each from a Poisson(3) distribution. The identity of each supplier is drawn uniformly with replacement from the other firms. The productivity parameter, $z(e)$, for each edge e is drawn from a Pareto$(0.2, -1.8)$ distribution. This parameterization is motivated by the Carvalho (2012) survey on Input-Output analysis. Note that drawing the supplier identities with replacement allows for multiple edges from a given firm. However, because the productivity parameters are realizations of continuous random variables, the edge with the higher z will always be chosen. These simulations consisted of the creation of $1,000$ potential networks, in 512 of which a solution to all three problems was found.

7.1. Removal of an Input Supplier

For each successful solution to the planner's problem, pairwise-stable equilibrium, and coordination-proof equilibrium determination, I delete each edge in use in equilibrium. I do this by removing that edge from the potential network and then find a new solution of each type. In $2,536$ of these edge deletion experiments, a new solution to all three allocations is found. I take the ratio of output after the edge is deleted to output before the edge is deleted. Label this relative output for each allocation: planner, pairwise-stable, and coordination-proof.

I measure the connectivity of each original equilibrium using the average shortest path distance. I do this by calculating the length of the directed path from each node to every other node and taking the average across all such paths. A larger average shortest path distance corresponds to a less connected network and a shorter average shortest path distance corresponds to a more connected network. For each edge deletion, i, I regress the relative output of each allocation on the average shortest path distance of the corresponding original equilibrium network. That is, I estimate the following regression equation.

$$\widehat{\text{relative output}}^{E} = \beta(\text{avg. shortest path distance})$$

for each $E \in \{$Planner's Solution, Pairwise-Stable, Coordination-Proof$\}$ using ordinary least squares. The results are reported in Table 1.

Table 1: Output and Connectivity

E	$\hat{\beta}$
Planner's Solution	0.3966
Pairwise-Stable	1.6061
Coordination-Proof	0.7337

Each of the three estimated regression coefficients is positive, indicating that a larger average shortest path distance is correlated with a larger relative output. This means that more connected equilibrium networks are correlated with larger drops in output after an edge is deleted, and this result holds over all three allocations.

Label the number of input suppliers available to j^* in the potential network as $\#\text{sup}_i$ and the number of customers j^* has in the original equilibrium network $\#\text{cust}_i$. The observation is one edge deletion experiment and each such edge deletion experiment defines a j^*. For each equilibrium type, $E \in \{$Planner's Solution, Pairwise-Stable, Coordination-Proof$\}$, I estimate

$$\widehat{\text{relative output}}^{E} = \gamma_1(\#\text{sup}) + \gamma_2(\#\text{cust})$$

using ordinary least squares. The results are reported in Table 2.

Table 2: Output and Centrality		
E	$\hat{\gamma}_1$	$\hat{\gamma}_2$
Planner's Solution	0.2626	0.0382
Pairwise-Stable	0.9249	0.4786
Coordination-Proof	0.4728	0.0998

The estimated regression coefficients on the number of available suppliers are positive for all three equilibrium definitions. This indicates that a larger number of available input suppliers is correlated with a smaller drop in output after an edge is deleted.

The estimated regression coefficients on the number of customers in the original equilibrium network are all positive as well, indicating the more customers j^* has when it loses its input supplier, the higher output will be afterwards. However, in the case of the coordination-proof equilibrium the confidence interval of $\hat{\gamma}_2$ includes zero, and this was true for every size of simulation. While it may be the case that in the planner's problem and in pairwise-stable networks, a larger number of customers is correlated with a smaller drop in aggregate output, this cannot be concluded for coordination-proof networks.

8. Conclusion

The key contributions of this paper are a new network model which features a finite number of firms and endogenous network determination, a refinement of the standard network equilibrium, and a better understanding of the role network connectivity and firm centrality play in the determination of aggregate outcomes. Simulation results indicate the following. First, on average, the more connected a production network is, the smaller the decrease in aggregate output will be when a firm loses an input supplier. Second, on average, the more alternative suppliers that firm has, the smaller the drop in output will be when that firm loses its input supplier.

