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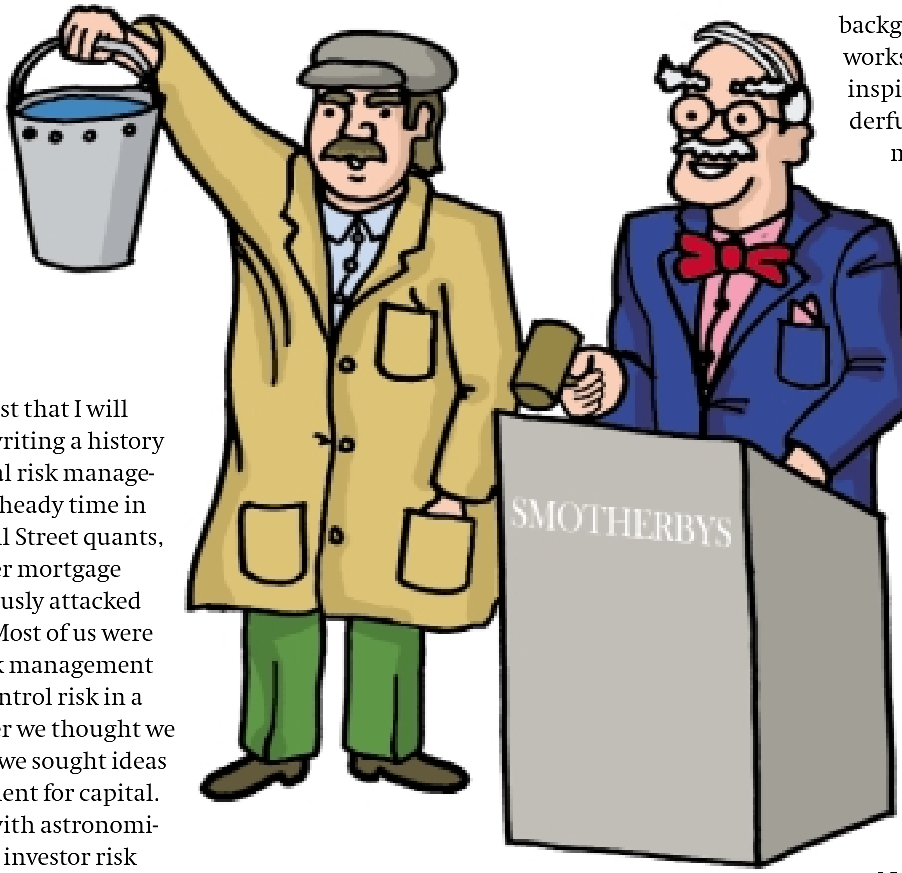
The Dakota Option Part I

The author harks back to a time when he had more brains than money



And our next lot, what remains of the iceberg that sank the titanic

Questions Mal Posées

One of the projects on my to-do list that I will probably never get around to is writing a history of the origins of modern financial risk management from 1988 to 1993. It was a heady time in which the first generation of Wall Street quants, fresh from apparent victories over mortgage securities and derivatives, vigorously attacked the problem of risk and capital. Most of us were not interested in the modern risk management problem, how to measure and control risk in a large financial institution. Rather we thought we had more brains than money, so we sought ideas that substituted clever management for capital. Some of us started hedge funds with astronomical leverage ratios and negligible investor risk (hopefully). Others worked on trading desks to introduce products and strategies that produced steady high profits without taking up limit (hopefully). We were a hopeful bunch. Along the way, we got VaR and RAROC and changed the way financial institutions are regulated and run.

What makes the history hard to write today, and impossible in the near future, is that many of the most productive people switched fields. When the field was wide-open with interesting problems and the potential for riches, lots of smart, greedy, creative original thinkers flocked to it. As it matured, many of them moved on to other frontiers. This is a typical pattern in intellectual development. But unlike some other fields, the early work is not preserved in publications or archived documents, or even Internet sites or email. Much of it took place on private dial-up computer bulletin boards, or in proprietary internal working groups of organizations that no longer exist.

Although we came from a wide variety of backgrounds, there were a few published works that everyone seemed to have been inspired by independently. One was the wonderful book *Ill-Posed Problems*[1] (or, as it was more commonly known, *Questions Mal Posées*, said in a bad French accent) by the amazing Russian mathematician Andrei Nikolaevich Tikhonov.

