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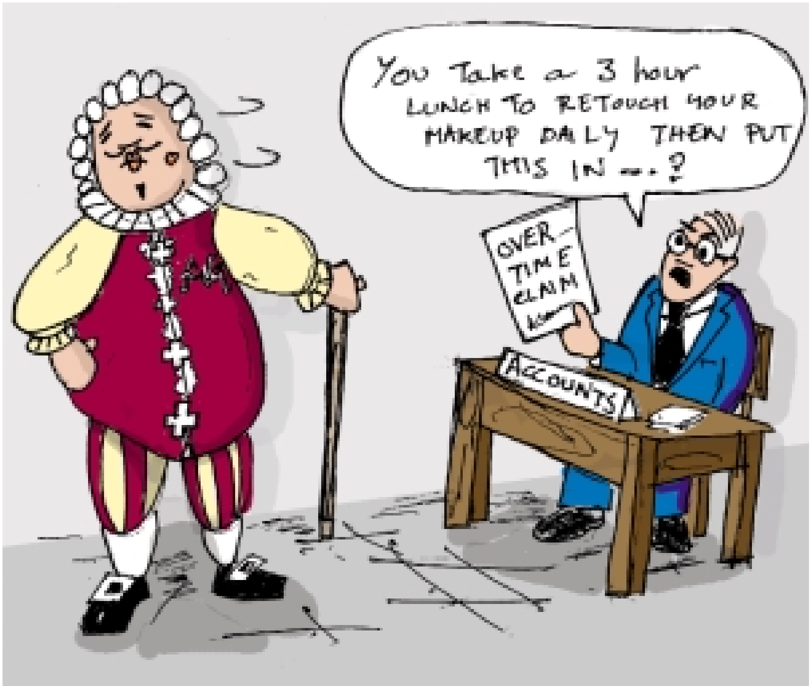
The Dakota Option Part III

Applying Dakotas to options that cannot be settled before expiry

Two issues back, I introduced the *Dakota option*, a cash settlement derivative with a defined payout at expiry, which can be exercised by the holder at any time before or after expiry for the Black-Scholes value of a vanilla European option with the same expiry payout. Last time I applied Dakotas to hedging employee stock options. Now I'm going to consider fair payment for options that cannot be settled at expiry.



The overtime problem was more tiresome than Gombaud had anticipated

The chevalier

This problem has a grand tradition in applied mathematics, dating back to Antoine Gombaud (1607-1684). Gombaud appears in every introductory statistics textbook, but is described variously as a "nobleman," "con man," "dandy," "aristocratic gambler" or "mathematician." Depending on the book he is identified as either "chevalier de Méré" or "self-styled chevalier de Méré."

In fact, Gombaud was the first and most important theorist of salons and his books were important for two centuries after his death. His current neglect can be blamed on the degeneration of his ideas into dandyism, and their contribution to the violent excess of revolutionary France. Democracy through transparent institutions run by dour middle-class Protestants won out over vibrant culture sustained by exclusive networks of intellectual Catholics imbued with je ne sais quoi. However, the Internet should revive interest in his works.

We can acquit Gombaud of two specific charges. He was not a con man; he was not even much of a gambler. He was a mathematician interested in probability (he would not have accepted that designation, however, one of his aphorisms is "a philosopher who descends to mathematics is like a suitor refused by the mistress, who seduces the housemaid instead"). He was accused of cheating for the same reason that blackjack card counters are accused today, he used his brain in competition with people who preferred stupid opponents. He also was not an impersonator. He wrote dialogs in which the character "chevalier de Méré" represented his opinions, a common literary form of the time. He did not represent himself as a chevalier (knight) in person.

The interrupted dice game and the extended dice game

In 1654, Gombaud was asked to arbitrate a dispute. Two gamblers had begun a common dice game of the time. Each player picked a number. A six-sided die was rolled repeatedly; the first person whose number came up seven times was the winner. This game lasts an average of 33 rolls, but will go twice as long or more every 500 games. In this case, the game was called off with one number having come up five times and the other four. The gambler who was ahead felt he was entitled to some payment, the question was how much?

Gombaud talked the matter over with his friend Blaise Pascal[1], who figured out the answer (6/16 of the stake) in five letters exchanged with Pierre de Fermat. To solve the problem, Pascal invented his triangle and Fermat reinvented Cardano's algebraic probability theory and communicated it to Christian Huygens, who published the first important work on probability, *De Ratiociniis in Ludo Aleae* ("On Reasoning in Games of Chance"). Then it's Huygens' student Leibnitz, the Bernoullis and all that. Probability theory was born. Over three hundred years later, the Black-Scholes model can be viewed as a mathematical elaboration of this problem, you know the future payoff of a bet and you want to compute the fair value today.