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9. Appendix A: The Planner's Problem

9.1. Proof of Theorem 3: Aggregate Output and Efficiency

Each firm j uses a number of supply chains in the production process. A supply chain is a string of firms, each producing intermediate goods for the next firm in the chain. Label the set of supply chains which lead to firm j as $\mathcal{S}_j$. Partition firm j's final output, y_j^0, by final output produced by each supply chain, $s \in \mathcal{S}_j$. That is, $\sum_{s\in\mathcal{S}_j} y_j^0(s) = y_j^0$. To make $y_j^0(s)$, firm j uses labor $l_j^0(s) \leq l_j$ and intermediate input $x_j^0(s) \leq x_j$. This $x_j^0(s)$ is produced by firm j's supplier using $l_j^1(s)$ and $x_j^1(s)$, this $x_j^1(s)$ is produced using $l_j^2(s)$ and $x_j^2(s)$, and so on up the supply chain. In general, I write $l_j^{k+1}(s)$ and $x_j^{k+1}(s)$ are the labor and intermediate input amounts used to make $x_j^k(s)$, along supply chain s to make firm j's output for final consumption. The lower subscript describes the good at the end of the supply chain and the superscript describes the step up the supply chain,

Let $\lambda_j^0(s)$ be the marginal social cost of producing $x_j^0(s)$ and let μ be the marginal social cost of labor, following the earlier notation. The optimal choices of $x_j^0(s)$ and $l_j^0(s)$ give:

$$\frac{\lambda_j^k(s)x_j^k(s)}{\alpha} = \frac{wl_j^k(s)}{1-\alpha}$$

$$\frac{\lambda_j^{k+1}(s)x_j^{k+1}(s)}{\alpha} = \frac{wl_j^{k+1}(s)}{1-\alpha}.$$

Using the technological constraint,

$$\lambda_j = \frac{1}{z(e_j)}\lambda_{s(e_j)}^{\alpha}\mu^{1-\alpha}$$

where $s(e_j)$ is the supplier of edge e_j. This means that the marginal social cost of producing product j is determined by the marginal social cost of firm j's supplier. This will be necessary for connecting efficiency across the entire network. The network structure gives $\lambda_j^{k+1}(s)x_j^{k+1}(s) = \alpha\lambda_j^k(s)x_j^k(s)$, so the optimality condition for the $(k+1)$th step up the supply chain becomes $\lambda_j^k(s)x_j^k(s) = \frac{wl_j^{k+1}(s)}{1-\alpha}$. Substituting this into the optimality condition for the kth step gives $l_j^{k+1}(s) = \alpha l_j^k(s)$. That is, the labor used in each step along the supply chain to make intermediate goods for $y_j^0(s)$ is a constant share of the labor used in the previous step.

Let $l_j(s)$ be the total labor used along supply chain s to make $y_j^0(s)$ such that $l_j(s) = \sum_{k=1}^{\infty} l_j^k(s) = \sum_{k=1}^{\infty} \alpha^k l_j^0(s) = \frac{l_j^0(s)}{1-\alpha}$. From the optimality condition for the final step in the supply chain,

$$\lambda_j y_j^0(s) = \frac{wl_j^0(s)}{1-\alpha} = wl_j(s).$$

Finally, write $y_j^0 = \sum_{s\in\mathcal{S}_j} y_j^0(s) = \sum_{s\in\mathcal{S}_j} \frac{w}{\lambda_j} l_j(s) = \frac{w}{\lambda_j}\sum_{s\in\mathcal{S}_j} l_j(s) = \frac{w}{\lambda_j} l_j$. From the definition of q_j, I can now write $y_j^0 = q_j l_j$.