A well-posed problem is one that has a unique solution that is a stable function of initial conditions. Interesting ill-posed problems typically violate both of these conditions and have an infinite number of solutions that are infinitely sensitive to initial conditions. This sounds similar to chaos theory, which also studies systems that are infinitely sensitive to initial conditions. The difference is chaos theory accepts the chaos, and tries to make useful predictions anyway. Ill-posed theory teaches that if you have an ill-posed problem, look for the nearest well-posed problem and solve it instead.

What happens to an option price after expiration?

The classic example of an ill-posed problem is the backward heat equation. Imagine an infinitely long perfect thermal conductor one-dimensional wire, encased in a perfect thermal insulator. In this system, the second partial derivative of temperature with distance is proportional to the first partial derivative of temperature with respect to time. If I tell you the temperature of the wire at all points in space at one time, it's not hard to solve for the temperature at any point in space at any future time. But it's an ill-posed problem to ask for the temperature at times in the past.

A question that gets asked every time I teach Introductory Option Pricing is, "isn't the Black – Scholes equation identical to the backward heat equation?" The answer is no. The Black – Scholes equation is the easy-to-solve forward heat equation, with underlying price taking the place of temperature. The second partial derivative of option price with respect to underlying price (the derivative of delta) is proportional to the first partial derivative with respect to time (theta). But the constant of proportionality has the opposite sign as the physical heat equation.

It's true the Black – Scholes pricing argument runs backward from known values at expiration to earlier prices. But that's the direction price diffuses. As we go back in time, the graph of option price versus underlying price gets smoother; with heat, as you go forward in time the temperature in the wire becomes a smoother function of position.

In general, it's easy to solve things in the direction of diffusion. At each step the functions become smoother. In the opposite direction things get kinkier and eventually discontinuous, at which point analytic formulae become undefined and numerical simulation routines generate error messages or give unreliable answers.

It's easy to predict what an ice sculpture will look like in the future, just weigh it and predict a puddle of the appropriate size. But imagine seeing a puddle and trying to figure out if it started as a copy of the Venus de Milo or just a lump of ice.

Another way to see the difficulty of pricing options after expiration is to use risk-neutral pricing. The payoff function of an option gives us the value at expiration, time T. Risk-neutral pricing says the value at any other time is the expected value of the payoff function at expiration discounted or compounded at the risk-free rate of interest.

For an arbitrary payoff function, I can represent it (or approximate it if the function is not analytic) with a MacLaurin series:

$$C(S, T) = \sum_{i=0}^{\infty} C^{(i)}(0, T)\frac{S^i}{i!}$$

where $C^{(i)}(0, T)$ is the ith derivative of the payoff function with respect to underlying price evaluated at $S = 0$. In the Black–Scholes world with risk-free interest rate r and underlying volatility σ this implies:

$$C(S, T) = \sum_{i=0}^{\infty} C^{(i)}(0, T)\frac{S^i}{i!}\exp\left((t - T)(i - 1)(r + i\sigma^2/2)\right).$$

The key is the $(t–T)$ in the exponential. If it is negative, that is if we are pricing before expiration, the higher derivative terms will have less influence at t than T, and $C(S, t)$ will be smoother than $C(S, T)$. But if $(t–T)$ is positive, even by a small amount, the series will diverge with the terms growing to infinity (except in some special circumstances such as all of the high order derivatives are zero). For large enough i, the $\exp(i^2)$ term will usually dominate everything else.

The well-posed versions

The first step in attacking an ill-posed problem is to find a similar well-posed problem. To make this meaningful, it's useful to consider why you want the answer in the first place. Physical or financial reality can suggest the appropriate change. For the purpose of this article, I'm going to suggest two applications, one simple but fanciful and one more complicated and practical.

The simple application is to help Bill hedge his employee stock options. He has just been granted a 10-year call on 100,000 shares of Megacomp stock at $100 per share. He is not allowed to short Megacomp stock, sell call options, buy put options or make any transaction whose contractual terms reference the price of Megacomp stock during his period of employment.

Bill calls his old girlfriend Emma who works at a structured equity derivatives desk. He's 45 years old and Megacomp has mandatory retirement at 65. Therefore he wants to write an option with the payoff dependent on the value of Megacomp stock in 20 years. This is allowed, because it depends on the price of the stock after he has retired. He asks Emma to figure out the payoff function in 20 years that implies an option value of $\max(S–100, 0)$ in 10 years, where S is the price of Megacomp stock in 10 years.