What if Gombaud had asked how to modify the stake for a game that continued beyond its original payout time instead of an interrupted game? Suppose that the score were seven to six, what would be the appropriate stake to continue the game until one number comes up nine times? If Gombaud had asked that, the history of applied mathematics might have been different and we might have had a Dakota option pricing model before Black-Scholes.

At first consideration, the overtime problem is the inverse to the interruption problem. When the score is five to four, the first player has 11/16 chance of getting to seven first so his expected winning is 11/16 − (1 − 11/16) = 6/16. Similarly, when the score is seven to six, the first player has 11/16 chance of getting to nine first. If he gives up the sure stake he receives if the game is terminated as originally agreed, the stake for the extended game must be 16/6 times the original amount. If the original game were to play to seven for \$3,000, the player ahead five to four must be paid \$1,125 to stop the game early and the player ahead seven to six must have the stake raised to \$8,000 to agree to extend the game to nine.

But the second problem is actually ill-posed . It has an infinite number of solutions. If the initial stake is S, the first player can receive any amount W for winning the extended game, as long as he pays $2.2 \times W - 3.2 \times S$ for losing.

The problem is also highly sensitive to initial conditions. Instead of assuming equal probabilities of winning each point, suppose the first player has probability p of winning. In the interrupted game problem, the appropriate payment is $6p^4 - 16p^3 + 12p^2 - 1$. This is a well-behaved function with range −1 to 1. The interrupted game solution is the inverse amount, which among other problems, becomes infinite at $p = 0.3857$.

The option game

An option is just a gambling game, with payoff defined at expiry. Black and Scholes showed how to compute the fair payment to get into or out of the game before expiry. We want to compute the fair adjustment to continue the game past expiry. Consider a European call option allowing purchase of one share of stock for \$100 at time T. At time T, due to some unforeseen circumstance, the markets are closed and no price information is available. Or there is a legal dispute about ownership of the option, or the nature of the underlying or whether or not the option was exercised. It is now $T + X$, the market is open, price information is available and all disputes are resolved. The stock price is S_{T+X}, what is the fair payment to settle the option?

One approach is to assign blame for the difficulties at expiry and resolve the dispute to the disadvantage of the party at fault. But what if neither or both parties are at fault? In any case, both parties might prefer in advance to settle things fairly regardless of fault in order to reduce risk and legal expense. Another approach is to reconstruct the situation at time T and figure out what should have happened. That also may not be possible or desired by the parties. The biggest objection to both these approaches is they have to be argued while S_{T+X} is known. It's much easier for two parties to agree on a fair solution before either one knows what S_{T+X} is going to be.

The definition of a "fair" solution in this case is one that both parties would accept at time T, if the markets were open, price information were available and there were no disputes. The value of the option at T is $\max(S_T - 100, 0)$. We want a function $C(S_{T+X})$ such that at time T, for all S_T, the risk-adjusted net present value of receiving $C(S_{T+X})$ at $T + X$ equals $\max(S_T - 100, 0)$. Unfortunately, the Parts I and II of this series demonstrated that there is no well-defined function C that meets this requirement.

A solution is possible if we relax the "for all S_T" condition. After all, if the option is clearly in or out of the money at time T, no dispute is likely. Suppose that the volatility over interval X is 1 per cent and interest rates are zero. Then the piecewise-linear function C defined below generates the correct values at time T to the nearest penny for $S_T \leq \$105$. This is five standard deviations in the money, which should be enough for practical purposes.

S_{T+X}	$C(S_{T+X})$
105	-\$35
104	\$44
103	-\$27
102	\$20
101	-\$8
100	\$2
99	-\$1
98	\$0
97	\$0

The trouble with the solution is the required payments at $T + X$ are large and erratic. Both parties must be prepared to pay amounts more than ten times any likely value of the original option. Small changes in S_{T+X} lead to large changes in payout. Moreover, if S_T is even a little higher that \$105, the solution breaks down. The value at $S_T =$ \$106 is \$23 instead of \$6, at $S_T =$ \$110 it's \$201 instead of \$10.

By loosening the restrictions further to allow errors of up to \$0.25, the following more reasonable piecewise linear function is a solution:

S_{T+X}	$C(S_{T+X})$
103	\$4
102	\$4
101	-\$4
100	\$4
99	-\$1
98	\$0
97	\$0

Discrete inquiry

To get a more general solution, we need to loosen the restrictions in another way, we must discretize the option. This is a different type of discretization than usual in option pricing. For binomial option pricing, we discretize in order to allow perfect hedging, since there are only two possible scenarios for each decision point, we can hedge any derivative with two instruments (typically the underlying and the risk-free asset). It is also common to make the problem discrete in order to program it on a computer or solve it by elementary mathematical methods. We discretize as an approximation, and make the discrete interval as small as possible or even let it go to a limit of zero.