From here, I use the solution to this planner's problem to show that $Y^0 = L\Big[\sum_{j=1}^{J} q_j^{\epsilon-1}\Big]^{\frac{1}{\epsilon-1}}$. Let $\Big[\sum_{j=1}^{J_t} q_j^{\epsilon-1}\Big]^{\frac{1}{\epsilon-1}} \equiv Q$. First, the first order condition with respect to firm y_j^0 is $\big(Y^0\big)^{\frac{1}{\epsilon}}(y_j^0)^{-\frac{1}{\epsilon}} = \lambda_j$. Rearranging this gives $\lambda_j = \Big(\frac{y_j^0}{Y^0}\Big)^{-\frac{1}{\epsilon}}$. This expression can be used to show that $\sum_{j=1}^{J_t} \lambda_j^{1-\epsilon} = 1$.

Rewriting the definition of firm efficiency for q_j allows me to write $\sum_{j=1}^{J_t}\big(\frac{w}{q_j}\big)^{1-\epsilon} = 1$. Then, using the definitions of q_j and Q, I can write $\frac{y_j^0}{Y^0} = \Big(\frac{q_j}{Q}\Big)^{\epsilon}$.

Finally, I use the labor constraint to show $L = \sum_{j=1}^{J_t}\Big(l_j\Big) = \sum_{j=1}^{J_t}\Big(\frac{y_j^0}{q_j}\Big) = Y^0 Q^{-\epsilon}\sum_{j=1}^{J_t} q_j^{\epsilon-1} = Y^0 Q^{-\epsilon} Q^{\epsilon-1} = \frac{Y^0}{Q}$. Rewritten, this is the expression I need: $Y^0 = QL$.

10. Appendix B: Equilibrium

10.1. Existence of a Solution to Each Agents' Problem

I use the Theorem of the Maximum to show that the individual agents' problems each have a solution. Recall that the optimization process proceeds as follows: taking the set of inputs that are used and their associated two-part prices as given, firms maximize profits by choosing the price for final output, p_j^o, the amount of final output, y_j^o, the amount of the intermediate good they use, x_j, and the labor they use, l_j. Then the firms choose their optimal input and the two-part prices they charge their potential customers, that is, the set of edges that are used in the production network, $\hat{E}$, and prices for each edge, $\{p(e), \tau(e)\}_{e\in\hat{E}}$.

The objective function is certainly continuous, so I focus here on the compactness and continuity of the constraint correspondence. Each firm must choose prices for each technique for which they are a supplier. These prices are drawn from compact and continuous sets. These prices in turn continuously determine the budget sets for the firms. Note that for each possible input choice, there is a set of four choice variables, p_j^o, y_j^o, x_j, and l_j, each of which is chosen from a compact and continuous correspondence determined by the budget sets.

The constraint correspondence is therefore the finite Cartesian product of compact and continuous correspondences and is therefore compact and continuous. Thus, the Theorem of the Maximum applies and each agent's problem has a solution.

Given that each agent has a solution to her maximization problem, and that in each time period there is a finite number of agents, Kakutani's Fixed Point Theorem applies and there exists a Mixed Strategy Nash Equilibrium in each time period.

11. Appendix C: Prices

11.1. Pricing at Marginal Cost

Proposition: The per-unit price, $p(e)$, that firm j charges to firm i along edge e is the marginal cost of firm j, MC_j.

Proof: Following Oberfield (2013), suppose for the purpose of contradiction that firm j charges some other price, $p(e) = \hat{p} \neq MC_j$. Label the associated contract $(\hat{p}, \hat{\tau})$.