Because the value of this contract is the same as the value of Bill's employee stock option in 10 years, it has to be equal at all times up to 10 years. In particular, the fee Bill will get for writing this option is equal to the value of his employee stock options. In 10 years, Bill's plan is to exercise his options if they are in the money and use the profit to buy back his hedge contract. If there is no profit, the hedge contract is worthless and can be canceled for nothing. If there is profit, it will exactly equal the amount needed to buy back the hedge contract. Bill doesn't care what the payoff function in 20 years is, since he will buy back the contract before that time. He just needs such a payoff function to exist to satisfy Megacomp's compliance officer.

Imagine seeing a puddle and trying to figure out if it started as a copy of the Venus de Milo or just a lump of ice

The Dakota option

For a more practical application, consider the problems that come up due to the discontinuity at option expiration. What if the market is closed at expiration, as it was from September 11 to September 16, 2001? If you hold to the legal terms of an option expiring during the period, you penalize the holder as she doesn't have the market information, and possibly the communications or systems support, anticipated at the time the option was written. If you allow her to postpone the exercise decision until the market reopens, you penalize the writer. Broad market closures like September 11 are thankfully rare, but individual markets can close or become highly illiquid for periods due to government actions, legal uncertainty, credit problems or natural disasters.

A more common problem is what to do if the holder forgets to exercise, or cannot communicate with the writer? Or if there is a legal dispute over what entity can make the exercise decision? Or if there is a legal dispute over the nature of the underlying security? For physical settlement options, the underlying may be impossible to deliver. For cash settlement options, the appropriate prices may be uncertain at expiration.

None of these problems are insurmountable, some are handled by market convention or ISDA standards, others are resolved by the counterparties as they arise. But options would be more attractive to both writers and holders if the problems went away because it would reduce uncertainty and legal expense. The obvious solution is to let option prices continue to evolve past expiration. In problematic cases it would be possible to postpone settlement of the option on terms fair to both sides. This is what we do for fixed income contracts. If I borrow money from you and fail to pay it back at the promised time, the debt continues to accrue interest. There may be additional penalties but the point is there is a generally accepted adjustment for the delayed settlement that is fair to both sides, additional adjustments use this as a starting point.

Therefore I define a Dakota option[2] as a cash settlement option in which the holder has the right to settle at a defined payoff function at a defined time T, and at any other time before or after T at the Black-Scholes value of a vanilla European option with the same payoff function at T.[3] A Dakota option is more valuable than an American option for the same reason an American is more valuable than a European. However, under strict Black–Scholes assumptions, including payout protection and constant volatility and interest rates, all three options have the same value. Dakota option reduces certain settlement problems, and therefore should be preferred by both holder and writer at its fair market price.

Pricing Dakota

The problem with my definition of Dakota options is for times greater than T, the Black–Scholes formula gets a negative square root and starts giving imaginary results. In order to make Dakota options practical, we need to solve the ill-posed problem of option dynamics after expiration.

Let's begin our attack with one of the tools for defusing ill-posed problems, discretization. Assume that time moves in discrete steps and the underlying price can only go up or down by 1 at each step. Further assume that the option price at underlying price S and time t, $C(S, t)$, equals the average of $C(S+1, t+1)$ and $C(S-1, t+1)$[4]

Let option prices continue to evolve past expiration. In problematic cases it would be possible to postpone settlement of the option on terms fair to both sides

We are given the values of $C(S, T)$ for all S. It's an easy matter to compute $C(S, t)$ for all $t < T$, it's simply:

$$C(S, t) = 2^{t-T} \sum_{i=0}^{T-t} \binom{T-t}{i} C(S - T + t + 2i, T).$$

If you're writing a computer program, it's easy to imagine solving for all the $C(S, T-1)$s by averaging $C(S+1, T)$ and $C(S-1, T)$, then using the $C(S, T-1)$s to solve for the $C(S, T-2)$s and so on.

When we try to go forward, the formula above doesn't work since neither the summation nor the combination function are defined for negative values. The computer program runs into a problem on the first number. Suppose we start with computing $C(100, T+1)$. We know that:

$$C(99, T) = \frac{C(98, T+1) + C(100, T+1)}{2}$$

and

$$C(101, T) = \frac{C(100, T+1) + C(102, T+1)}{2}$$

but these cannot be solved for $C(100, T+1)$ without knowing some other value of C at $T+1$.

In fact, we can pick any value we want for $C(100, T+1)$ and then solve for all other values of C at $T+1$ for even underlying prices. This is the non-unique part of the ill-posed backward heat equation.