In this case, we are redefining the option itself, to get a security we can solve forward. We need the discrete interval to be large enough to tame the dynamics of forward evolution. We rely on financial intuition to tell us that the value of the redefined option is similar to the actual option, so that our solution for the discrete option is a good solution for the actual option.

In this case it is convenient to restrict the underlying stock price to multiples of $4. However, we cannot simply round to the nearest $4, that would give unstable dynamics. Whenever Brownian motion crosses a barrier, it crosses it an infinite number of times. If we rounded to the neared $4, when the price crossed $2 it would switch from $0 to $4 an infinite number of times. Instead the discrete stock price will start at $100 (the same as the stock) and not move at all until the continuous stock price goes under $96 or above $104. At that time the discrete stock will switch to $96 or $104, and not change unless the continuous stock crosses another $4 boundary. The continuous stock price's infinite ups and downs from just below $104 to just above $104 do not affect the discrete stock price. Notice that the discrete stock price is a path-dependent function of the continuous stock price, which means that the option on the discrete stock price is a path-dependent option. This is the trick that will allow us to evolve the option price forward beyond expiry.

We cannot use Black-Scholes to compute the value of the option on the discrete stock price. However, we can easily compute the Black-Scholes option value if we round the option payoff to the nearest multiple of $4. An at-the-money one-year option with 32 per cent volatility and zero interest and payout rates is $12.71 for the continuous payout and $12.70 for the rounded payout. So moving from continuous to discrete at $4 intervals does not matter much. From a year away, the difference between the rounded and hysteretic discrete options should not be significant to an investor. The latter will just add a very nearly unbiased gamble to the option payoff.

Although the difference between the continuous option and the discrete option is small at time of origination, it can be large near expiry. For example, if the stock price is $102 the continuous option is worth a little more than $2, but the discrete option will be worth a little more than $0 or a little less than $4 depending on whether the stock was below $100 more recently than it was above $104.

Now some mathematical sleight of hand...

Now for some mathematical sleight of hand. We assume the market shuts down at some point before option expiry. At that point we have no price information about the stock. It remains closed after expiry for a period in which the stock has 1 per cent volatility. Upon reopening, the stock price is S_{T+X}. We want to find the fair settlement value for the option.

The following shows possible stock prices and continuous call values at time T, then possible stock prices and settlement prices at time $T+X$. Notice that for any S_T, C_T is equal to the average of the C_{T+X}s for $S_{T+X} = S_T + 1$ and $S_{T+X} = S_T - 1$. Since interest rates are zero and the volatility is $1 (I am using arithmetic steps instead of the more usual geometric for simplicity, but the result does not depend on it), this means that an investor is indifferent between receiving C_T at T or C_{T+X} at $T+X$.

S_T	C_T	S_{T+X}	C_{T+X}
110	10	110	10
109	9	109	8
108	8	108	8
107	7	107	8
106	6	106	6
105	5	105	4
104	4	104	4
103	3	103	4
102	2	102	2
101	1	101	0
100	0	100	0
99	0	99	0
98	0	98	0
97	0	97	0
96	0	96	0
95	0	95	0
94	0	94	0
93	0	93	0
92	0	92	0
91	0	91	0
90	0	90	0

C_{T+X} is merely the option price rounded to the nearest $4, with the difference split for stock prices exactly in the middle. Therefore the proper settlement for the hysteretic discrete option is simply the rounded discrete option payment. The market closure wiped out the path information needed for the hysteretic discrete option price, which is exactly compensated by the addition time to expiry.

How useful is this result? It depends on the balance between the discretization interval and the amount of volatility between expiry and market reopening. This is obviously unknown at the time the option is written. If the volatility is large relative to the total volatility over the life of the option, the discretization interval is large enough to affect the option value and the argument breaks down. Also, even if the discretization is negligible initially, it will be important near expiry both due to stock price and volatility changes. This, in turn, means the discretized settlement terms for market closure will no longer be fair, they will affect the trading price of the option.

I think these ideas are most useful for options that are not intended to trade frequently on relatively illiquid underlyings, for which market disruptions are common but reasonably short. I admit I have never found an application I would present as practical, but neither have I given up hope. At this point Dakota options are an interesting academic exercise. Playing with them can improve your applied mathematics skills and deepen your understanding of options before expiry. A breakthrough theorem that provided a general pricing insight would be a major accomplishment.

FOOTNOTE & REFERENCE

1. The intellectual firepower assembled for such a minor problem, and the enormously productive result, gives strong support to Gombaud's ideas of salon interaction.

■ Tikhonov, A.N. and V.Y.Arsenin (1977), *Solutions of Ill-posed Problems* Winston, Washington