Consider the deviation from $(\hat{p}, \hat{\tau})$, $p(e) = \widetilde{p} = MC_j$ and $\tau(e) = \widetilde{\tau} = \hat{\tau} + (\hat{p} - MC_j)x(e) + K$, where

$$K = \frac{1}{2}\Big\{(\widetilde{p_i^0} - \widetilde{c}_i)\widetilde{y_i^0} - (\hat{p_i^0} - \widetilde{c}_i)\hat{y_i^0} + \sum_{e \in D_i}(p(e) - \widetilde{c}_i)[x_{b(e)}(\widetilde{y_i^0}) - x_{b(e)}(\hat{y_i^0})]\Big\}$$

and $\widetilde{c}_i = \frac{1}{z(e)} MC_j^{\alpha} w^{1-\alpha}$, the marginal cost of i given the deviation. I will show that both firm j and firm i are made better off by this deviation and thus $(\hat{p}, \hat{\tau})$ is not optimal.

First, $\widetilde{\pi}_j - \hat{\pi}_j = K > 0$.

Next, $\widetilde{\pi}_i - \hat{\pi}_i = \widetilde{p_i^0}\widetilde{y_i^0} - \hat{p_i^0}\hat{y_i^0} + \sum_{e \in D_i} p(e)[x_{b(e)}(\widetilde{y_i^0}) - x_{b(e)}(\hat{y_i^0})] - [\widetilde{c}_i\widetilde{y}_i - \hat{c}_i\hat{y}_i] - [\widetilde{\tau} - \hat{\tau}]$. By using, (i) $p(e)x(e) = \alpha c_i y_i$ and (ii) $\frac{c_i}{\widetilde{c}_i} = \left(\frac{p(e)}{c_j}\right)^{\alpha}$, I can write:

$$\widetilde{\pi}_i - \hat{\pi}_i =$$

$$(\widetilde{p_i^0} - \widetilde{c_i})\widetilde{y_i^0} - (\hat{p_i^0} - \widetilde{c_i})\hat{y_i^0} + \sum_{e \in D_i}(p(e) - \widetilde{c_i})[x_{b(e)}(\widetilde{y_i^0}) - x_{b(e)}(\hat{y_i^0})] - K$$

$$+\big[\big((1-\alpha) + \frac{MC_j}{p(e)}\alpha\big)\big(\frac{p(e)}{MC_j}\big)^{\alpha} - 1\big]\widetilde{c_i}\hat{y}_i.$$

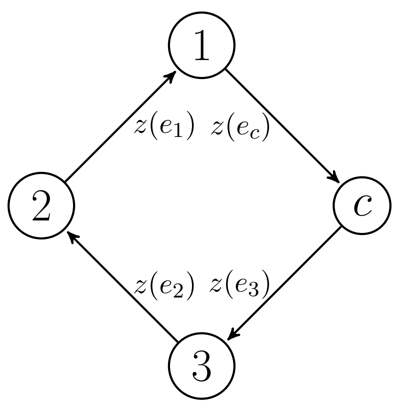
By the definition of K, the first line of the equation is positive. Jensen's Inequality gives that $[(1-\alpha) + x\alpha]x^{-\alpha} \geq 1$ for $\alpha \in [0, 1]$. Applying this to the second line of the equation gives that $[(1-\alpha) + \frac{MC_j}{p(e)}\alpha](p(e)/MC_j)^{\alpha} \geq 1$, so the second line of the equation is non-negative. As a result, $\widetilde{\pi_i} - \hat{\pi_i}$ is postive and both firm i and firm j are made better off by firm j charging MC_j.

11.2. Derivation of Price Expressions

Both of the following derivations are driven by the fact that the price that each firm pays for each unit of the input they use is the marginal cost of the supplier of that input, as proved above.

1. Price Paid by Firms on a Cycle

   For a cycle of length c, there are c firms and c edges. Label these firms $1, \ldots, c$. Without loss of generality, we find the price of firm c and label the supplier c uses as 1, the supplier 1 uses as 2 and so on.