To see the extreme dependence on initial conditions, consider a specific example of a vanilla call option: $C(S, T) = \max(S-100, 0)$. Looking at the left equation above, and remembering that $C(99, T) = 0$, it's clear that $C(98, T+1) = -C(100, T+1)$. A little more thought should demonstrate that $C(96, T+1) = C(100, T+1)$ and we're going to set up an infinite wave unless we decide $C(100, T+1) = 0$. Once we do that we get the happy result that $C(S, T+1) = C(S, T)$ for all even $S > 100$. We'll also set $C(99, T+1)$ to zero to avoid an infinite wave for the odd numbers.

For $T+2$ we're going to keep to the same trick and make $C(99, T+2)$ and $C(100, T+2)$ zero. If we keep going this way we will discover that $C(S, T+i) = 0$ for $S \leq 100 + i$. But $C(101+i, T+i) = 2i$. For $S > 101 + i$ the values of $C(S, T+i)$ get larger in magnitude and alternate in sign. For different choices of $C(100, T+i)$ and $C(99, T+i)$ we get different results, but the same general pattern of a region of stability from $C(100-i, T+i)$ to $C(100+i, T+i)$ for all i, but large alternating values outside that region.

What does this mean in practice? Consider Bill's Megacomp option. Let's price it using 1,000 steps of \$1 up or down per year, corresponding to a volatility of about 32 per cent, about right for a large company stock. The payoff function at time T+10,000 (ten years after Bill's employee stock options expire) that will give the appropriate values at time T is 0 for $S \leq \$10100$. But if $S = \$10101$ Bill will be required to pay 2^{10000} dollars. At $S = \$10102$, Bill pays 2^{10001} dollars, but at $S =$ \$10103 he *receives* 9997×2^{10000} dollars. At higher values of S, Bill will pay or receive even larger amounts.

Clearly this option is economic nonsense. Discretization did not help Bill, unless his compliance officer is an ivory tower type. But in Part II we'll see how discretization paired with its usual partner bounding (or B&D as we ill-posed types like to day) can help price Dakota options for our second application.

To solve Bill's problem, we'll have to take a different approach. Instead of forcing a call option payout function to evolve, we approximate it with functions that evolve naturally. In the Black-Scholes world there are three natural evolvers: a constant amount, the underlying and a security whose value is the logarithm of the underlying price. We won't be able to fit $\max(S-100, 0)$ exactly with constant, S and $\ln(S)$, but we'll let Bill cash out a good portion of his stock option value.

THE NAME DAKOTA

Up to the early 1970s, when modern option pricing models were developed, all traded options could be exercised at any time up to expiration. However, the early models priced options that could be exercised only at maturity. It was difficult enough to publish serious financial work in those days without saying that your model couldn't quite price the options everyone traded.

Myron Scholes came across a reference to some turn of the century Swiss forward contracts that allowed one party to cancel at delivery in return for paying a fixed liquidated damage fee. With a little imagination, this became the European option, and papers were submitted with European option pricing models. American options, the kind everyone used at the time, were a parallel formation.

The geographic naming convention has continued with Bermuda options (halfway between Europe and America, in geography and culture), Asian and Himalayan. The tradition has not always been honored, for example we have digital, one-touch and lookback options.

"Dakota" is supposed to convey something more American than "American" since Dakota options can be exercised not only before expiration, but after. The Dakota Territory was the size of the UK, France and Germany combined, home to the Sioux (Dakota) native Americans. It was the last country-sized stronghold of native Americans, where the Sioux lived unconfined in reservations. It was not fully conquered until the battle/massacre at Wounded Knee in December 1890, the last battle of the Indian wars.

Dakota option as a cash settlement option in which the holder has the right to settle at a defined payoff function at a defined time *T*, and at any other time before

ENDNOTES

1 A.N. Tikhonov and V.Y. Arsenin. *Solutions of Ill-Posed Problems*. Winston, Washington, 1977.

2 "Option" may be misleading because a Dakota can require the holder to make payments if it is held after expiration. In that sense it is more like a future.

3 The use of Black – Scholes with a fixed volatility gives the Dakota Option implied volatility optionality. If implied volatility goes up, the holder can sell the Dakota call in the market for the increased price. If implied volatility goes down, the holder can settle the option with the writer at the fixed volatility. Variants could use other option pricing models and adjust for changes in volatility or other parameters. Another technical point is that Dakota options should be payout protected.

4 This amounts to an assumption that interest and payout rates are zero. Binomial option pricing usually assumes multiplicative moves rather than additive. I have simplified things to make the math easier, but the general argument applies to general discrete option pricing models.