   The price that c pays for its input is $p(e_c) = \frac{1}{z(e_1)}p(e_1)^{\alpha}w^{1-\alpha}$, where $p(e_1)$ is the price firm 1 pays for its input. This price is $p(e_1) = \frac{1}{z(e_2)}p(e_2)^{\alpha}w^{1-\alpha}$, where $p(e_2)$ is the price firm 2 pays for its input. Continuing in this way I can write the price that firm $c-1$ pays as $p(e_{c-1}) = \frac{1}{z(e_c)}p(e_c)^{\alpha}w^{1-\alpha}$. Substituting each price expression into the previous one gives:

$$p(e_c) = p(e_c)^{\alpha^c}\Big[\frac{1}{z(e_1)}\frac{1}{z(e_2)^{\alpha}}\cdots\frac{1}{z(e_c)^{\alpha^{c-1}}}\Big]w^{(1-\alpha)+\alpha(1-\alpha)+\ldots\alpha^{c-1}(1-\alpha)}.$$

Solving for $p(e_c)$ gives:

$$p(e_c) = \left(\left[\frac{1}{z(e_1)}\frac{1}{z(e_2)^{\alpha}}\cdots\frac{1}{z(e_c)^{\alpha^{c-1}}}\right]w^{(1-\alpha)+\alpha(1-\alpha)+\ldots\alpha^{c-1}(1-\alpha)}\right)^{\frac{1}{1-\alpha^c}}$$

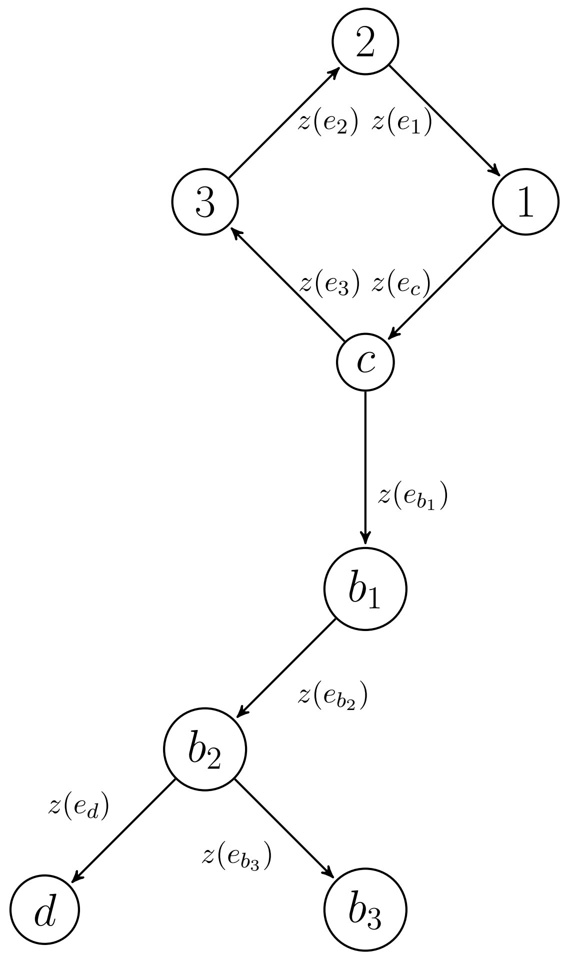
$$= w^{\frac{1-\alpha}{1-\alpha^c}\sum_{k=1}^{c}\alpha^{k-1}}\prod_{i=1}^{c}\left(\frac{1}{z(e_{i+1})}\right)^{\frac{\alpha^{i-1}}{1-\alpha^c}}$$

$$= w^{\frac{1-\alpha}{1-\alpha^c}\frac{\alpha^c-1}{\alpha-1}}\prod_{i=1}^{c}\left(\frac{1}{z(e_{i+1})}\right)^{\frac{\alpha^{i-1}}{1-\alpha^c}}$$

$$= w\prod_{i=1}^{c}\left(\frac{1}{z(e_{i+1})}\right)^{\frac{\alpha^{i-1}}{1-\alpha^c}}.$$

2. Price Paid by Firms on a Branch

   Each connected component of the network has one cycle, and potentially many branches emanating from that cycle. Thus, each branch has a root node on the cycle. Because each firm pays the marginal cost of its supplier, the cost of any branch firm can be traced back and written in terms of the price of this root node. Let firm d be d edges down the branch from the node where $d > 1$.



The price that d pays for its input is the marginal cost of its supplier. The price the supplier pays is the marginal cost of his supplier and so on up to the root node, whose price was found in the above derivation. Label $MC_r = \frac{1}{z(e_r)} p_r^{\alpha} w^{1-\alpha}$. Then the price that firm d pays is

$$p(e_d) = MC_r^{\alpha^{d-1}} \left[\prod_{i=1}^{d-1} \frac{1}{z(e_{b_i})^{\alpha^{d-i-1}}} \right] w^{(1-\alpha) \sum_{k=0}^{d-2} \alpha^k}$$

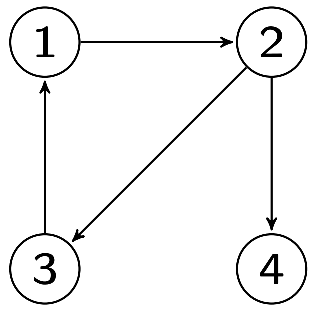
$$= MC_r^{\alpha^{d-1}} \left[\prod_{i=1}^{d-1} \frac{1}{z(e_{b_i})^{\alpha^{d-i-1}}} \right] w^{(1-\alpha) \frac{\alpha^d - 1}{\alpha - 1}}$$

$$= MC_r^{\alpha^{d-1}} \left[\prod_{i=1}^{d-1} \frac{1}{z(e_{b_i})^{\alpha^{d-i-1}}} \right] w^{1-\alpha^d}.$$

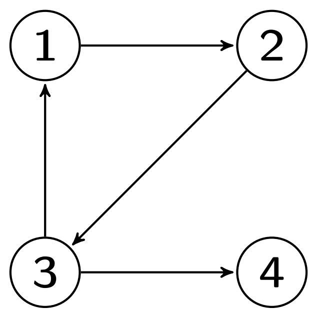
For a firm that is only one edge away, for example firm b_1 in the figure, the price that firm pays is the marginal cost of the root node, MC_r.

12. Figures

Figure 1: Network N_1 is 1-adjacent to N_2 and vice versa.

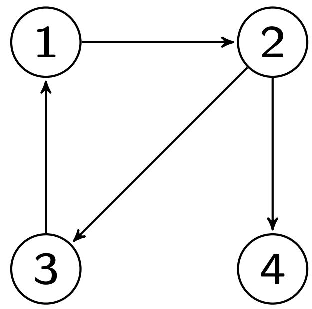


(a) N_1

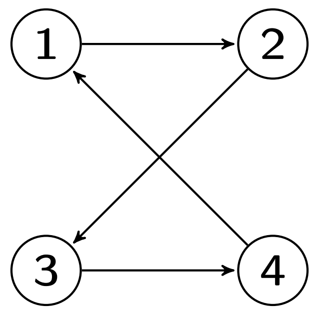


(b) N_2

Figure 2: Network N_3 is 2-adjacent to N_4 and vice versa.



(a) N_3



(b) N_4

Figure 3: The gray nodes form a cycle; the white nodes form a branch.

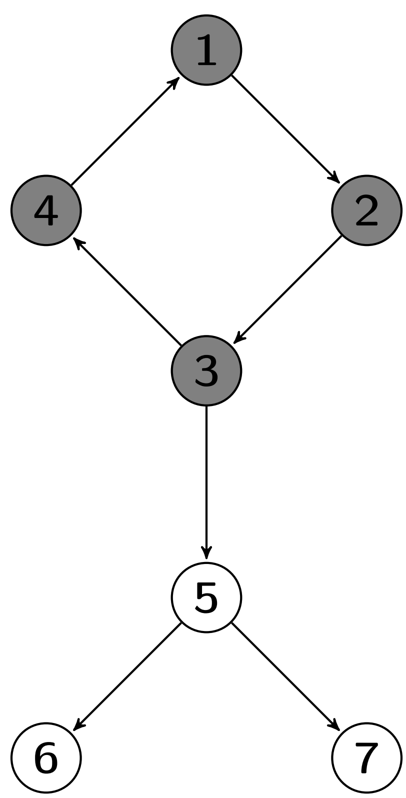
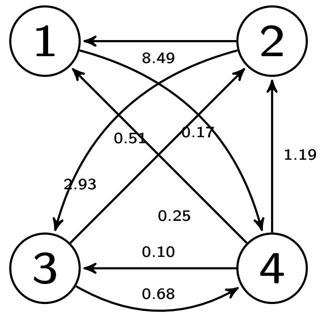
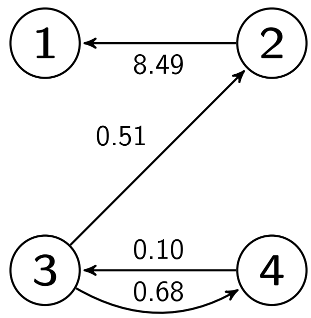


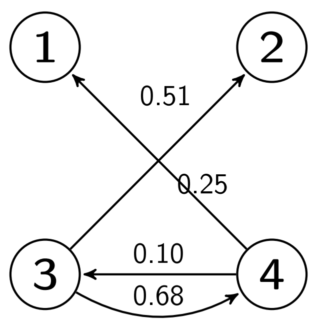
Figure 4: Deleting an edge and holding the other edges fixed is not neccessarily an equilibrium



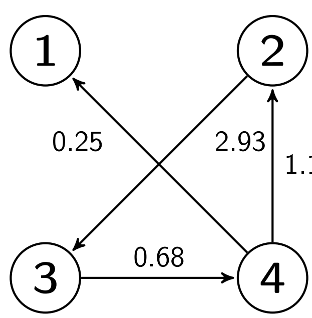
(a) Potential Network



(b) Original Equilibrium Network

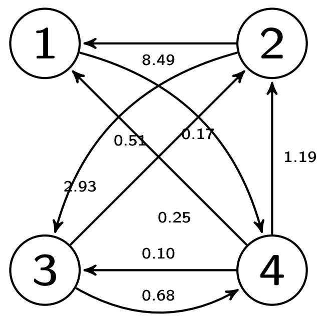


(c) Holding other edges fixed

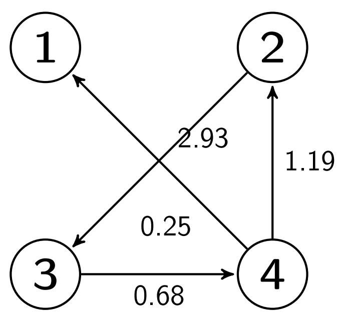


(d) The equilibrium network when the edge is deleted

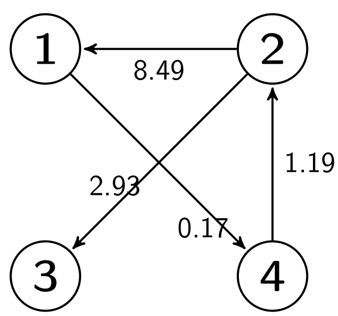
Figure 5: Output increases when the edge from firm 4 to firm 3 is deleted.



(a) Potential Network

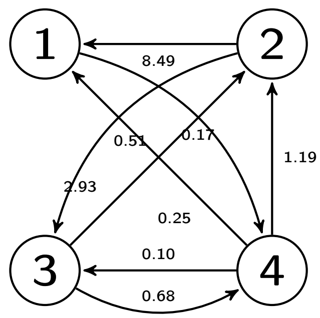


(b) Original Equilibrium Network, Output = 0.1430

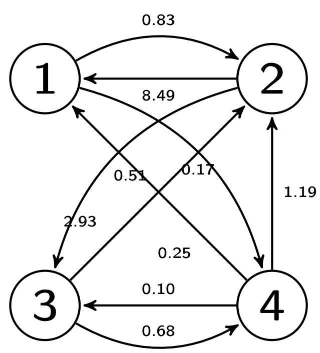


(c) New Equilibrium, Output= 0.1904

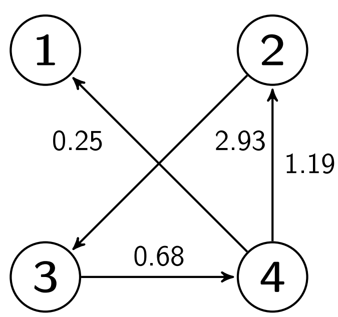
Figure 6: Output goes down by more when firm 2 has more alternative suppliers.



(a) Potential Network #1, Firm 2 has two available suppliers.

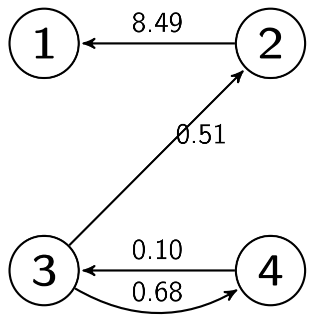


(b) Potential Network #2, Firm 2 has three available suppliers.

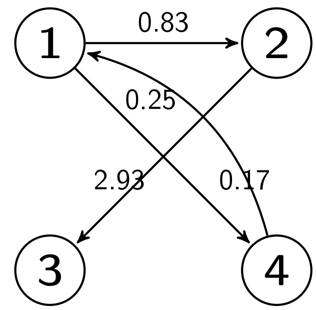


(c) Original Equilibrium Network, Output = 0.1430

Figure 7: Output goes down by more when firm 2 has more alternative suppliers.

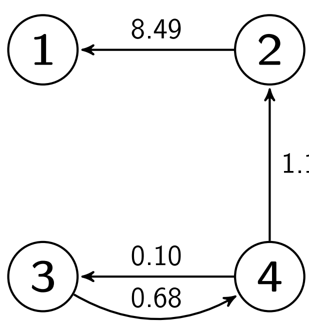


(a) New Equilibrium for Potential Network #1, Output= 0.0371

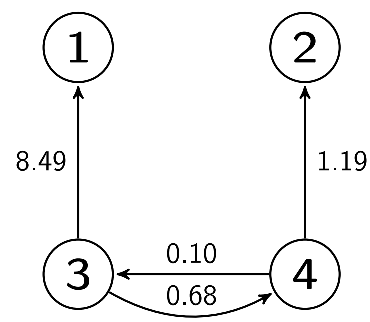


(b) New Equilibrium for Potential Network #2, Output= 0.0356

Figure 8: Output goes down less for the more connected network.

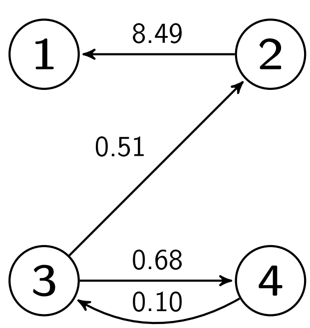


(a) Original Equilibrium #1, Output= 0.0452, Avg. Path Distance= 1.6875

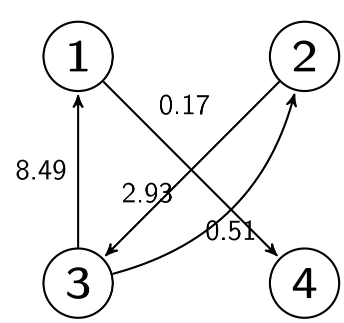


(b) Original Equilibrium #2, Output= 0.0445, Avg. Path Distance= 1.5

Figure 9: Output goes down less for the more connected network.



(a) New Equilibrium #1, Output= 0.0371



(b) New Equilibrium #2, Output= 0.1854